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CUTS IN CYCLICALLY ORDERED SETS

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1. PRELIMINARY REMARKS

An ordered set is a pair \((G, <)\) where \(G\) is a set and \(<\) is an order on \(G\), i.e. an irreflexive and transitive binary relation on \(G\). We write briefly \(G\) instead of \((G, <)\) if the order \(<\) is given. If \(<\) is an order on \(G\), then the dual relation \(<^* = >\) is an order on \(G\). An element \(x \in G\) is called the least element of \((G, <)\) if \(x < y\) for any \(y \in G - \{x\}\); the greatest element is defined dually. If \((G, <)\) is an ordered set and \(H \subseteq G\), then \(< \cap H^2\) is an order on \(H\); this order is denoted by \(<_H\) or, briefly, also \(<\), and the subset \(H = (H, <)\) is called an ordered subset of the ordered set \(G = (G, <)\). An order \(<\) on a set \(G\) is linear iff \(x < y\) or \(y < x\) for any \(x, y \in G\), \(x \neq y\); in this case \((G, <)\) is called a linearly ordered set.

1.1. Definition. Let \((G, <_G), (H, <_H)\) be ordered sets with \(G \cap H = \emptyset\). An ordinal sum \(G \oplus H\) of ordered sets \(G, H\) is the set \(G \cup H\) with the binary relation \(<\) defined by \(x < y\) iff either \(x, y \in G\), \(x <_G y\) or \(x, y \in H\), \(x <_H y\) or \(x \in G, y \in H\).

It is known ([1]; but it is trivial to prove it) that \(<\) is an order on \(G \cup H\) so that \(G \oplus H\) is an ordered set. Further, the operation \(\oplus\) is associative so that the symbol \(G_1 \oplus G_2 \oplus \ldots \oplus G_n\) is defined, whenever \(G_1, \ldots, G_n\) are pairwise disjoint ordered sets.

1.2. Definition. Let \((G, <)\) be a linearly ordered set. A subset \(I \subseteq G\) is called an interval in \(G\) iff there exist subsets \(A, B\) of \(G\) with \(G = A \oplus I \oplus B\). A subset \(A \subseteq G\) is called an initial interval in \(G\) iff there exists a subset \(B\) of \(G\) with \(G = A \oplus B\). A final interval is defined dually.

The following assertion is known; however, it is not difficult to prove it directly:

1.3. Theorem. Let \((G, <)\) be a linearly ordered set. A subset \(I \subseteq G\) is an interval in \(G\) iff it has the following property: \(x, y \in I, z \in G, x < z < y \Rightarrow z \in I\). A subset \(A \subseteq G\) is an initial interval in \(G\) iff it has the following property: \(x \in A, y \in G, y < x \Rightarrow y \in A\). A subset \(B \subseteq G\) is a final interval in \(G\) iff it has the following property: \(x \in B, y \in G, x < y \Rightarrow y \in B\).
1.4. Definition. Let $G$ be a set, $T$ a ternary relation on $G$. This relation is called:

- asymmetric, iff $(x, y, z) \in T \Rightarrow (z, y, x) \in T$,
- cyclic, iff $(x, y, z) \in T \Rightarrow (y, z, x) \in T$,
- transitive, iff $(x, y, z) \in T, (x, z, u) \in T \Rightarrow (x, y, u) \in T$,
- linear, iff $x, y, z \in G, x \neq y \neq z \neq x \Rightarrow (x, y, z) \in T$ or $(z, y, x) \in T$.

1.5. Definition. Let $G$ be a set, $C$ a ternary relation on $G$ which is asymmetric, cyclic and transitive. Then $C$ is called a cyclic order on $G$ and the pair $(G, C)$ is called a cyclically ordered set. If, moreover, card $G \geq 3$ and $C$ is linear, it is called a linear cyclic order on $G$ and $(G, C)$ is called a linearly cyclically ordered set or a cycle.

If we define a dual relation $T^*$ to a ternary relation $T$ by $(x, y, z) \in T^* \iff (z, y, x) \in T$, then the following remark obviously holds:

1.6. Remark. If $C$ is a cyclic order on a set $G$, then $C^*$ is a cyclic order on $G$.

1.7. Theorem. Let $(G, C)$ be a cyclically ordered set, let $x \in G$. For any $y, z \in G$ put $y <_{C,x} z$ iff either $(x, y, z) \in C$ or $x = y \neq z$. Then $<_{C,x}$ is an order on $G$ with the least element $x$.

Proof. [4], 3.1.

1.8. Remark. Analogously we can define, for a cyclically ordered set $(G, C)$ and $x \in G : y <_{C,x} z \iff$ either $(y, z, x) \in C$ or $y \neq z = x$. Then $<_{C,x}$ is an order on $G$ with the greatest element $x$.

1.9. Lemma. If $C$ is a linear cyclic order on a set $G$, then $<_{C,x}$ is a linear order on $G$.

Proof. Trivial; see also [4], 3.4.

1.10. Theorem. Let $(G, <)$ be an ordered set. Define a ternary relation $C_<$ on $G$ by $(x, y, z) \in C_<$ iff either $x < y < z$ or $y < z < x$ or $z < x < y$. Then $C_<$ is a cyclic order on $G$.

Proof. [4], 3.5.

1.11. Lemma. Let $(G, <)$ be a linearly ordered set with card $G \geq 3$. Then $C_<$ is a linear cyclic order on $G$.

Proof. Trivial; see also [4], 3.7.

1.12. Lemma. Let $<$ be an order on a set $G$. Then $C_{<^*} = C_*^\preceq$.

Proof. Trivial.
2. DEFINITION OF A CUT

From now on, we shall deal only with linearly cyclically ordered sets. For the sake of brevity, we shall omit the adjective “linear”; thus, “cyclically ordered set” means always “linearly cyclically ordered set”.

A cut in a linearly ordered set is defined as a couple of its subsets. An analogue in a cyclically ordered set is impossible. Intuitively, a “section” of an oriented circle determines a linear ordering of points of that circle. This is a motivation for the following

2.1. Definition. Let \((G, C)\) be a cyclically ordered set. A cut on this set is a linear order \(<\) on \(G\) with the property \(x < y < z \Rightarrow (x, y, z) \in C\).

In 2.5 we shall see that cuts exist. Now we derive some simple properties of theirs.

2.2. Lemma. Let \((G, C)\) be a cyclically ordered set, let \(<\) be a cut on \((G, C)\), let \(x, y, z \in G\), \((x, y, z) \in C\). Then either \(x < y < z\) or \(y < z < x\) or \(z < x < y\).

Proof. Any of the remaining possibilities \(z < y < x\), \(y < x < z\), \(x < z < y\) implies \((z, y, x) \in C\) by definition of a cut, which contradicts \((x, y, z) \in C\).

2.3. Theorem. Let \((G, C)\) be a cyclically ordered set, let \(<\) be a linear order on \(G\). The order \(<\) is a cut on \((G, C)\) if and only if \(C_< = C\).

Proof. 1. Let \(<\) be a cut on \((G, C)\) and let \((x, y, z) \in C_<\). Then either \(x < y < z\) or \(y < z < x\) or \(z < x < y\), which implies (by the definition of a cut) \((x, y, z) \in C\).

Thus \(C_< \subseteq C\). As \(C_<\) is a linear cyclic order by 1.11 and \(C\) is linear, we have \(C_< = C\).

2. Let \(C_< = C\). If \(x, y, z \in G\), \(x < y < z\), then \((x, y, z) \in C_< = C\). Thus \(<\) is a cut on \((G, C)\).

2.4. Theorem. Let \((G, <)\) be a linearly ordered set with \(\text{card}\ G \geq 3\). Then there exists just one cyclic order \(C\) on \(G\) such that \(<\) is a cut on \((G, C)\).

Proof. Existence: Put \(C = C_<\). By 1.11, \(C\) is a cyclic order on \(G\) and by 2.3, \(<\) is a cut on \((G, C)\).

Unicity: Let \(C_1, C_2\) be cyclic orders on \(G\) for which \(<\) is a cut. Let \((x, y, z) \in C_1\).

By 2.2 we have either \(x < y < z\) or \(y < z < x\) or \(z < x < y\), which implies \((x, y, z) \in C_2\) by 2.1. Thus \(C_1 \subseteq C_2\) and as the both relations \(C_1, C_2\) are linear, we obtain \(C_1 = C_2\).

2.5. Theorem. Let \((G, C)\) be a cyclically ordered set, let \(x \in G\). Then \(<_{c,x}\) is a cut on \((G, C)\).

Proof. By 1.9, \(<_{c,x}\) is a linear order on \(G\). Let \(u, v, w \in G\), \(u <_{c,x} v <_{c,x} w\).

First assume \(x \in \{u, v, w\}\). Then \((x, u, v) \in C\), \((x, v, w) \in C\), thus \((v, w, x) \in C\), \((v, x, u) \in C\) and by transitivity of \(C\), \((v, w, u) \in C\) and hence \((u, v, w) \in C\). If \(x \in \{u, v, w\}\), then \(x = u\) and as \(v <_{c,x} w\), we have \((x, v, w) \in C\), i.e. \((u, v, w) \in C\).
Thus we always have \( u <_{c,x} v <_{c,x} w \Rightarrow (u, v, w) \in C \) and \( <_{c,x} \) is a cut on \((G, C)\).

Dually, we can prove:

2.6. Remark. Let \((G, C)\) be a cyclically ordered set, let \(x \in G\). Then \( <_{c,x} \) is a cut on \((G, C)\).

The both orders \( <_{c,x}, <_{c,x} \) are thus cuts on \((G, C)\) and by their definitions, \( <_{c,x} \) has the least element, \( <_{c,x} \) the greatest element. Other cuts with this property do not exist, for:

2.7. Theorem. Let \((G, C)\) be a cyclically ordered set, let \(<\) be a cut on \((G, C)\) with the least element \(x\). Then \(< = <_{c,x}\).

Proof. Let \(y, z \in G - \{x\}, y < z\). Then \(x < y < z\) and, by definition of a cut, \((x, y, z) \in C\). Hence \(y <_{c,x} z\). Further, \(x\) is the least element in both \((G, <)\) and \((G, <_{c,x})\). We have shown that \(< \subseteq <_{c,x}\) and as the both orders are linear, we have \(< = <_{c,x}\).

Of course, dually we have:

2.8. Remark. Let \((G, C)\) be a cyclically ordered set, let \(<\) be a cut on \((G, C)\) with the greatest element \(x\). Then \(< = <_{c,x}\).

3. PROPERTIES OF CUTS

3.1. Definition. Let \((G, C)\) be a cyclically ordered set, let \(<\) be a cut on \((G, C)\). This cut is called:

a. jump, iff \((G, <)\) has both the least and the greatest element,

a. gap, iff \((G, <)\) has neither the least nor the greatest element,

Dedekind, iff \((G, <)\) has just one of the boundary elements.

3.2. Definition. A cyclically ordered set \((G, C)\) is called dense iff there exists no jump on \((G, C)\).

As one can expect, it holds:

3.3. Theorem. A cyclically ordered set \((G, C)\) is dense iff it has the following property: \(x, y \in G, x \neq y \Rightarrow there exists \ z \in G with (x, z, y) \in C\).

Proof. 1. Assume that for any \(x, y \in G, x \neq y\) there exists \(z \in G\) with \((x, z, y) \in C\) and let \(<\) be a jump on \((G, C)\) with the least element \(y\) and the greatest element \(x\). By 2.7. we obtain \(< = <_{c,y}\) and by the assumption an element \(z \in G\) exists with \((x, z, y) \in C\). Then \((y, x, z) \in C\) which implies \(x <_{c,y} z\), i.e. \(x < z\) and this is a contradiction, for \(x\) is the greatest element in \((G, <)\). Thus, \((G, C)\) contains no jumps and it is dense.

2. Let elements \(x, y \in G, x \neq y\) exist so that \((x, z, y) \in C\) holds for no \(z \in G\). Then \(<_{c,y}\) is a cut on \((G, C)\) with the least element \(y\); we show that \(x\) is its greatest element. When an element \(z \in G\) exists with \(x <_{c,y} z\), then \((y, x, z) \in C\) and also
(x, z, y) ∈ C which contradicts our assumption. Thus <c,y is a jump on (G, C) and (G, C) is not dense.

3.4. Definition. Let (G, C) be a cyclically ordered set, let x, y ∈ G, x ≠ y. The ordered pair (x, y) is called a pair of consecutive elements in (G, C) iff there exists no z ∈ G with (x, z, y) ∈ C.

Note that by 3.3, (G, C) is dense iff it contains no pair of consecutive elements.

3.5. Lemma. Let (G, C) be a cyclically ordered set, let (x, y) be a pair of consecutive elements in (G, C) and let < be any cut on (G, C). Then just one of the following possibilities occurs:
1. y is the least and x is the greatest element in (G, <);
2. y covers x in (G, <).

Proof. If < = <c,y or < = <c,x, then by the same argument as in the proof of 3.3 we find that (1) holds. In all the other cases x is not the greatest element in (G, <). Suppose y < x; then there exists z ∈ G with y < x < z, which implies (y, x, z) ∈ C and (x, z, y) ∈ C, a contradiction. Hence x < y and there exists no z ∈ G with x < z < y, for otherwise (x, z, y) ∈ C. This means that y covers x in (G, <).

3.6. Theorem. Let (G, C) be a cyclically ordered set and let <1, <2 be two distinct cuts on (G, C). Then there exist nonempty disjoint subsets A, B of G such that A ∪ B = G, <1|A = <2|A, <1|B = <2|B and (G, <1) = A ⊕ B, (G, <2) = B ⊕ A.

Proof. First observe that <2 = <1 is impossible for in that case C<2 = C<1, by 1.12, while necessarily C<1 = C = C<2 by 2.3. Thus there exist elements x, y ∈ G such that x <1 y, x <2 y so that there exist nonempty subsets H ⊆ G with <1|H = <2|H. Denote this property of subsets of G by (P). If S is a chain (with respect to set inclusion) of (P)-subsets of G, then the set-theoretic union ∪S is a (P)-subset; so, by Zorn’s lemma, there exists a maximal (P)-subset A ⊆ G. We show that A is an interval in (G, <1). Let x, y ∈ A, z ∈ G, x <1 z <1 y. Then (x, z, y) ∈ C so that either x <2 z <2 y or z <2 y <2 x or y <2 x <2 z. The second and the third cases are impossible, since x <2 y. Thus x <2 z <2 y. Let u ∈ A be any element with u <1 z. If u = x, then u <2 z. If u <1 x, then u <1 x <1 z, thus (u, x, z) ∈ C, which implies either u <2 x <2 z or x <2 z <2 u or z <2 u <2 x. The second case is impossible, since u <2 x (u, x ∈ A and <1|A = <2|A), the third one is also impossible, since x <2 z. If x <1 u, then u <1 z <1 y, thus (u, z, y) ∈ C and hence either u <2 z <2 y or z <2 y <2 u or y <2 u <2 z. The second and the third cases are impossible, since u <2 y and z <2 y. We have shown u <1 z ⇒ u <2 z. By a similar argument we find u ∈ A, z <1 u ⇒ z <2 u. It follows that A ∪ {z} is a (P)-subset and the maximality of A implies z ∈ A. Note that for the same reason A is an interval also in (G, <2).
As \( A \) is an interval in \((G, <_1)\), we have \( x \in G - A, x <_1 y \) for some \( y \in A \) \( \Rightarrow x <_1 z \) for each \( z \in A \); the same holds for \(<_2\). This yields:

\( x \in G - A, x <_1 y \) for some \( y \in A \) \( \Rightarrow y <_2 x \). (\( * \))

Otherwise there would exist \( x \in G - A, y \in A \) with \( x <_1 y, x <_2 y \) and then \( x <_1 z, x <_2 z \) for each \( z \in A \), thus \( A \cup \{ x \} \) is a (P)-subset, which contradicts the maximality of \( A \).

Suppose now that \( A \) is neither an initial nor a final interval in \((G, <_1)\). Then \((G, <_1) = (H, <_1) \oplus (A, <_1) \oplus (K, <_1)\) with \( H \neq \emptyset, K \neq \emptyset \). Choose \( x \in H, y \in A, z \in K \). Then \( x <_1 y <_1 z \) and (\( * \)) implies \( z <_2 y <_2 x \). This is a contradiction, for \( x <_1 y <_1 z \) implies \( (x, y, z) \in C \) and \( z <_2 y <_2 x \) implies \( (y, x, z) \in C \). Thus \( A \) is an initial or a final interval in \((G, <_1)\) and for the same reason it is an initial or a final interval also in \((G, <_2)\).

Put \( B = G - A; B \) is a final or an initial interval both in \((G, <_1)\) and in \((G, <_2)\), and we show that \( <_1|_B = <_2|_B \). Assume the existence of elements \( x, y \in B \) with \( x <_1 y, y <_2 x \). Choose any \( z \in A \); if \( A \) is an initial interval in \((G, <_1)\), then \( z <_1 \) \( x <_1 y \) and from (\( * \)) we have \( y <_2 x <_2 z \). This is a contradiction, for \( z <_1 x <_1 y \) implies \( (x, x, y) \in C \) and \( y <_2 x <_2 z \) implies \( (y, x, z) \in C \). If \( A \) is a final interval in \((G, <_1)\), then \( x <_1 y <_1 z \) and \( z <_2 y <_2 x \), which leads to a contradiction as well.

Assume that \( A \) is an initial interval both in \((G, <_1)\) and in \((G, <_2)\). Then \((G, <_1) = (A, <_1) \oplus (B, <_1), (G, <_2) = (A, <_2) \oplus (B, <_2)\) and as \((A, <_1) = (A, <_2), (B, <_1) = (B, <_2), (G, <_1) = (G, <_2) = (B, <_2) \oplus (A, <_2)\). If \( A \) is a final interval in \((G, <_1)\), it is an initial interval in \((G, <_2)\) and the given equality holds after interchanging the sets \( A, B \).

3.7. Remark. The sets \( A, B \) from 3.6 are unique.

Proof. Assume \((G, <_1) = A \oplus B, (G, <_2) = B \oplus A\) and, at the same time, \((G, <_1) = A_1 \oplus B_1, (G, <_2) = B_1 \oplus A_1\). As \( A, A_1 \) are initial intervals of the linearly ordered set \((G, <_1)\), either \( A \subseteq A_1 \) or \( A_1 \subseteq A \) holds; let the first possibility occur. Suppose \( A \neq A_1 \); if we choose arbitrary elements \( x \in A_1 - A \) and \( y \in B_1 \), then \( x <_2 y \) in \((B, <_2) \oplus (A, <_2)\) and \( y <_2 x \) in \((B_1, <_2) \oplus (A_1, <_2)\). This is a contradiction and hence \( A = A_1 \).

3.8. Lemma. Let \( G \) be a set with \( \text{card} \ G \geq 3 \). Let \( <_1, <_2 \) be linear orders on \( G \) such that there exist disjoint subsets \( A, B \) of \( G \) with \( A \cup B = G, <_1|_A = <_2|_A, <_1|_B = <_2|_B \) and \((G, <_1) = A \oplus B, (G, <_2) = B \oplus A\). Then there exists just one cyclic order \( C \) on \( G \) such that \( <_1, <_2 \) are cuts on \((G, C)\).

Proof. The uniqueness follows from 2.4. For the existence it suffices to prove \( C_{<_1} = C_{<_2}\). Let \((x, y, z) \in C_{<_1}\). Then either \( x <_1 y <_1 z \) or \( y <_1 z <_1 x \) or \( z <_1 x <_1 y \). We investigate only the first case; the second and the third one are
similar. We have the following possibilities:
\[ x, y, z \in A \Rightarrow x \leq y \leq z \Rightarrow (x, y, z) \in C_{<_{_{2}}} ; \]
\[ x, y \in A, z \in B \Rightarrow z \leq x \leq y \Rightarrow (x, y, z) \in C_{<_{_{2}}} ; \]
\[ x \in A, y, z \in B \Rightarrow y \leq z \leq x \Rightarrow (x, y, z) \in C_{<_{_{2}}} ; \]
\[ x, y, z \in B \Rightarrow x \leq y \leq z \Rightarrow (x, y, z) \in C_{<_{_{2}}} . \]
Thus we have shown \( C_{<_{_{1}}} \subseteq C_{<_{_{2}}} \) and as both cyclic orders \( C_{<_{_{1}}}, C_{<_{_{2}}} \) are linear, we conclude \( C_{<_{_{1}}} = C_{<_{_{2}}} \).

3.9. Corollary. Let \( G \) be a set with \( \text{card } G \geq 3 \), let \( <_{_{1}}, <_{_{2}} \) be distinct linear orders on \( G \). Then \( C_{<_{_{1}}} = C_{<_{_{2}}} \) holds if and only if there exist nonempty disjoint subsets \( A, B \) of \( G \) with \( A \cup B = G \), \( <_{_{1}}|_{A} = <_{_{2}}|_{A} \), \( <_{_{1}}|_{B} = <_{_{2}}|_{B} \) and \( (G, <_{_{1}}) = (G, <_{_{2}}) = A \oplus B, (G, <_{_{2}}) = B \oplus A \).

Proof. If \( C_{<_{_{1}}} = C_{<_{_{2}}} \), then, by 2.3, \( <_{_{1}}, <_{_{2}} \) are two distinct cuts on a cyclically ordered set \((G, C)\) where \( C = C_{<_{_{1}}} = C_{<_{_{2}}} \). By 3.6, the orders \( <_{_{1}}, <_{_{2}} \) have the desired properties. Conversely, if the condition of Corollary is satisfied, then, by 3.8 and 2.3, \( C_{<_{_{1}}} = C_{<_{_{2}}} \) holds.

If \((G, <)\) is a linearly ordered set and \( x \in G \), then we denote by \((G, <)_{x}\) or, briefly, \( G_{x} \), the open initial interval in \((G, <)\) determined by the element \( x \), i.e. \( G_{x} = \{ y \in G; y < x \} \).

3.10. Lemma. Let \((G, C)\) be a cyclically ordered set, let \( < \) be a cut on \((G, C)\) and let \( x \in G \). Then \((G, <_{c,x}) = (G - (G, <)_{x}) \oplus (G, <)_{x} \).

Proof. If \( x \) is the least element in \((G, <)\), then \( < = <_{c,x} \) and the formula holds, since \((G, <)_{x} = 0 \). Otherwise \( <, <_{c,x} \) are distinct cuts on \((G, C)\) and by 3.6 there exist nonempty disjoint subsets \( A, B \) of \( G \) with \( A \cup B = G \), \( <_{A} = <_{c,x}|_{A} \), \( <_{B} = = <_{c,x}|_{B} \) and \((G, <) = A \oplus B, (G, <_{c,x}) = B \oplus A \). Then \( A \) is an initial interval in \((G, <)\) and \((G, <_{c,x}) = B \oplus A \) implies that \( B \) has the least element \( x \). Thus \( A = = (G, <)_{x} \) and \( B = G - (G, <)_{x} \).

3.11. Theorem. Let \((G, C)\) be a cyclically ordered set and let \( <_{_{1}}, <_{_{2}}, <_{_{3}} \) be three pairwise distinct cuts on \((G, C)\). Then there exist three nonempty pairwise disjoint subsets \( A, B, D \) of \( G \) such that \( A \cup B \cup D = G \), \( <_{_{1}}|_{A} = <_{_{2}}|_{A} = <_{_{3}}|_{A} \), \( <_{_{1}}|_{B} = <_{_{2}}|_{B} = <_{_{3}}|_{B} \), and either \((G, <_{_{1}}) = A \oplus B \oplus D, (G, <_{_{2}}) = B \oplus D \oplus A, (G, <_{_{3}}) = D \oplus A \oplus B \) or \((G, <_{_{1}}) = A \oplus B \oplus D, (G, <_{_{2}}) = B \oplus D \oplus A, (G, <_{_{3}}) = D \oplus A \oplus B \) holds.

Proof. By 3.6 there exist nonempty disjoint subsets \( A_{1}, B_{1}, D_{1} \) of \( G \) with \( A_{1} \cup B_{1} = G, <_{_{1}}|_{A_{1}} = <_{_{2}}|_{A_{1}}, <_{_{1}}|_{B_{1}} = <_{_{2}}|_{B_{1}}, (G, <_{_{1}}) = A_{1} \oplus B_{1}, (G, <_{_{2}}) = B_{1} \oplus A_{1} \), and there exist nonempty disjoint subsets \( A_{2}, B_{2}, D_{2} \) of \( G \) with \( A_{2} \cup B_{2} = G, <_{_{1}}|_{A_{2}} = <_{_{3}}|_{A_{2}}, <_{_{1}}|_{B_{2}} = <_{_{3}}|_{B_{2}}, (G, <_{_{1}}) = A_{2} \oplus B_{2}, (G, <_{_{3}}) = B_{2} \oplus A_{2} \). As \( A_{1}, A_{2} \) are initial
intervals of the linearly ordered set \((G, <)\), we have either \(A^1 \subseteq A^2\) or \(A^2 \subseteq A^1\). The inclusion here is proper, for if \(A^1 = A^2\), then \(B^1 = B^2\) so that \(<2 = <3\), which contradicts our assumption.

1. Let \(A^1 \subset A^2\). Consider the sets \(A^1, A^2 - A^1, B_1\). As \(<1|A^1 = <2|A^1, <1|A^2 = <3|A^2, A^1 \subset A^2\), we have \(<1|A^1 = <2|A^1 = <3|A^1\). Further, \(<1|A^2 - A^1 = <3|A^2 - A^1\), and as \(A^2 - A^1 \subseteq B_1\), we have \(<1|A^2 - A^1 = <2|A^2 - A^1\). Thus \(<1|A^2 - A^1 = <2|A^2 - A^1 = <3|A^2 - A^1\). Finally, we have \(B_2 \subseteq B_1\) and hence \(<1|B^2 = <2|B^2 = <3|B^2\). Consequently, \(<1|B^2 = <2|B^2 = <3|B^2\). Now, we have

\[
(G, <1) = A^1 \oplus (A^2 - A^1) \oplus B_2,
(G, <2) = (A^2 - A^1) \oplus B_2 \oplus A^1,
(G, <3) = B_2 \oplus A^1 \oplus (A^2 - A^1).
\]

2. Let \(A^2 \subset A^1\). By an analogous reasoning we find

\[
<1|A^2 = <2|A^2 = <3|A^2, <1|A^1 - A^2 = <2|A^1 - A^2 = <3|A^1 - A^2,
<1|B^1 = <2|B^1 = <3|B^1
\]

and

\[
(G, <3) = (A^1 - A^2) \oplus B_1 \oplus A^2,
(G, <2) = B_1 \oplus A^2 \oplus (A^1 - A^2),
(G, <1) = A^2 \oplus (A^1 - A^2) \oplus B_1.
\]

4. CYCLIC ORDERING OF CUTS

4.1. Definition. Let \((G, C)\) be a cyclically ordered set, let \(<1, <2, <3\) be three pairwise distinct cuts on \((G, C)\). Put \((<1, <2, <3) \in \mathcal{C}\) iff there exist three nonempty pairwise disjoint subsets \(A, B, D\) of \(G\) such that \(A \cup B \cup D = G, <1|A = <2|A = <3|A, <1|B = <2|B = <3|B, <1|D = <2|D = <3|D,\) and \((G, <1) = A \oplus B \oplus D, (G, <2) = B \oplus D \oplus A, (G, <3) = D \oplus A \oplus B.\)

4.2. Theorem. Let \((G, C)\) be a cyclically ordered set and let \(\mathcal{C}\) be the set of all cuts on \((G, C)\). Then \(\mathcal{C}\) is a cyclic order on the set \(\mathcal{C}\).

Proof. Suppose that there exist pairwise distinct cuts \(<1, <2, <3\) on \((G, C)\) with \((<1, <2, <3) \in \mathcal{C}, (<3, <2, <1) \in \mathcal{C}\). Then there exist nonempty pairwise disjoint subsets \(A, B, D\) of \(G\) with \(A \cup B \cup D = G, <1|A = <2|A = <3|A, <1|B = <2|B = <3|B, <1|D = <2|D = <3|D, (G, <1) = A \oplus B \oplus D, (G, <2) = B \oplus D \oplus A, (G, <3) = D \oplus A \oplus B,\) and nonempty pairwise disjoint subsets \(A^1, B^1, D^1\) of \(G\) with \(A^1 \cup B^1 \cup D^1 = G, <1|A^1 = <2|A^1 = <3|A^1, <1|B^1 = <2|B^1 = <3|B^1, <1|D^1 = <2|D^1 = <3|D^1, (G, <3) = A^1 \oplus B^1 \oplus D^1, (G, <2) = B^1 \oplus D^1 \oplus A^1, (G, <1) = D^1 \oplus A^1 \oplus B^1.\) Then \(B \oplus D \oplus A = B^1 \oplus D^1 \oplus A^1 = (G, <2)\), and
hence either $B \subseteq B_1$ or $B_1 \subseteq B$. Let $B \subseteq B_1$; if $B \subseteq B_1$, choose $x \in B$, $y \in (B_1 - B) \cap D$. Then $(G, \prec_3) = A_1 \oplus B_1 \oplus D_1$ implies $x <_3 y$ and $(G, \prec_3) = D \oplus A \oplus B$ implies $y <_3 x$. This is a contradiction. Analogously $B_1 \subseteq B$ is impossible and thus $B = B_1$. Now we have $(G, \prec_1) = A \oplus B \oplus D$, $(G, \prec_1) = D_1 \oplus A \oplus B$, which implies $D = 0$, $D_1 = 0$ and this is a contradiction. The relation $\mathcal{C}$ is thus asymmetric.

Assume $<_1, <_2, <_3, <_4 \in \mathcal{C}$, $(<_1, <_2, <_3) \in \mathcal{C}$, $(<_1, <_3, <_4) \in \mathcal{C}$. Then there exist nonempty disjoint subsets $A, B, D$ of $G$ with $A \cup B \cup D = G$, $<_1|_A = <_2|_A = <_3|_A$, $<_1|_B = <_2|_B = <_3|_B$, $<_1|_D = <_2|_D = <_3|_D$, $(G, <_1) = A \oplus B \oplus D$, $(G, <_2) = B \oplus D \oplus A$, $(G, <_3) = D \oplus A \oplus B$, and nonempty disjoint subsets $A_1, B_1, D_1$ of $G$ with $A_1 \cup B_1 \cup D_1 = G$, $<_1|_{A_1} = <_3|_{A_1} = <_4|_{A_1}$, $<_1|_{B_1} = <_3|_{B_1} = <_4|_{B_1}$, $<_1|_{D_1} = <_3|_{D_1} = <_4|_{D_1}$, $(G, <_1) = A_1 \oplus B_1 \oplus D_1$, $(G, <_3) = B_1 \oplus D_1 \oplus A_1$, $(G, <_4) = D_1 \oplus A_1 \oplus B_1$. As $A \oplus B \oplus D = A_1 \oplus B_1 \oplus D_1 = (G, <_1)$, we have either $A \subseteq A_1$ or $A_1 \subseteq A$. The equality $A = A_1$ is impossible, for in that case $D \oplus A \oplus B = B_1 \oplus D_1 \oplus A_1 = (G, <_3)$, which implies $B = 0$, a contradiction. Suppose $A_1 \subseteq A$; if we choose $x \in A_1$, $y \in A - A_1$, then $(G, <_3) = D \oplus A \oplus B$ implies $x <_3 y$ and $(G, <_3) = B_1 \oplus D_1 \oplus A_1$ implies $y <_3 x$. This is a contradiction and thus $A \subseteq A_1$. Further, we have either $A_1 \subseteq A \oplus B$ or $A \oplus B \subseteq A_1$. If $A_1 \subseteq A \oplus B$, choose $x \in A$, $y \in B - A_1$. Then $(G, <_3) = D \oplus A \oplus B$ implies $x <_3 y$ and $(G, <_3) = B_1 \oplus D_1 \oplus A_1$ implies $y <_3 x$, which is impossible. If $A \oplus B \subseteq A_1$, choose $x \in A \oplus B$, $y \in A - (A \oplus B)$. Then $(G, <_3) = B_1 \oplus D_1 \oplus A_1$ implies $x <_3 y$ and $(G, <_3) = D \oplus A \oplus B$ implies $y <_3 x$, which is a contradiction. Thus $A_1 = A \oplus B$ and from $A \oplus B \oplus D = A_1 \oplus D = A_1 \oplus B_1 \oplus D_1 = (G, <_1)$ we have $D = B_1 \oplus D_1$. Now, we have $(G, <_1) = A \oplus (B \oplus B_1) \oplus D_1$, $(G, <_2) = (B \oplus B_1) \oplus D_1 \oplus A$, $(G, <_4) = D_1 \oplus A \oplus (B \oplus B_1)$. This implies $(<_1, <_2, <_4) \in \mathcal{C}$ and the relation $\mathcal{C}$ is transitive. It follows directly from the definition that $\mathcal{C}$ is cyclic. Finally, if $<_1$, $<_2$, $<_3 \in \mathcal{C}$ are pairwise distinct, then 3.11 implies either $(<_1, <_2, <_3) \in \mathcal{C}$ or $(<_3, <_2, <_1) \in \mathcal{C}$. Thus $\mathcal{C}$ is linear and it is a cyclic order on $\mathcal{C}$.

4.3. Lemma. Let $(G, C)$ be a cyclically ordered set and let $<_1, <_2, <_3 \in \mathcal{C}$. Then $(<_1, <_2, <_3) \in C$ holds if and only if there exist elements $x, y, z \in G$ with $x <_1 y <_1 z$, $y <_2 z <_2 x$, $z <_3 x <_3 y$.

Proof. Let $(<_1, <_2, <_3) \in \mathcal{C}$. If $A, B, D$ are subsets of $G$ with the properties from 4.1, choose $x \in A$, $y \in B$, $z \in C$. Then $x <_1 y <_1 z$, $y <_2 z <_2 x$, $z <_3 x <_3 y$. Conversely, let there exist elements $x, y, z \in G$ with $x <_1 y <_1 z$, $y <_2 z <_2 x$, $z <_3 x <_3 y$. Then the cuts $<_1, <_2, <_3$ are pairwise distinct and thus there exist subsets $A, B, D$ of $G$ with the properties from 3.11. Elements $x, y, z$ must lie in the distinct sets $A, B, D$, since the orders $<_1, <_2, <_3$ coincide on these sets. If the second case from 3.11 occurred, we should obtain in all possible situations always a contradiction. Thus the first case of 3.11 occurs and $(<_1, <_2, <_3) \in \mathcal{C}$.

4.4. Theorem. Let $(G, C)$ be a cyclically ordered set and let $x, y, z \in G$, $x \neq y \neq z \neq x$. Then $(x, y, z) \in C$ holds if and only if $(<_c x, <_c y, <_c z) \in \mathcal{C}$. 330
Proof. \((x, y, z) \in C\) implies \(x <_{c,x} y <_{c,x} z, y <_{c,y} z <_{c,y} x, z <_{c,z} x <_{c,z} y\) and from 4.3 we have \((<_{c,x}, <_{c,y}, <_{c,z}) \in \mathcal{C}\). Conversely, let \((<_{c,x}, <_{c,y}, <_{c,z}) \in \mathcal{C}\) and assume \((x, y, z) \in C\). Then \((z, y, x) \in C\) and from the first step of the proof we have \((<_{c,z}, <_{c,y}, <_{c,x}) \in \mathcal{C}\), which is a contradiction. Thus \((x, y, z) \in C\).

4.5. Corollary. Let \((G, C)\) be a cyclically ordered set. Then \((\{<_{c,x}; x \in G\}, \mathcal{C})\) is a cyclically ordered set isomorphic with \((G, C)\).

Proof. \((\{<_{c,x}; x \in G\}, \mathcal{C})\) is — as a subset of \((G, \mathcal{C})\) — cyclically ordered. The mapping \(G \to \{<_{c,x}; x \in G\}\) assigning to any \(x \in G\) the cut \(<_{c,x}\) is evidently a bijection; by 4.4 it is an isomorphism.

5. COMPLETION BY CUTS

5.1. Definition. A cyclically ordered set is called complete, iff it contains no gaps.

Note that "complete" has another meaning here than in [4].

5.2. Theorem. Let \((G, C)\) be a cyclically ordered set. Then the cyclically ordered set \((\mathcal{G}, \mathcal{C})\) is complete.

Proof. Let \(<\) be a cut on \((\mathcal{G}, \mathcal{C})\). Define a linear order \(<\) on \(G\) by \(x < y \iff \langle_{c,x} < \langle_{c,y}\). The relation \(<\) is indeed a linear order on \(G\), for \(<\) is a linear order on \(\mathcal{G}\), thus also on \(\{<_{c,x}; x \in G\}\) and as a consequence of the bijection \(x \to <_{c,x}, <\) is a linear order. We show that \(<\) is a cut on \((G, C)\). Let \(x, y, z \in G\), \(x < y < z\). Then \(<_{c,x} < <_{c,y} < <_{c,z}\), thus \((<_{c,x}, <_{c,y}, <_{c,z}) \in \mathcal{C}\) and by 4.4, \((x, y, z) \in C\). Thus \(<\) \(\in \mathcal{G}\).

Suppose that \(<\) is neither the least nor the greatest element in \((\mathcal{G}, <)\). Then there exist \(<_1, <_2 \in \mathcal{G}\) such that \(<_1 < <_2 < <_2\). This implies \((<_1, <, <_2) \in \mathcal{C}\) and by 4.1 there exist nonempty disjoint subsets \(A, B, D\) of \(G\) such that \(A \cup B \cup D = G\), \(<_1|_A = <_1|_A = <_2|_A, <_1|_B = <_2|_B, <_1|_D = <_2|_D\) and \((G, <_1) = A \oplus B \oplus D, (G, <_2) = B \oplus D \oplus A, (G, <) = D \oplus A \oplus B\). Choose elements \(x \in A, y \in B\). We show that \(<_{c,x} < <_{c,y}\). If \(<_{c,x} = <_1\) and \(<_{c,y} = <_2\), then the desired relation holds. Let \(<_{c,x} \neq <_1\). Then \(A = A_x \oplus (A - A_x)\) and as \((G, <_1)_x = A_x\), 3.10 implies \((G, <_1) = A_x \oplus (A - A_x) \oplus (B \oplus D)\). Consequently, \((<_1, <_{c,x}, <) \in \mathcal{C}\) and hence either \(<_1 < <_{c,x} < <_1\) or \(<_{c,x} < <_1 < <_1\). The second and the third case are impossible for \(<_1 < <_1\). Thus \(<_1 < <_{c,x} < <_1\). Analogously, if \(<_{c,y} \neq <_2\), then we find \(< <_{c,y} < <_2\). Thus in all cases we have \(<_1 \leq <_{c,x} < <_1 < <_{c,y} \leq <_2\) and hence \(<_{c,x} < <_{c,y}\).

But \((G, <) = B \oplus D \oplus A\) and \(y \in B, x \in A\), thus \(y < x\). This contradicts the definition of the order \(<\).
5.3. **Corollary.** Let \((G, C)\) be a cyclically ordered set. Then there exists a complete cyclically ordered set \((H, D)\) containing an isomorphic subset with \((G, C)\).

**Proof** follows from 5.2 and 4.5.

5.4. **Lemma.** Let \((G, C)\) be a cyclically ordered set, let \(x \in G\). Then \((<c,x, <c,x'> <c,x', <c,x'>)\) is a pair of consecutive elements in \((G, C)\).

**Proof.** Let \(<\) be any cut on \((G, C)\) distinct from both \(<c,x\) and \(<c,x'>. By 3.6 there exist nonempty disjoint subsets \(A, B\) of \(G\) with \(A \cup B = G\), \(<c,x\) \(A = \{A\}\), \(<c,x\) \(B = \{\}\) and \((G, <c,x) = A \oplus B\). As \(<c,x\) \(A \neq {}\), we have \(A \neq \{x\}\). Now we have \((G, <c,x) = \{x\} \oplus (A - \{x\}) \oplus B\), \((G, <c,x) = (A - \{x\}) \oplus B \oplus \{x\}\), \((G, <) = B \oplus \{x\} \oplus (A - \{x\})\) so that \((<c,x, <c,x', <c,x') \in \mathcal{G}\). Thus \((<c,x, <c,x', <c,x') \in \mathcal{G}\) holds for no cut \(< \in \mathcal{G}\) and, therefore, \((<c,x, <c,x')\) is a pair of consecutive elements in \((G, C)\).

Note that 5.4 implies that \((\mathcal{G}, C)\) is never dense.

5.5. **Notation.** Let \((G, C)\) be a cyclically ordered set. Denote \(\mathcal{G}_r = \{< \in \mathcal{G}; <\text{ is a gap}\} \cup \{<c,x; x \in G\}\); the elements of \(\mathcal{G}_r\) will be called regular cuts.

\(\mathcal{G}_r\) thus contains all jumps and all gaps in \((G, C)\) and from Dedekind cuts it contains only those which have the least element. As a subset of \(\mathcal{G}\), \((\mathcal{G}_r, C)\) is a cyclically ordered set and by 4.5, \(x \rightarrow <c,x\) is an isomorphic embedding of \((G, C)\) into \((\mathcal{G}_r, C)\).

5.6. **Theorem.** Let \((G, C)\) be a cyclically ordered set. Then the cyclically ordered set \((\mathcal{G}_r, C)\) is complete.

**Proof.** Let \(<\) be a cut on \((\mathcal{G}_r, C)\). This cut in a natural way determines a cut on \((\mathcal{G}, C)\), which we denote by the same symbol \(<\) : any cut from \(\mathcal{G} - \mathcal{G}_r\) is of the form \(<c,x; for such a cut we put \(<c,x <c,x'\); if \(y \in G\), \(y \neq x\), then \(<c,y <c,x <c,x'\). It is not difficult to show that \(<\) is indeed a cut on \((\mathcal{G}, C)\). By 5.2 there exists a cut \(< < \in \mathcal{G}\) which is either the least or the greatest element in \((\mathcal{G}, <)\). If \(< \in \mathcal{G}_r\), then \(\mathcal{G}_r\) has either the least or the greatest element. If \(< \in \mathcal{G}_r\), then \(< <c,x\) for some \(x \in G\). In this case, by 5.4, \((<c,x, <c,x')\) is a pair of consecutive elements in \((\mathcal{G}, C)\) and by 3.5 we have: (1) either \(<c,x\) is the least and \(<c,x'\) the greatest element in \((\mathcal{G}, <)\), (2) or \(<c,x'\) covers \(<c, x\) in \((\mathcal{G}, <)\). If (1) holds, then \(<c,x\) is the greatest element in \((\mathcal{G}_r, <)\). If (2) holds, then \(<c,x'\) cannot be the least element in \((\mathcal{G}, <)\), for it covers \(<c, x\). Therefore \(<c,x\) is the greatest element in \((\mathcal{G}, <)\) and then \(<c,x\) is the greatest element in \((\mathcal{G}_r, <)\). Thus no cut on \((\mathcal{G}_r, C)\) is a gap and \((\mathcal{G}_r, C)\) is complete.

If \((G, C)\) is a cyclically ordered set, then \((\mathcal{G}_r, C)\) will be called its completion by cuts.

5.7. **Theorem.** Let \((G, C)\) be a cyclically ordered set. If \((G, C)\) is dense, then \((\mathcal{G}_r, C)\) is dense.
Proof. Let \( <1, <2 \in \mathcal{G}, <1 \neq <2 \). If \( <1 = <c, x, <2 = <c, y \) for some \( x, y \in G \), then \( x \neq y \) and by the assumption, there exists \( z \in G \) such that \( (x, z, y) \in C \). Then 4.4 yields \( (<1, <c, z, <2) \in \mathcal{C} \). Assume now that at least one of the cuts \( <1, <2 \) is a gap. By 3.6 there exist nonempty disjoint subsets \( A, B \) of \( G \) with \( A \cup B = G, <1|_A = <2|_A, <1|_B = <2|_B \) and \( (G, <1) = A \oplus B, (G, <2) = B \oplus A \). The subset \( A \) is necessarily infinite: otherwise \( A \) would have both the least and the greatest element and then \( (G, <1) \) would have the least, \( (G, <2) \) the greatest element. Choose any element \( x \in A \) which is neither its least nor its greatest element and put \( < = <c, x \). Then \( A = A_x \oplus (A - A_x) \) and by 3.10 we have \( (G, <1) = A_x \oplus (A - A_x) \oplus B, (G, <) = (A - A_x) \oplus B \oplus A_x, (G, <2) = B \oplus A_x \oplus (A - A_x) \). This implies \( (<1, <, <2) \in \mathcal{C} \). Thus \( (\mathcal{G}, \mathcal{C}) \) is dense.

5.8. Definition. A cyclically ordered set \( (G, C) \) is called continuous if any cut on \( (G, C) \) is Dedekind.

In other words, \( (G, C) \) is continuous if it is dense and complete. From 5.6 and 5.7 we directly obtain

5.9. Theorem. Let \( (G, C) \) be a dense cyclically ordered set. Then its completion by cuts \( (\mathcal{G}, \mathcal{C}) \) is continuous.

5.10. Corollary. For any dense cyclically ordered set \( (G, C) \) there exists a continuous cyclically ordered set \( (H, D) \) and an isomorphic embedding of \( (G, C) \) into \( (H, D) \).

5.11. Definition. Let \( (G, C) \) be a cyclically ordered set, let \( H \subseteq G \). \( H \) is called dense in \( (G, C) \) if for any elements \( x, y \in G \) there exists \( z \in H \) with \( (x, z, y) \in C \).

Note that if \( (G, C) \) contains a dense subset, then \( (G, C) \) itself is dense.

Let \( (G, C) \) be a cyclically ordered set, let \( (\mathcal{G}, \mathcal{C}) \) be its completion by cuts. Let us identify the set \( G \) with its image by the canonical isomorphism given in 4.5, i.e. let us identify the element \( x \in G \) with the element \( <c, x \in \mathcal{G} \). Thus any cyclically ordered set is a subset of a complete cyclically ordered set.

5.12. Theorem. Let \( (G, C) \) be a dense cyclically ordered set. Then \( G \) is dense in \( (\mathcal{G}, \mathcal{C}) \).

Proof. In the proof of 5.7 we have shown that for any distinct elements \( <1, <2 \in \mathcal{G} \), there exists \( x \in G \) such that \( (<1, <c, x, <2) \in \mathcal{C} \), i.e., after identifying the elements \( y \in G \) with the cuts \( <c, y \), \( (<1, x, <2) \in \mathcal{C} \). Thus \( G \) is dense in \( (\mathcal{G}, \mathcal{C}) \).

References


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