

Zdzisław Jackiewicz; Marian Kwapisz

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A NOTE ON THE STABILITY OF θ -METHODS
FOR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

ZDZISŁAW JACKIEWICZ, Fayetteville, MARIAN KWAPISZ, Gdańsk

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1. INTRODUCTION

The majority of stability analyses of numerical methods for Volterra integral equations of the second kind

$$(1) \quad y(t) = g(t) + \int_0^t k(t, s, y(s)) ds, \quad t \geq 0,$$

were based on the simple linear test equation

$$(2) \quad y(t) = g(t) + \lambda \int_0^t y(s) ds, \quad t \geq 0,$$

where λ is a complex number and $\operatorname{Re}(\lambda) < 0$ (see, for example, [1-3]). It is the purpose of this note to present a stability analysis of some methods for the numerical integration of (1) based on the more general test equation

$$(3) \quad y(t) = g(t) + \lambda \int_0^t k(s) y(s) ds, \quad t \geq 0,$$

where $\operatorname{Re}(\lambda) < 0$ and the functions g and k satisfy certain conditions which will be given later.

Denote by $h > 0$ a fixed step size and define the grid $\{t_i\}_{i=0}^{\infty}$ by $t_i = ih, i = 0, 1, \dots$. We are interested in the following class of the so called θ -methods:

$$(4) \quad y_n = g_n + \lambda h \left[(1 - \theta) \sum_{i=0}^{n-1} k(t_n, t_i, y_i) + \theta \sum_{i=1}^n k(t_n, t_i, y_i) \right],$$

$i = 0, 1, \dots, \theta \in [0, 1]$ ($\sum_{i=0}^{-1} = 0, \sum_{i=1}^0 = 0$). Here, $g_n = g(t_n)$ and y_n is the approximation to $Y(t_n)$, where Y is the solution of (1). For $\theta = 0$, $\theta = \frac{1}{2}$, and $\theta = 1$ these are direct quadrature methods based on the left rectangular rule, the trapezoidal rule, and the right rectangular rule, respectively. It is easy to check that the local discretiza-

tion error of (4)

$$\eta_\theta(t_n, h) := Y(t_n) - g_n - \lambda h \left[(1 - \theta) \sum_{i=0}^{n-1} k(t_{n-1}, t_i, Y(t_i)) + \theta \sum_{i=1}^n k(t_n, t_i, Y(t_i)) \right]$$

satisfies $\eta_\theta(t_n, h) = O(h)$ for $\theta \neq \frac{1}{2}$ and $\eta_\theta(t_n, h) = O(h^2)$ for $\theta = \frac{1}{2}$ uniformly in t_n as $h \rightarrow 0$. Consequently, this method is convergent with order one for $\theta \neq \frac{1}{2}$ and with order two for $\theta = \frac{1}{2}$, i.e.

$$y_n - Y(t_n) = \begin{cases} O(h), & \theta \neq \frac{1}{2}, \\ O(h^2), & \theta = \frac{1}{2}, \end{cases}$$

as $n \rightarrow \infty$, $nh = t_n$ (see [1, 4]).

In the next section we examine the behaviour of the solution Y of (3) and the approximate solution $\{y_n\}_{n=0}^\infty$ when the method (4) is applied to (3), for a fixed step size $h > 0$. It turns out that both Y and $\{y_n\}_{n=0}^\infty$ are bounded and the last bound is uniform in h and θ .

2. STABILITY ANALYSIS

We have the following bound on the solution Y of the equation (3).

Theorem 1. Assume that $|g(t)| \leq G < \infty$ and $k(t) \geq 0$ for $t \geq 0$. Assume also that $\operatorname{Re}(\lambda) < 0$. Then the solution Y of (3) satisfies $|Y(t)| \leq G(1 - |\lambda|/\operatorname{Re}(\lambda))$ for $t \geq 0$.

Proof. Putting $z(t) = \int_0^t k(s) y(s) ds$, the problem (3) can be written as

$$z'(t) = \lambda k(t) z(t) + k(t) g(t), \quad t \geq 0,$$

$$z(0) = 0.$$

The solution Z of this equation is given by

$$Z(t) = \int_0^t k(s) g(s) \exp\left(\lambda \int_s^t k(\tau) d\tau\right) ds, \quad t \geq 0.$$

Let $\lambda = a + bi$. It follows that

$$\begin{aligned} |Z(t)| &\leq -(G/a) \int_0^t (-a) k(s) \exp\left(a \int_s^t k(\tau) d\tau\right) ds \leq \\ &\leq -(G/a) \left[1 - \exp\left(a \int_0^t k(\tau) d\tau\right) \right] \leq -G/a. \end{aligned}$$

Taking into account that the solution Y of (3) is given by $Y(t) = \lambda Z(t) + g(t)$, we obtain $|Y(t)| \leq G(1 - |\lambda|/\operatorname{Re}(\lambda))$, $t \geq 0$, which is our claim.

Remark. It is impossible to bound the solution Y of (3) by a constant independent

of λ . To see this, let us consider the problems

$$y(t) = \lambda_n \int_0^t y(s) ds + \sin(t), \quad t \geq 0,$$

$$y(t) = i \int_0^t y(s) ds + \sin(t), \quad t \geq 0,$$

where $\lambda_n \rightarrow i$ as $n \rightarrow \infty$, $\operatorname{Re}(\lambda_n) < 0$, with solutions Y_n and Y , and observe that Y is unbounded and $Y_n \rightarrow Y$ as $n \rightarrow \infty$ uniformly on any compact interval $[0, T]$, $T > 0$.

Our next theorem establishes a bound on the solution $\{y_n\}_{n=0}^\infty$ of the equation (4) applied to (3).

Theorem 2. *In addition to the conditions given in Theorem 1, assume that $0 < \omega \leq k(t) \leq \Omega < \infty$ for $t \geq 0$. Then there exists $h_\theta > 0$ and a constant $M \geq 0$ independent of h, λ and θ such that $|y_n| \leq M(1 - |\lambda|/\operatorname{Re}(\lambda))$, $n = 0, 1, \dots$, for $h \in (0, h_\theta]$ and $\theta \in [0, 1]$.*

Proof. The method (4) applied to (3) yields

$$(5) \quad y_n = g_n + \lambda h \left[(1 - \theta) \sum_{i=0}^{n-1} k_i y_i + \theta \sum_{i=1}^n k_i y_i \right],$$

$n = 0, 1, \dots$, where $k_i = k(t_i)$. Subtracting y_{n+1} and y_n we obtain

$$y_{n+1} = \frac{1 + (1 - \theta) \lambda h k_n}{1 - \theta \lambda h k_{n+1}} y_n + \frac{g_{n+1} - g_n}{1 - \theta \lambda h k_{n+1}},$$

$n = 0, 1, \dots$. This is a recurrent equation of the first order, its solution being given by

$$y_n = \left(\prod_{i=0}^{n-1} \frac{1 + (1 - \theta) \lambda h k_i}{1 - \theta \lambda h k_{i+1}} \right) y_0 + \sum_{i=0}^{n-1} \frac{g_{i+1} - g_i}{1 - \theta \lambda h k_{i+1}} \prod_{j=i+1}^{n-1} \frac{1 + (1 - \theta) \lambda h k_j}{1 - \theta \lambda h k_{j+1}},$$

$n = 0, 1, \dots$ (see [5]). Hence,

$$(6) \quad |y_n| \leq \prod_{i=0}^{n-1} \frac{1 + (1 - \theta) \lambda h k_i}{|1 - \theta \lambda h k_{i+1}|} |y_0| + 2G \sum_{i=0}^{n-1} \prod_{j=i+1}^{n-1} \frac{1 + (1 - \theta) \lambda h k_j}{|1 - \theta \lambda h k_{j+1}|},$$

$n = 0, 1, \dots$. On the other hand, the equation (5) can be written as $y_n = \lambda h z_n + g_n$, where

$$z_n = (1 - \theta) \sum_{i=0}^{n-1} k_i y_i + \theta \sum_{i=1}^n k_i y_i,$$

$n = 0, 1, \dots$. Subtracting z_{n+1} and z_n and eliminating y_{n+1} and y_n from the resulting

equation we obtain

$$z_{n+1} = \frac{1 + (1 - \theta) \lambda h k_n}{1 - \theta \lambda k_{n+1}} z_n + \frac{(1 - \theta) k_n g_n + \theta k_{n+1} g_{n+1}}{1 - \theta \lambda h k_{n+1}},$$

$n = 0, 1, \dots$. Hence, noting that $z_0 = 0$ we get

$$z_n = \sum_{i=0}^{n-1} \frac{(1 - \theta) k_i g_i + \theta k_{i+1} g_{i+1}}{1 - \theta \lambda h k_{i+1}} \prod_{j=i+1}^{n-1} \frac{1 + (1 - \theta) \lambda h k_j}{1 - \theta \lambda h k_{j+1}},$$

$n = 0, 1, \dots$. In view of this relation and the relationship between y_n and z_n we obtain

$$(7) \quad |y_n| \leq |\lambda h| \Omega G \sum_{i=0}^{n-1} \prod_{j=i+1}^{n-1} \frac{|1 + (1 - \theta) \lambda h k_j|}{|1 - \theta \lambda h k_{j+1}|} + G,$$

$n = 0, 1, \dots$. For any $\xi \in [\omega, \Omega]$ (and fixed λ , $\operatorname{Re}(\lambda) < 0$) let us consider the functions

$$\phi_\xi(h, \theta) := |1 + (1 - \theta) \lambda h \xi|, \quad \psi_\xi(h, \theta) := |1 - \theta \lambda h \xi|.$$

For any fixed $\theta \in [0, 1)$ the function ϕ_ξ attains its minimum value $|\operatorname{Im}(\lambda)|/|\lambda|$ for $h_\xi = -\operatorname{Re}(\lambda)/((1 - \theta) \xi |\lambda|^2)$. In view of the assumptions on the function k it is clear that there exists $h_\theta > 0$ such that for all $h \in (0, h_\theta]$ the following inequality holds:

$$\phi_{k_j}(h, \theta) \leq \phi_\omega(h, \theta) < 1.$$

The largest value of h_θ for which this inequality is satisfied for any function k satisfying the assumptions of the theorem can be computed from the condition $\phi_\omega(h_\theta, \theta) = = \phi_\Omega(h_\theta, \theta)$, which leads to the formula

$$h_\theta = -2 \operatorname{Re}(\lambda)/((1 - \theta) |\lambda|^2 (\omega + \Omega)).$$

It is also clear that $\psi_{k_{j+1}}(h, \theta) \geq \psi_\omega(h, \theta) > 1$ for any $h > 0$ and $\theta \in (0, 1]$. Hence, putting $q_\omega(h, \theta) = \phi_\omega(h, \theta)/\psi_\omega(h, \theta)$ and $Q = \max\{2G, \Omega G\}$, in view of (6), (7), and the relation $q_\omega(h, \theta) < 1$, which holds for any $\theta \in [0, 1]$ and $h \in (0, h_\theta]$ ($h_1 = \infty$), we obtain

$$(8) \quad |y_n| \leq \begin{cases} Q \frac{|\lambda h|}{1 - q_\omega(h, \theta)} + G, & h \leq 1/|\lambda|, \\ Q \frac{1}{1 - q_\omega(h, \theta)} + G, & h > 1/|\lambda|, \end{cases}$$

$n = 0, 1, \dots$. Let us set $D := \{(h, \theta): \theta \in [0, 1], h \in (0, h_\theta]\}$ and define the (continuous) function $\eta_\omega : D \rightarrow [0, \infty)$ by

$$\eta_\omega(h, \theta) := \begin{cases} \frac{|\lambda h|}{1 - q_\omega(h, \theta)}, & h \leq 1/|\lambda|, \\ \frac{1}{1 - q_\omega(h, \theta)}, & h > 1/|\lambda|. \end{cases}$$

We will show that η_ω is bounded on D . We have

$$\frac{1}{1 - q_\omega(h, \theta)} \leq \frac{2|1 - \theta\lambda h\omega|^2}{|1 - \theta\lambda h\omega|^2 - |1 + (1 - \theta)\lambda h\omega|^2},$$

hence,

$$\eta_\omega(h, \theta) \leq \begin{cases} \frac{2|\lambda| (1 - 2\theta ah\omega + \theta^2|\lambda|^2 h^2\omega^2)}{-2a\omega + |\lambda|^2 h\omega^2(2\theta - 1)}, & h \leq 1/|\lambda|, \\ \frac{2(1 - 2\theta ah\omega + \theta^2|\lambda|^2 h^2\omega^2)}{-2ah\omega + |\lambda|^2 h^2\omega^2(2\theta - 1)}, & h > 1/|\lambda|. \end{cases}$$

Here, $a = \operatorname{Re}(\lambda)$. Next we define θ^* by $|\lambda h_{\theta^*}| = 1$, i.e.

$$\theta^* = 1 + 2 \operatorname{Re}(\lambda)/(|\lambda|(\omega + \Omega)).$$

We may assume without loss of generality that $\omega + \Omega > 4$, hence $\theta^* > \frac{1}{2}$. We consider the following cases:

1. $\theta \in [0, \frac{1}{2}]$. Then there exists a constant $M_1 \geq 0$ such that

$$\eta_\omega(h, \theta) \leq \frac{2|\lambda| (1 - 2\theta ah_\theta\omega + \theta^2|\lambda|^2 h^2\omega^2)}{-2a\omega + |\lambda|^2 h_\theta\omega^2(2\theta - 1)} \leq -M_1(|\lambda|/a).$$

2. $\theta \in (\frac{1}{2}, 1]$ and $h \in (0, \min\{h_\theta, 1/|\lambda|\})$. Then there exists $M_2 \geq 0$ such that

$$\eta_\omega(h, \theta) \leq \frac{2|\lambda| (1 - 2a\omega/|\lambda| + \omega^2)}{-2a\omega + |\lambda|^2 h\omega^2(2\theta - 1)} \leq -M_2(|\lambda|/a).$$

3. $\theta \in [\theta^*, 1]$ and $h \in (1/|\lambda|, h_\theta]$. Then

$$\begin{aligned} \eta_\omega(h, \theta) &\leq \frac{2}{-2ah\omega + |\lambda|^2 h^2\omega^2(2\theta - 1)} + \frac{-4a\omega}{-2a\omega + |\lambda|^2 h\omega^2(2\theta - 1)} + \\ &+ \frac{|\lambda|^2 \omega^2 h}{-2a\omega + |\lambda|^2 h\omega^2(2\theta - 1)} \leq -(|\lambda|/a\omega) + 2 + (1/(2\theta^* - 1)). \end{aligned}$$

However, we have $(1/(2\theta^* - 1)) \leq (\omega + \Omega)/(\omega + \Omega - 4)$, hence

$$\eta_\omega(h, \theta) \leq -M_3(|\lambda|/a) + M_4$$

for some nonnegative constants M_3 and M_4 .

Combining all these inequalities and taking into account (8) we immediately see that there exists a constant $M \geq 0$ independent of h, θ , and λ such that

$$|y_n| \leq M(1 - |\lambda|/\operatorname{Re}(\lambda)),$$

$n = 0, 1, \dots$. Thus the theorem is proved.

Remark. It follows from the proof of this theorem that the approximate solution $\{y_n\}_{n=0}^{\infty}$ given by (5) for $\theta = 1$ is bounded for any $h > 0$. This property is similar to the A -stability property of numerical methods for ordinary differential equations.

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Author's addresses: Prof. *Zdzisław Jackiewicz*, Department of Mathematical Sciences SE 301, University of Arkansas, Fayetteville, AR 72701, USA; Prof. *Marian Kwapisz*, Department of Mathematics, University of Gdańsk, 80-952 Gdańsk, Wita Stwosza 57, Poland.