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A THEOREM OF H. STEINHAUS AND (R)-DENSE SETS OF POSITIVE INTEGERS

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0. H. Steinhaus observed (see [5], p. 155) that the prime number theorem immediately implies the following property of the primes: for every real $x > 0$ one can find a sequence $q_1 < q_2 < \ldots$ of primes such that the ratio q_n/n tends to x. Motivated by this observation we introduce the following definition:

A subset $A = \{a_1, a_2, \ldots\}$ of the set of positive integers has the property (S) provided every real positive number x is a limit point of the sequence a_n/n.

In [9] the following property of subsets of the set of positive integers was introduced: such a subset A is called (R)-dense provided the set $R(A)$ of all rationals of the form a/b with $a, b \in A$ is dense on the positive half-axis. If this is the case we shall simply say that A has the property (R).

In this paper we study the relation between the properties (R) and (S) and obtain some results about the classes T_R and T_S of all sets having the property (R) or (S), respectively, and also about the difference of these classes.

We shall use the following notation: $A(x)$ will denote the counting function of the set A, i.e.

$$A(x) = \sum_{a \leqq x, a \in A} 1 .$$

1. We start with some simple results concerning the properties (R) and (S).

Proposition 1. *Every set with the property (S) has also the property (R).*

Proof. If the set $A = \{a_1 < a_2 < \ldots\}$ has the property (S) then for every given positive x we can find a sequence $n_1, n_2, \ldots$ of elements of A such that n_k/k tends to x when k tends to infinity. Thus

$$\lim_{k \to \infty} \frac{n_{a_k}}{a_k} = x .$$

As all numbers n_{a_k}/a_k belong to $R(A)$, the property (R) results. □

Note, however, that the properties (R) and (S) are not equivalent. Indeed, the next proposition implies that if A is the union of intervals $[2^{2^k}, k \,.\, 2^{2^k}]$ then it does not have property (S). Nonetheless, it is obvious that this set has the property (R).

Proposition 2. *A set* $A = \{a_1 < a_2 < \dots\}$ *has the property* (S) *if and only if*

$$(1) \qquad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \,.$$

Proof. Assume that A has the property (S) and that, contrary to our assertion, one has

$$\limsup_{n\to\infty} (a_{n+1}/a_n) > 1 \,.$$

Then one can find a sequence $n_1 < n_2 < \dots$ and a positive ε such that none of the intervals $I_k = (a_{n_k}, (1 + \varepsilon)\, a_{n_k})$ contains an element of A. Because A has the property (S) we can find a sequence a_{i_j} of its elements such that for sufficiently large j we have

$$(x - \varepsilon/4)\, j < a_{i_j} < (x + \varepsilon/4)\, j$$

with $x = 1 + \varepsilon/2$. This immediately yields

$$j < (1 + \varepsilon/4)\, j < a_{i_j} < (1 + 3\varepsilon/4)\, j < (1 + \varepsilon)\, j$$

and taking $j = a_{n_k}$ we see that the interval I_k contains elements of A, at least for large k, a contradiction.

Assume now that (1) is satisfied. Then it easily follows that for every positive ε and sufficiently large real T the interval $(T, (1 + \varepsilon)\, T)$ contains at least one element of A. If now x is a positive number and $\eta > 0$ is given then for all large integers N we have at least one element of A in the interval $((x - \eta)\, N, (1 + \eta)(x - \eta))\, N)$, say a_{i_N}. It follows that

$$|a_{i_N}/N - x| < \eta x$$

and this easily implies that the set A has the property (S). □

Corollary 1. *If a set* $A = \{a_1 < a_2 < \dots\} \subset N$ *has bounded differences, i.e. there exists a constant* B *such that for all* n *one has* $a_{n+1} - a_n < B$, *then* A *has the property* (S). □

Corollary 2. *If the counting function* $A(x)$ *of a set* $A \subset N$ *satisfies*

$$A(x) = (c + o(1))\,.\, x^\alpha L(x)$$

where c, α *are positive constants and* $L(x)$ *is slowly oscillating, i.e.* $L(2x)/L(x)$ *tends to unity when* $x \to \infty$, *then* A *has the property* (S).

Proof. One easily sees that for any positive constant B we have

$$\lim_{x\to\infty} A(Bx)/A(x) = B^\alpha \,,$$

hence one can choose B in such a way that this limit exceeds 2. Then for sufficiently large x we have $A(Bx) > 2\,A(x)$ and it follows that for large n the inequality $a_{n+1} \leqq \leqq Ba_n$ holds. This in turn implies that

$$L(a_{n+1})/L(a_n)$$

tends to unity and it suffices to observe that in view of

$$\frac{n+1}{n} = \frac{A(a_{n+1})}{A(n)} = \frac{(c+o(1))\,a_{n+1}\,L(a_{n+1})}{(c+o(1))\,a_n\,L(a_n)}$$

the ratio a_{n+1}/a_n tends to unity and so we may apply Proposition 2. □

The last corollary implies, in particular, that every set with a positive density has the property (S). However, one can construct (see [8]) sets with positive lower density which do not have the property (R) and so in view of Proposition 1 do not have the property (S). In the same paper it was also proved that sequences of upper density 1 have the property (R). Note, however, that they need not have the property (S). In fact, the union of intervals $[(2k-1)^{2k-1}, (2k)^{2k}]$ $(k = 1, 2, \ldots)$ has its upper density equal to 1, nevertheless it fails to satisfy (1) and so cannot have the property (S).

2. Metrical theory of the classes T_S and $T_R \setminus T_S$.

For every infinite set A of positive integers let us define the dyadic value $\varrho(A)$ of A by

$$\varrho(A) = \sum_{i=1}^{\infty} \varepsilon(i)\,2^{-i},$$

where $\varepsilon(i)$ denotes the characteristic function of A. (Cf. [4], p. 17–18 and [11].) The function $\varrho(A)$ gives a one-to-one map of the family U of all infinite subsets of the positive integers onto the interval $(0, 1] = I$. If L is a subfamily of U then we denote by $\varrho(L)$ the set $\{\varrho(A) : A \in L\}$. In this section we shall consider metrical and topological properties of the sets $\varrho(T_S)$ and $\varrho(T_R \setminus T_S)$. The fundamental properties of the set $\varrho(T_R)$ were deduced already in [9] where it was shown, in particular, that it is a homogeneous $F_{\sigma\delta}$-set, residual in I, and of Lebesgue measure one. For the Hausdorff measure of the set $I \setminus \varrho(T_R)$ the inequality $\dim(I \setminus \varrho(T_R)) \geqq \frac{1}{2}$ was obtained there.

(We recall that a subset M of I is called homogeneous provided its density is the same in every subinterval of I, i.e. for every such subinterval J the exterior measure of $J \cap M$ equals $d.m(J)$, where d is a constant independent of J, and $m(J)$ denotes the length of J. One can show that either $d = 0$ or $d = 1$ (see e.g. [2], [10]).)

The following theorem describes the fundamental metrical and topological properties of the set $\varrho(T_S)$:

Theorem 1. (i) *The set $\varrho(T_S)$ is a homogeneous $F_{\sigma\delta}$-set in I*).*

*) Note that the Conference on Number Theory in Čaradice 1979 A. Schinzel showed that $\varrho(T_S)$ is a $G_{\delta\sigma\delta}$-set.

(ii) *The set $\varrho(T_S)$ is of the first Baire category in I.*

(iii) *The set $I \setminus \varrho(T_S)$ is of measure zero and its Hausdorff dimension is equal to one.*

Proof. For $n = 1, 2, 3, \ldots$ and $j = 0, 1, \ldots, 2^n - 1$ put

$$I_n^{(j)} = (j\,2^{-n}, (j+1)\,2^{-n}]$$

and observe that all numbers in a given interval $I_n^{(j)}$ have the same sequence of the first n dyadic digits (we shall consider only those dyadic expansions which contain infinitely many digits equal to 1). If those digits are $\varepsilon_1, \ldots, \varepsilon_n = \bar{\varepsilon}$, then we shall say that the interval $I_n^{(j)}$ corresponds to the sequence $\bar{\varepsilon}$. To show that the set $\varrho(T_S)$ is homogeneous it is sufficient to prove that if

$$x = \sum_{k=1}^{\infty} \varepsilon_k(x)\, 2^{-k} \in \varrho(T_S)$$

then for all $m = 1, 2, \ldots$

$$x_m = \sum_{k=1}^{\infty} \varepsilon_{k+m}\, 2^{-k} \in \varrho(T_S)$$

(see [2]).

But it is easy to observe that if $x = \varrho(A)$, $A \in T_S$ then the set $A_m = A \setminus \{1, 2, \ldots, m\}$ as well as the set A'_m consisting of all differences $a - m$, where $a \in A_m$, belong to T_S, and clearly $\varrho(A'_m) = x_m$. Thus $\varrho(T_S)$ is in fact a homogeneous set.

Now define for $k \geqq 1$, $1 \leqq m < n$ the set $A(k, n, m)$ as the union of those intervals $I_n^{(j)}$ which correspond to sequences $\varepsilon_1, \ldots, \varepsilon_n$ with the following property: if $m + 1 \leqq r_1 < \ldots < r_t \leqq n$ are the indices bigger than m for which $\varepsilon_{r_i} = 1$, then

$$|r_{s+1}/r_s - 1| \leqq 1/k$$

holds for $s = 1, 2, \ldots, t$.

Each of the sets $A(k, n, m)$ is closed in the set $E = I \setminus D$, where D denotes the set of all dyadic rationals. Moreover, it is easy to infer, with the use of Proposition 2, that

$$E \cap \varrho(T_S) = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=1+m}^{\infty} A(k, n, m)\,, \tag{2}$$

which implies that $E \cap \varrho(T_S)$ is a $F_{\sigma\delta}$ set in E as well as in I. Because of

$$\varrho(T_S) = D \cup (E \cap \varrho(T_S))$$

the assertion (i) results.

To prove (ii) note that D is countable and so it is sufficient to prove that the set $E \cap \varrho(T_S)$ is of the first category, and to prove this it is enough (using (2)) to show that

$$B(m) = \bigcap_{n=m+1}^{\infty} A(1, n, m)$$

is for $m = 1, 2, 3, \ldots$ a nowhere dense set in I, i.e. for every interval $J \subset I$ there exists a subinterval of it disjoint with $B(m)$. (Cf. [3], p. 43.) If J is such an interval, choose an integer $n \geqq 1 + m$ such that for a certain j the interval $I_n^{(j)}$ is contained in J and, moreover, in its corresponding sequence $\varepsilon_1, \ldots, \varepsilon_n$ not all terms $\varepsilon_{m+1}, \varepsilon_{m+2}, \ldots, \varepsilon_n$ vanish. Denote by M the maximal index $r \leqq n$ for which $\varepsilon_r = 1$, consider the sequence

$$\varepsilon_1, \ldots, \varepsilon_n, 0, \ldots, 0, 1$$

of length $n + 2M$, and denote by J' the corresponding interval. Then $J' \subset I_n^{(j)} \subset J$ and obviously J' is disjoint with $B(m)$.

The first part of (iii) is easy. In fact, by Corollary 2 to Proposition 2 the family U of all subsets on N with density $\frac{1}{2}$ is a subfamily of T_S, thus $\varrho(U) \subset \varrho(T_S)$ but $\varrho(U)$ coincides with the set of all dyadically normal numbers and so by a theorem of E. Borel (see e.g. [4], pp. 190–193) is of measure one. Hence $\varrho(T_S)$ is of measure one.

To evaluate the Hausdorff dimension consider for a given number $a \in (0, 1)$ the set $A = \bigcup_{k=1}^{\infty} A_k$, where for $k = 1, 2, \ldots, A_k$ denotes the set of integers contained in the interval $[2^k, 2^k + [a\, 2^k]]$. One easily sees that the lower density of A is equal to a, and Proposition 2 implies that A does not have the property (S). This shows that no subset of A can have this property, and so if we denote by W the family of all infinite subsets of A, then

$$\varrho(W) \subset I \setminus \varrho(T_S) .$$

Now we note that Theorem 1 of [6] implies that if

$$\varrho(A) = \sum_{k=1}^{\infty} \varepsilon_k\, 2^{-k}$$

then the Hausdorff dimension of W equals

$$\dim W = \liminf_{n \to \infty} \frac{\log \prod_{k=1}^{n} (\varepsilon_k + 1)}{n \log 2}$$

and a short computation leads to

$$\dim W = a .$$

This implies $\dim (I \setminus \varrho(T_S)) \geqq a$, but a was an arbitrary positive number less than 1 and so finally we get

$$\dim (I \setminus \varrho(T_S)) = 1$$

as asserted. □

We obtained an analogous result for the set $\varrho(T_R \setminus T_S)$, however, we have been unable to obtain the exact value of its Hausdorff dimension.

Theorem 2. (i) *The set $\varrho(T_R \setminus T_S)$ is a homogeneous $G_{\delta\sigma\delta}$-set as well as a $F_{\sigma\delta\sigma}$-set in I.*

(ii) *The set $\varrho(T_R \setminus T_S)$ is a residual set in I.*

(iii) *The Lebesgue measure of* $\varrho(T_R \setminus T_S)$ *equals zero and its Hausdorff dimension is* $\geqq \frac{1}{6}$.

Proof. The homogeneity of $\varrho(T_R \setminus T_S)$ follows in the same way as that of $\varrho(T_S)$ proved in Theorem 1.

In [9] it was shown that $\varrho(T_R)$ is a $F_{\sigma\delta}$-set and Theorem 1 (i) implies in view of

$$\varrho(T_R \setminus T_S) = \varrho(T_R) \cap (I \setminus \varrho(T_S))$$

our assertion (i). To prove (ii) it suffices to apply part (ii) of the previous theorem and the fact, established in [9], that $\varrho(T_R)$ is residual in I. Moreover, the first assertion of (iii) follows immediately from part (iii) of Theorem 1.

Now we prove the assertion concerning the Hausdorff measure. For $k = 1, 2, \ldots$ let A_{2k} be the set of all integers from the interval $[2^{2k}, 2^{2k+1}]$, and denote by A their union. We first show that the set A has the property (R). To do this, note first that for $k = 1, 2, \ldots$; $j = 0, 1, \ldots$; and $0 \leqq r \leqq 4^{k+j}$ the numbers

$$(4^{k+j} + r)/4^k = 4^j + r/4^k$$

belong to $R(A)$ and their set is dense in the union of intervals

$$\bigcup_{n=0}^{\infty} [4^n, 2 \cdot 4^n] .$$

Moreover, for $k = 1, 2, \ldots; j = 0, 1, 2, \ldots$ and $0 \leqq r \leqq 4^{k+j}$ the numbers

$$(4^{k+j} + r)/2 \cdot 4^k = 4^j/2 + r/2 \cdot 4^k$$

lie in $R(A)$ and their set is dense in

$$\bigcup_{n=0}^{\infty} [4^n/2, 4^n] .$$

Thus $R(A)$ is dense in the half-line $[\frac{1}{2}, \infty)$, and taking inverses we obtain that $R(A)$ is dense on the positive half-axis. This shows that A has the property (R).

Now let $2m_1 - 1 < 2m_2 - 1 < \ldots$ be an arbitrary sequence of odd integers about which we assume

$$m_{r+1} - m_r \geqq 2 \quad (r = 1, 2, \ldots) \tag{3}$$

and

$$\lim_{r \to \infty} (m_{r+1} - m_r) = \infty . \tag{4}$$

Now define, for $m = 1, 2, \ldots$,

$$A_{2m-1} = \begin{cases} \emptyset \text{ if } m \in \{m_1, m_2, \ldots\} \\ \{2^{2m-1} + 1, 2^{2m-1} + 2, \ldots, 2^{2m} - 1\} \quad \text{otherwise} \end{cases}$$

and put

$$A^* = A \cup \bigcup_{m=1}^{\infty} A_{2m-1} .$$

Clearly the set A^* has the property (R) and, as the quotients of its two consecutive terms are infinitely often equal to 2, it cannot have, in view of Proposition 2, the property (S).

Denote now by F the family of all sets C satisfying

$$A \subset F \subset A^* .$$

Then $F \subset T_R \setminus T_S$ and so to obtain the last assertion of our theorem it is sufficient to show that the Hasudorff dimension of $\varrho(F)$ is $\geqq \frac{1}{6}$. This we shall now establish.

We shall apply the following result, which follows from Theorem 2.7 of [7]:

Let D be a subset of the set N of all positive integers and let $E = N \setminus D$. For every $d \in D$ let $\varepsilon_d \in \{0, 1\}$ be given. If Z denotes the set of all numbers

$$x = \sum_{k=1}^{\infty} \varepsilon_k(x)\, 2^{-k} \in I \quad \{\varepsilon_k(x) = 0, 1\}$$

for which $\varepsilon_d(x) = \varepsilon_d$ holds for $d \in D$, then the Hasudorff dimension of Z equals the lower density of the set E.

We apply this to the case when $D = A \cup \bigcup_{m=1}^{\infty} A_{2m-1}$ and put for $d \in D$

$$\varepsilon_d = \begin{cases} 1 & \text{if} \quad d \in A \\ 0 & \text{otherwise} . \end{cases}$$

Then clearly $\varrho(F) = Z$ and, since an elementary computation shows that the lower density of E is $\geqq \frac{1}{6}$, our assertion follows. □

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