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ON BASIC CONCEPTS OF NON-COMMUTATIVE TOPOLOGY

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A non-commutative generalization of locally compact Hausdorff spaces was independently offered by Ch. A. Akemann [1–3], R. Giles and H. Kummer [8; 9]. They used the description of such topological spaces in terms of the bounded continuous functions algebra as the matter to be extended to the non-abelian situation. In that way a non-commutative or "quantum" topology associated with a C*-algebra was defined as a certain family of projections in its atomic W*-envelope, and the C*-algebra was interpreted as an algebra of "continuous" elements in accordance with this topology.

In the present paper we give an intrinsic axiomatic definition of a general non-commutative topology in terms of the lattice of all projections in an arbitrary atomic W*-algebra B. The system of axioms connects the properties of non-commutative topology with order, Jordan and C*-structures on B. For all that two key ideas are pursued: firstly, to generalize to non-abelian case the description of any topology by means of the bounded lower semicontinuous functions cone and, secondly, to provide that the set of "continuous" elements in B be a C*-algebra. Moreover, we give an effective characterization of compactness and show that a non-commutative topology is locally compact iff it is the Akemann-Giles topology associated with a certain C*-algebra.

1. Preliminaries. Let us consider, together with any set X, the commutative W*-algebra B(X) of all complex-valued bounded functions on X. The field of all subsets of X may be naturally identified with the atomic boolean algebra PrB(X) of all projections in B(X) (each subset of X is assigned its characteristic function); the points of X are in one to one correspondence with the atoms in the lattice PrB(X).

Now, if there is a topology τ on X, it may be regarded as some family of projections in B(X). Namely, τ = PrB(X) ∩ L(X), where L(X) is the convex cone of all lower semi-continuous bounded functions on X. In particular, if τ is completely regular, let C(τ) denote the C*-algebra of all continuous bounded functions on X and let C(τ)^m denote the set of suprema in B(X) (i.e. pointwise suprema) of all bounded increasing nets of real-valued elements in C(τ), then τ = PrB ∩ C(τ)^m.

It easily follows from the spectral theory that an arbitrary commutative atomic
$W^*$-algebra $M$ is isomorphic to the $W^*$-algebra $B(X)$, where $X$ is the set of all minimal projections in $M$ (i.e. atoms in the atomic boolean algebra $\text{Pr}M$). In the light of these facts the lattice $\text{Pr}B$ of all projections in an arbitrary non-commutative atomic $W^*$-algebra $B$ can be regarded as a non-commutative analogue of the concept of set; minimal projections in $B$ play the role of points, any projection in $B$, being the supremum of minimal ones, plays the role of a subset. Finally, a “non-commutative topology” arises as the appropriate family of projections in a non-commutative atomic $W^*$-algebra.

This outlook at a non-commutative generalization of topology was the basis for the Akemann-Giles analogue of a locally compact Hausdorff space. In what follows we construct a non-commutative analogue of a general topological space and show that the Akemann-Giles construction coincides with ours in the locally compact case.

For the general theory of $C^*$- and $W^*$-algebras we shall make systematic use of books [6] and [12].

1.1. $q$-sets. An atomic $W^*$-algebra $B$ together with the set of all minimal projections therein will be called a $q$-set, the elements of $\text{Pr}B$ $q$-subsets and the atoms in $\text{Pr}B$ $q$-points. Union and intersection of $q$-sets are to be taken in the lattice $\text{Pr}B$. Two $q$-sets $e$ and $f$ will be called disjoint iff $ef = 0$. (The terminology is lifted from [8]).

Any $C^*$-algebra $A$ is associated with a $q$-set as follows. The second conjugate space $A^{**}$ is a $W^*$-algebra; let $Z_A$ be the supremum of all minimal projections in $A^{**}$, then $Z_A$ belongs to the center of the algebra $A^{**}$ [1; p. 278]. Set $B_A = Z_A A^{**}$, then $B_A$ is the atomic $W^*$-algebra and we can consider $A$ as the weakly dense sub-algebra of $B_A$, since $A \subset A^{**}$ and $A \to Z_A A$ is an isomorphism (see [3], p. 1). If $A$ is abelian with the spectrum $X$, then the points of $X$ are in one to one correspondence with the minimal non-zero projections of $A^{**}$ so that $B_A \approx B(X)$ and the Gelfand isomorphism give $A \approx C_0(X) \subset B(X)$.

1.2. Notation. For any subset $E \subset B$ put $E^+ = \{a \in E \mid a \geq 0\}$, $E^s = \{a \in E \mid a = a^*\}$, $E_1 = \{a \in E \mid \|a\| \leq 1\}$. Recall that the self-adjoint part $B^s$ of a $W^*$-algebra $B$ is an ordered space, in which every norm bounded increasing net has a supremum. Let $E^m$ denote the set of suprema in $B^s$ of all norm-bounded increasing nets of elements of $E^*$; put $E_m = -(E^m)$.

1.3. The Akeman-Giles $Q$-topology. For any $C^*$-algebra $A$ consider $A \subset B_A$ as above. The family of $q$-sets $\tau_A = \text{Pr}B_A \cap (A^+)^m$ will be called the Akemann-Giles $q$-topology on $B_A$. The elements of $\tau_A$ are called $q$-open $q$-sets, the elements of $1 - \tau_A = \{1 - e \mid e \in \tau_A\}$ are called $q$-closed. The pair $(B_A, \tau_A)$ will be called the $q$-spectrum of the $C^*$-algebra $A$ and denote as $q$ spec $A$. If $A$ is commutative, then $\tau_A$ is the usual topology on the spectrum of $A$ and therefore is Hausdorff and locally compact. In the non-abelian case $\tau_A$ has similar properties. Namely, a $q$-topology $\tau_A$
is Hausdorff: i.e., given disjoint $q$-points $x$ and $y$ there exist disjoint open $q$-sets $e$ and $f$ with $x \leq e$ and $y \leq f$ [8; III.6]. After Akemann a $q$-set $p \in \Pr B_A$ is called $q$-compact if $p$ is closed and there exists $a \in A^+_\tau$ with $p \leq a$ [3; II.1 and II.5]. The $q$-topology $\tau_A$ is locally compact: i.e., for any $q$-point $x$ there exist an open $q$-set $e$ and a compact $q$-set $p$ with $x \leq e \leq p$ (this follows from [3; III.1]). In the abelian case, when $A \approx C_0(X) \subset B(X)$, these concepts of $q$-compactness are equivalent to the usual definitions.

**1.4. Gelfand-Akemann-Giles theorem.** An element $a \in B_A^+$ is called $\tau_A$-continuous if each spectral projection of $a$ which corresponds to an open subset of the real numbers is also an open $q$-set. A $\tau_A$-continuous element $a$ is called vanishing at $\infty$ if each spectral projection of $a$ corresponding to a closed subset of the real numbers which do not contain 0 is $q$-compact [3; I.1 and III.3].

Denote by $C(\tau_A)^+$ the set of all $\tau_A$-continuous elements of $B_A^+$ and by $C_0(\tau_A)^+$ the set of all elements vanishing at $\infty$. Set $C(\tau_A) = C(\tau_A)^+ + iC(\tau_A)^+$ and $C_0(\tau_A) = C_0(\tau_A)^+ + iC_0(\tau_A)^+$. The elements of $C(\tau_A)$ and $C_0(\tau_A)$ will also be called $\tau_A$-continuous and vanishing at $\infty$, respectively.

**Theorem** [3; 5; 8]. A $C^*$-algebra $A$ is exactly the algebra of all $\tau_A$-continuous elements of $B_A$ vanishing at $\infty$, i.e. $A = C_0(\tau_A)$; the set of all continuous elements of $B_A$ coincides with the $C^*$-algebra of all multipliers of $A$, i.e. $C(\tau_A) = M(A) \equiv \{ b \in B_A \mid Ab + bA \subset A \}$.

Throughout the whole paper $B$ will always be used to denote an arbitrary atomic $W^*$-algebra, and $\tau \subset \Pr B$ a family of projections (i.e. $q$-sets) in $B$. We start to discuss the individual axioms which will connect the properties of $\tau$ with various structures on $B$.

**2. Order axioms.** It is easy to see that any Akemann-Giles $q$-topology contains 0 and 1 and that a union of open $q$-sets is also open. Akemann has shown that in contradistinction to the usual topology, the intersection of two open $q$-sets is not necessarily open (see a counterexample in [I]). The first three axioms describe the corresponding properties of the general $q$-topology.

**AXIOM A1.** $0, 1 \in \tau$;

**AXIOM A2.** $(e) \subset \tau \Rightarrow \forall e \in \tau$, i.e. “union of open $q$-sets is open”.

**AXIOM A3.** $e, f \in \tau$, $[e, f] = 0 \Rightarrow e \wedge f \in \tau$, i.e. “intersection of two commuting open $q$-sets is open”.

**Remark** that actually axiom A3 is an order condition since $[e, f] = 0 \Leftrightarrow e = (e \wedge f) \vee (e \wedge (1 - f))$. Any Akemann-Giles $q$-topology satisfies all these axioms [I].

We have noticed in § 1 that in the commutative case a topology of a topological space $X$ may be described algebraically by the equality $\tau = \Pr B(X) \cap L(X)$. Our fourth axiom will be a non-commutative version of this description, therefore we need an appropriate definition of the class of lower semicontinuous (LSC) elements in $B$. 

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Definition 2.1. The set $E \subset B^s$ is lower monotone closed (LMC) if $E = E^m$; $E$ is upper monotone closed (UMC) if $E = E_m$. The minimal LMC set in $B^s$ containing $E$ is called the lower monotone closure of $E$ and is denoted by $L(E)$. The upper monotone closure of $E$ is similarly defined and denoted by $U(E)$.

Lemma 2.2. If $E$ is a convex cone in $B^s$, then $L(E)$ is a convex cone, too. If, besides, $E \supseteq \mathbb{R} \cdot 1$, then the convex cone $L(E)$ is norm-closed in $B^s$.

Proof. Take $\lambda, \mu \in \mathbb{R}^+$ and $a \in E$, then

\[ E \subset M(a) \equiv \{ b \in L(E) \mid \lambda a + \mu b \in L(E) \} \subset L(E) \]

and the set $M(a)$ is LMC. So $M(a) = L(E)$ and for each $a \in E$ and $b \in L(E)$ we have $\lambda a + \mu b \in L(E)$. This implies that for each $a \in L(E)$ (*) is correct and we get similarly $M(a) = L(E)$. The last equality shows that $L(E)$ is a convex cone. Let $E \supseteq \mathbb{R} \cdot 1$. If a sequence $(a_n) \subset L(E)$ converges to an element $b \in B^s$ we may suppose that $\|a_n - b\| \leq 2^{-n}$. Then the increasing sequence $a_n = a_n - 2^{-n+1} \cdot 1$ is contained in $L(E)$ and also converges to $b$. Hence $b = \bigvee_n a_n \in L(E)$ and $L(E)$ is norm-closed.

Put $A^+(\tau) = \{ \sum_{i=1}^n \lambda_i e_i \mid \lambda_i \geq 0, e_i \in \tau, n \in \mathbb{N} \}$ and consider $A(\tau) = A^+(\tau) + \mathbb{R} \cdot 1$, the minimal convex cone containing $\tau$ and "constants". We now define the class of lower semicontinuous (LSC) elements in $B^s$ as $L(\tau) = L(A(\tau))$. By Lemma 2.2, $L(\tau)$ is a norm-closed convex cone in $B^s$. If $\tau$ were the usual topology on a set $X$, $L(\tau)$ would be the class of all lower semicontinuous real-valued functions on $X$.

Similarly we may define the class $U(\tau)$ of upper semicontinuous (USC) elements in $B^s$ by setting

\[ A^+(1 - \tau) = \{ \sum_{i=1}^n \lambda_i f_i \mid \lambda_i \geq 0, f_i \in 1 - \tau \}, \]

\[ A(1 - \tau) = A^+(1 - \tau) + \mathbb{R} \cdot 1 \quad \text{and} \quad U(\tau) = U(A(1 - \tau)). \]

For any subset $E \subset B^s$ we shall denote by $\bar{E}$ the norm-closure of $E$ in $B^s$.

Lemma 2.3. If $A$ is a C*-algebra and $\tau_A$ is the Akemann-Giles topology associated with $A$, then

\[ L(\tau_A) = \bar{A}^m, \]

where $\bar{A} = A + C \cdot 1 \subset B_A$ is the C*-algebra obtained by adjoining the unit 1 of $B_A$ to $A$.

Proof. We have $A(\tau_A) \subset \bar{A}^m \subset \bar{A}^m$ since $\tau_A \subset (A^+)^m$. By [4; 3.3] the convex cone $\bar{A}^m$ is LMC, therefore we get $L(\tau_A) \subset \bar{A}^m$. To show the converse inclusion notice that by virtue of the Gelfand-Akemann-Giles theorem (1.4) we have $\bar{A} \subset C(\tau_A)$ and so by the spectral theory $\bar{A}^m \subset A(\tau_A)^m$. This implies $\bar{A}^m \subset L(\tau_A)$, which means $\bar{A}^m \subset L(\tau_A)$ as desired.

The proof of the last lemma was based on the assertion in [4], which was stated
there in terms of $A^{**}$, but applying theorems [9; 3.8 and 4; 2.6] we get the result in terms of $B_A$.

From the papers [9; 4] by Pedersen and Akemann we know that there is an isometric map of $\overline{A}^m$ onto the set $L^0(S(A))$ of all bounded LSC affine functions on the state space $S_A = \{\varphi \in A^{**} \mid \|\varphi\| = 1\}$ provided with the weak* topology of $A^*$. Through Lemma 2.3 we may therefore identify the convex cone $L(\tau_A)$ with the cone $L^0(S(A))$ (see also § 5 below).

The next definition coincides in essence with that in 1.1, and in the commutative case it is the usual definition of continuous functions.

**Definition 2.4.** An element $a \in B^s$ is called $q$-continuous, if for each open set $I \subset \mathcal{R}$ the spectral projection $E_a(I)$ belongs to $\tau$, $a = \int \lambda dE_\lambda$ being the spectral representation of $a$. Let $C(\tau)^s$ denote the set of all $q$-continuous elements of $B^s$ and $C(\tau) = C(\tau)^s + iC(\tau)^s$. The elements of $C(\tau)$ will also be called $q$-continuous.

**Proposition 2.5.** If a family $\tau \subset \text{PrB}$ satisfies axioms $A_1 - A_3$, then

$$C(\tau)^s = \bigcup \{ C^s \mid C \subset C(\tau), C \text{ is commutative } C^*\text{-subalgebra} \}.$$ 

**Proof.** Let $a \in C(\tau)^s$ and let $C^*(1, a)$ be the commutative $C^*$-subalgebra of $B$, generated by $a$ and the unit 1. By the Gelfand representation theorem $C^*(1, a)$ is isomorphic to the $C^*$-algebra $C(S)$ of all continuous functions on the spectrum of $a$, and any $b \in C^*(1, a)^s$ is associated with the continuous real-valued function $f_b$ on $S$ with $E_b(I) = E_a(f_b^{-1}(I))$ for each $I \subset \mathcal{R}$. This shows that for any $b \in C^*(1, a)^s$ and any open set $I \subset \mathcal{R}$ the $q$-set $E_b(I) \in \tau$, which means that $C^*(1, a)^s \subset C(\tau)^s$ and completes the proof.

It follows from the spectral theory that $C(\tau)^s \subset L(\tau) \cap U(\tau)$, i.e. that any continuous element in $B^s$ is both lower and upper semicontinuous. Set

$$Q(\tau)^s = L(\tau) \cap U(\tau) \quad \text{and} \quad Q(\tau) = Q(\tau)^s + iQ(\tau)^s.$$ 

The elements of $Q(\tau)$ will be called $q$-quasicontinuous. By Lemma 2.2, $Q(\tau)$ is a norm-closed linear subspace of $B$. In the commutative case, when $\tau$ is the usual topology, $Q(\tau)$ certainly coincides with $C(\tau)$. But it is not so for Akemann-Giles topologies.

**Proposition 2.6.** Let $A$ be a $C^*$-algebra and $(B_A, \tau_A) = q \text{ spec } A$. Then

$$Q(\tau_A) = \{ x \in B_A \mid a \times b \in A \forall a, b \in A \} \equiv Q(A),$$

i.e. $Q(\tau_A)$ is the space of all quasimultiples of $A$.

**Proof.** It follows from Lemma 2.3 that $Q(\tau_A) = \overline{A}^m \cap \overline{A}^m$; by [4; 4.1] this intersection is exactly the space $Q(A)$ of all quasimultiples of $A$ in $B_A$.

By the Gelfand-Akemann-Giles theorem, $C(\tau_A) = M(A)$ where $M(A)$ is the $C^*$-algebra of all multiples of $A$ in $B_A$; in the paper [4] an example is given of a $C^*$-algebra $A$ for which $Q(A) \neq M(A)$ and $Q(A)$ is not a Jordan algebra.
AXIOM A4. $\tau = \Pr B \cap L(\tau)$.

Definition 2.7. A family $\tau \subseteq \Pr B$, which obeys the axioms A1–A4, is called a q-topology. The elements of $\tau$ are q-open q-sets the elements of $1 - \tau$ are q-closed q-sets. Given any q-set $e \in \Pr B$, its q-closure is $\bar{e} = \bigwedge\{ f \mid f \text{ is q-closed and } e \subseteq f \}$; similarly its q-interior is $\overset{\circ}{e} = \bigvee\{ g \mid g \text{ is open and } e \supseteq g \}$. The pair $(B, \tau)$ is called the q-topological space.

It follows from Lemma 2.3 and [4; 3.6] that any Akemann-Giles q-topology satisfies axiom A4. In the commutative case that axiom follows from A1–A3, but if $B$ is non-abelian, A4 is independent of them. Axioms A1–A3 imply neither that $C(\tau)$ is a linear subspace of $B$ nor that it is norm-closed.

Theorem 2.8. If $\tau$ is a q-topology, then

$$C(\tau)^* = \{ a \in B^* \mid a^n \in Q(\tau)^* \text{ for all } n \in \mathbb{N} \}.$$  

Proof. If $a \in C(\tau)^*$ then by Proposition 2.5 $a^n \in C(\tau)^*$ for all $n \in \mathbb{N}$ and so $C(\tau)^* \subseteq \{ a \in B^* \mid a^n \in Q(\tau)^* \text{ for all } n \in \mathbb{N} \}$. Conversely, let $b \in B^*$, $b^n \in Q(\tau)^*$ for all $n \in \mathbb{N}$ and denote by $C^*(1, b)^+$ the $C^*$-subalgebra of $B$ generated by $b$ and the unit 1. then through the Stone-Weierstrass theorem we get $C^*(1, b)^+ \subset Q(\tau)^+$. Since for any open $l \in \mathcal{R}$ such that $E_b(l) \neq 0$ there exists an increasing sequence $(b_n) \subseteq C^*(1, b)^+$ with $E_b(l) \subseteq \bigvee_n b_n$, it implies that $E_b(l) \in (Q(\tau)^*)^n \subseteq L(\tau)$. Thus in virtue of axiom A4 $E_b(l)$ is open, whence $b \in C(\tau)^*$.  

Corollary 2.9. If $\tau$ is a q-topology then $C(\tau)^*$ is a norm-closed subset of $B^*$.

Recall that $B^*$ is a real Jordan algebra and $B$ is a Jordan $C^*$-algebra ($JC^*$-algebra) with multiplication $a \circ b = \frac{1}{2}(ab + ba)$.

Proposition 2.10. Let $\tau$ be a q-topology. If $C(\tau)^+$ is a convex cone in $B^*$ then $C(\tau)^*$ is a norm-closed real Jordan subalgebra of $B^*$ and $C(\tau)$ is a $JC^*$-subalgebra of $B$.

Proof. By Proposition 2.5, given $a, b \in C(\tau)^*$ we have $a + \|a\|, b + \|b\| \in C(\tau)^+$ so $(a + b) + (\|a\| + \|b\|) \in C(\tau)^+$ whence $a + b \in C(\tau)^*$. Moreover we get again by Proposition 2.5 $a^2 \in C(\tau)^*$ whenever $a \in C(\tau)^*$ and thus $a \circ b = \frac{1}{2}((a + b)^2 - (a - b)^2) \in C(\tau)^*$ whenever $a, b \in C(\tau)^*$. Together with Corollary 2.9 this proves the first assertion. Take a sequence $(a_n) \subseteq C(\tau)$ which uniformly converges to $b \in B$. Setting $\text{Re} x = \frac{1}{2}(x + x^*)$, $\text{Im} x = \frac{1}{2}(x - x^*) \in B^*$ for any $x \in B$ we see that $\|\text{Re} a_n - \text{Re} b\| \to 0$ and $\|\text{Im} a_n - \text{Im} b\| \to 0$. Since $C(\tau)^*$ is closed, we obtain that $\text{Re} b$, $\text{Im} b \in C(\tau)^*$ which implies $b \in C(\tau)^*$.

Lemma 2.11. Let $\tau$ be a q-topology. Suppose $X \subseteq Q(\tau)^+$ is a convex cone satisfying (i) $X \supseteq \mathbb{R}^+ \cdot 1$; (ii) $a^{1/2} \in Q(\tau)^*$ for any $a \in X$.

Then $X \subseteq C(\tau)^*$.  

Proof. Take $a \in X$. By Theorem 2.8 it suffices to prove that $a^n \in Q(\tau)^+$ whenever $n \in \mathbb{N}$. This assertion is trivial for $n = 1$ and we assume that it has been proved for all values of $n < m$. Consider $\alpha > \|a\|$; since by (i) $1 + \alpha^{-1} a \in X$ we see from (ii)
that the element
\[(1 + \alpha^{-1} a)^{1/2} = 1 + \frac{1}{2} \alpha^{-1} a + \frac{1}{2!} \left( \frac{1}{2} - 1 \right) \alpha^{-2} a^2 + \frac{1}{3!} \left( \frac{1}{2} - 1 \right) \left( \frac{1}{2} - 2 \right) \alpha^{-3} a^3 + \ldots \]
belongs to $Q(\tau)^+$. So $a^m$ is the uniform limit
\[a^m = \lim_{m \to \infty} \alpha^m \left( \frac{1}{m} \right)^{-1} \left( (1 + \alpha^{-1} a)^{1/2} - \sum_{k=0}^{m-1} \left( \frac{1}{k} \right) \alpha^{-k} a^k \right)\]
of elements in $Q(\tau)^x$. Because $Q(\tau)^x$ is closed, we get the lemma.

3. Algebraic regularity of $q$-topology. Next we introduce a condition connecting $\tau$ with the Jordan algebra structure on $B$.

AXIOM A5. If $a \in A^+(\tau)$, then $a^{1/2} \in L(\tau)$; if $a \in A^+(1 - \tau)$, then $a^{1/2} \in U(\tau)$.

An arbitrary Akemann-Giles $q$-topology satisfies this axiom. Indeed, if $\tau = \tau_A$ for a $C^*$-algebra $A$, then $\tau \subset (A^+)^m \equiv A^+(\tau) \subset (A^+)^m$ and $A^+(\tau)^{1/2} \subset (A^+)^m$, since $(A^+)^m \subset A^+$ and $t^{1/2}$ is an operator monotone function on $B^+$. From all that we obtain $(A^+(\tau))^{1/2} \subset L(\tau)$. Similarly we can conclude that $(A^+(1-\tau))^{1/2} \subset U(\tau)$.

**Theorem 3.1.** If a $q$-topology $\tau$ satisfies axiom A5, then $C(\tau)^\prime$ is a norm-closed real Jordan subalgebra of $B$ and $C(\tau)$ is a JC*-subalgebra of $B$.

**Proof.** Let $X$ denote the convex cone $A^+(\tau) \cup A^+(1-\tau)$. In view of Proposition 2.10 it suffices to show that $C(\tau)^\prime = X$. The inclusion $C(\tau)^+ \subset X$ follows from the spectral theory. The inverse inclusion is valid since the cone $X$ satisfies all the conditions of Lemma 2.11 (we have $a^{1/2} \in Q(\tau)^x$ whenever $a \in X$ in virtue of axiom A5 and norm-continuity of the operator function $t^{1/2}$).

Axiom A5 may be weakened to get a necessary and sufficient condition for the set $C(\tau)$ to be a JC*-subalgebra of $B$. Such a weak variant, being equivalent to the original axiom A5 for completely regular $q$-topologies (see Definition 3.4 below), will concern only the part of $q$-topology $\tau$, which can be reproduced by means of $C(\tau)$.

Let us define the \textit{regularization} $\tau_{reg}$ of a $q$-topology $\tau$ by setting $\tau_{reg} = \Pr B \cap (C(\tau)^+)^m$ (in general the projections family $\tau_{reg}$ need not be a $q$-topology).

**Lemma 3.2.** If $\tau \in \Pr B$ satisfies axioms A1-A3, then $C(\tau) = C(\tau_{reg})$.

**Proof.** Clearly, $C(\tau_{reg}) \subset C(\tau)$. Conversely, by Proposition 2.5, given $a \in C(\tau)^\prime$ the JC*-subalgebra $C^*(1, a) \subset C(\tau)$, so for any open $I \subset \mathbb{R}$ the spectral projection $E_a(I)$, being a supremum of an increasing sequence of elements in $C^*(1, a)$, belongs to $\tau_{reg}$. Thus we get $C(\tau)^\prime = C(\tau_{reg})^\prime$ and, consequently, $C(\tau) \subset C(\tau_{reg})$.

AXIOM A5$. If $a \in A^+(\tau_{reg})$, then $a^{1/2} \in L(\tau)$, if $a \in A^+(1-\tau_{reg})$ then $a^{1/2} \in U(\tau)$.

**Theorem 3.3.** A $q$-topology $\tau$ satisfies axiom A5 if $C(\tau)$ is a JC*-subalgebra of $B$. 

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Proof. Necessity follows from Proposition 2.9 and the equality
\[ C(\tau)^+ = A^+(\tau^{\text{reg}}) \cap A^+(1 - \tau^{\text{reg}}), \]
which can be easily deduced from Lemma 3.2 and axiom A5$^c$ in a manner similar to the proof of Theorem 3.1.

Sufficiency. If \( C(\tau) \) is a Jordan algebra, then \( C(\tau)^+ \) is a convex cone and
\[ A^+(\tau^{\text{reg}}) \subset (C(\tau)^+)^m. \]
Indeed, whenever \( a = \sum_{i=1}^n \lambda_i e_i, \quad \lambda_i \geq 0, \quad e_i \in \tau^{\text{reg}}, \)
there exist increasing sequences \( (b_{ik})_{k=1}^\infty \subset C(\tau)^+, \quad i = 1, 2, \ldots, n, \)
with \( e_i = \bigvee_k b_{ik} \) and \( a = \bigvee_k c_k, \)
where \( c_k = \sum_{i=1}^n \lambda_i b_{ik} \) is an increasing sequence in \( C(\tau)^+ \). Now by Proposition 2.5 we have \( (C(\tau)^+)^{1/2} \subset C(\tau)^+ \) and the operator monotonicity of the function \( t^{1/2} \) gives \( ((C(\tau)^+)^m)^{1/2} \subset (C(\tau)^+)^m \subset L(\tau) \).
Similarly we can conclude that \( A^+(1 - \tau^{\text{reg}})^{1/2} \subset U(\tau) \).
Hence axiom A5$^c$ holds.

Definition 3.4. A \( q \)-topology \( \tau \) is called completely regular, if for any \( q \)-point \( x \) disjoint from a closed \( q \)-set \( f \) there exists an element \( a \in C(\tau)^+ \) with \( ax = x \) and \( af = 0 \) (this means that \( a \) takes the value \( 1 \) at \( x \) and the value \( 0 \) on \( f \)).

Any Akemann-Giles \( q \)-topology is completely regular [8; 47].

Theorem 3.5. If a completely regular \( q \)-topology \( \tau \) satisfies axiom A5$^c$, then \( \tau = \tau^{\text{reg}} \).

Proof. Given any open \( q \)-set \( e \in \tau \) put
\[ I(e) = \{(1 + a)^{-1} \cdot a \mid a \in C(\tau)^+, \quad a \leq \lambda e \text{ for some } \lambda > 0\}. \]
Whenever \( (1 + a_i)^{-1} \cdot a_i \in I(e) \), \( i = 1, 2, \) we have \( \bar{a} = (1 + (a_1 + a_2)^{-1} \cdot (a_1 + a_2)) \in I(e) \) and \( (1 + a_i)^{-1} \cdot a_i \leq \bar{a} \) for \( i = 1, 2 \) (since by [6; 16.8] the function \( (1 + t)^{-1} \) is antimonotone and so the function \( (1 + t)^{-1} \cdot t = 1 - (1 + t)^{-1} \) is operator monotone). This means that \( I(e) \) is a directed set and there is a supremum \( e_1 \) of \( I(e) \) in \( B' \). To complete the proof we shall show that \( e = e_1 \). The implication \( a \leq \lambda e \Rightarrow (1 + a)^{-1} \cdot a \leq e \) gives \( e_1 \leq e \). Inasmuch \( \tau \) is completely regular, for any \( q \)-point \( x \leq e \) there exists \( a_x \in C(\tau)^+ \) with \( a_x x = x \) and \( a_x \leq e \). So for each natural \( n \) we have \( (1 + na_x)^{-1} na_x \in I(e) \) whence \( e_1 \geq (1 + na_x)^{-1} na_x \geq (1 + nx)^{-1} .nx \) and \( e_1 \geq x \). Finally, we have \( e_1 \geq \bigvee_{x \leq e} x = e \) and \( e_1 = e \).

Corollary 3.6. A complete regular \( q \)-topology \( \tau \) satisfies axiom A5 iff the set \( C(\tau) \) is a JC*-subalgebra of \( B \).

4. Symmetry: the sixth axiom. Every unitary \( u \in B \) (i.e. such that \( uu^* = 1 = u^*u \)) yields an * automorphism \( \varphi_u : a \to u^*au \) of the \( W^* \)-algebra \( B \), which induces an automorphism of the lattice \( PrB \) onto itself. If \( \tau \) is a \( q \)-topology in \( B \) and a unitary element \( u \) is \( \tau \)-continuous, it is very natural to require \( \varphi_u \) to be a "homeomorphism" of \( \tau \). Such requirement seems to be independent of axioms A1 - A5 and so it becomes our last axiom.
AXIOM A6. $u^*\tau u \subset \tau$ for any unitary $u \in C(\tau)$.
As a matter of fact we need only the weakened variant of A6 like that in § 3.
AXIOM A6°. $u^*\tau^{reg} u \subset \tau$ for any unitary $u \in C(\tau)$.

**Theorem 4.1.** Let $\tau$ be a q-topology. Then $C(\tau)$ is a $C^*$-subalgebra of $B$ iff $\tau$ satisfies axioms A5° and A6°.

**Proof.** Necessity. In view of Theorem 3.3 we need to check axiom A6° only. Take a unitary $u \in C(\tau)$, then $u^* C(\tau)^+ u = C(\tau)^+$ since $\varphi_u$ is an * automorphism and $C(\tau)$ is a $C^*$-algebra. Axiom A6° holds because

$$u^*\tau^{reg} u = u^*(C(\tau)^+)^m \cap \text{Pr}B = (C(\tau)^+)^m \cap \text{Pr}B = \tau^{reg}.$$  

Sufficiency. By Theorem 3.3, $C(\tau)$ is a JC*-subalgebra of $B$ and we have to prove that $i[a, b] = i(a b - b a) \in C(\tau)^s$ whenever $a, b \in C(\tau)^s$ for this implies that $a b = a \circ b + \frac{1}{2} [a, b] \in C(\tau)$. For any $t > 0$ consider the unitary element.

$$u_t = \exp(itb) \equiv \sum_{n=0}^{\infty} \frac{(it)^n}{n!} b^n \in C(\tau)^s.$$  

Then $(1/t) (u^*_t a u_t - a) = a_t$ uniformly converges to $i[a, b]$ as $t$ tends to 0. Since for any unitary $u \in C(\tau)$ we have by the spectral theory, Lemma 3.2 and by virtue of axiom A6°

$$u^* C(\tau)^s u = u^* C(\tau)^s u \subset C(u^*\tau^{reg} u)^s \subset C(\tau)^s,$$

the elements $a_t$ belong to $C(\tau)^s$ and thus $i[a, b] \in C(\tau)^s$.

**Definition 4.2.** A q-topology $\tau$ is a $T_1$ q-topology, if for any two disjoint q-points $x$ and $y$ there exists an open q-set $e \in \tau$ with $e \supseteq x$ and $e y = 0$. It also means that any q-point is $\tau$-closed.

**Proposition 4.3.** Let a $T_1$ q-topology $\tau$ satisfy axioms A5° and A6°, then $\tau$ is completely regular iff $\tau = \tau^{reg}$.

**Proof.** By virtue of Theorem 3.5 it is enough to prove sufficiency. Since $\tau$ is $T_1$, elements of $C(\tau)$ distinguish normal pure states of $B$ and so the $C^*$-algebra $C(\tau)$ is weakly dense in $B$. For any $e \in \tau$ consider the $C^*$-subalgebra $A(e) = \{ a \in C(\tau) \mid e a e = a \}$. By [8; 4.2 and 4.5] $A(e)$ is weakly dense in the $W^*$-algebra $eB e$, so by the transitivity theorem [8; 2.7] for any q-point $x \leq e$ there exists $a \in A(e)^+$ (i.e. $a \leq e$) with $a x = x$.

We now define a $C^*$-topology as a q-topology which obeys axioms A5 and A6. Any Akemann-Giles q-topology is a $C^*$-topology.

5. **Compactness.** Ch. Akemann introduced the notion of q-compactness in terms of the $C^*$-algebra $A$ (see 1.3), but an intrinsic q-topological description of q-compact q-sets has not been given. Nevertheless, Akemann showed in [3] that the intersection
condition for a \( q \)-set \( p \) (for any decreasing net \((q_n)\) of \( \tau_A \)-closed \( q \)-sets \( \forall p \wedge (\bigwedge_{i=1}^{n} q_{a_i}) \neq 0 \) implies \( p \wedge (\bigwedge_{i=1}^{n} q_{a_i}) \neq 0 \)) and the regularity after Effros [7] follow from the \( q \)-compactness of \( p \), it being unknown whether these conditions are sufficient. We make use of a multiplicative version of the intersection condition.

Let us define a \( q \)-topological space to be a pair \((B, \tau)\) where \( \tau \) is a \( q \)-topology. A \( q \)-set \( p \in \text{Pr}B \) is called regular if for any open \( q \)-set \( e \in \tau \) \( \|pe\| = \|\bar{p}e\| \) (\( \bar{p} \) is \( \tau \)-closure of \( p \)).

**Definition 5.1.** Let \((B, \tau)\) be a \( q \)-topological space. A \( q \)-set \( p \in \text{Pr}B \) is called quasi-compact if for any decreasing net \((a_n) \subseteq U(\tau)^+ \) with \( b = \bigwedge_{a_n} b \in U(\tau)^+ \), inf \( \|pb_n p\| = \|pbp\| \). If \((B, \tau)\) is a completely regular \( T_1 \) \( q \)-topological space, then a \( q \)-set \( p \in \text{Pr}B \) is called compact if \( p \) is quasicompact and regular. \((B, \tau)\) is called a compact \( q \)-space if the unit 1 is a compact \( q \)-set.

**Proposition 5.2.** If a completely regular \( T_1 \) \( q \)-topology satisfies axiom A5, then any compact \( q \)-set \( p \) is closed.

**Proof.** Suppose, on the contrary, that \( p \neq \bar{p} \) and consider a \( q \)-point \( x \) with \( x \leq \bar{p} - p \). By Theorem 3.5 there exists an increasing net \((a_n) \subseteq C(\tau)^+ \) with \( 1 - x = \bigvee_{a_n} a_n \). Put \( b = 1 - a_n \), then \( x \leq \bigwedge_{a_n} b \subseteq C(\tau)^+ \subseteq U(\tau)^+ \). For all natural \( n \) and all \( z \) we have \( x \leq E_{b_n}(1 - 1/n, \infty) = e_{s_n} \in \tau \) and \( b_n \geq (n - 1)/n \). Since \( p \) is regular we see that

\[
1 \geq \|pb_n p\| \geq \frac{n - 1}{n} \|pe_{s_n} p\| = \frac{n - 1}{n} \|\bar{pe}_{s_n} \bar{p}\| \geq \frac{n - 1}{n} \|\bar{p}x \bar{p}\| = \frac{n - 1}{n} \|\bar{x}\| = \frac{n - 1}{n} \frac{n}{n}
\]

and finally \( \|pb_p p\| = 1 \). Since \( p \) is compact, this implies that \( \|px p\| = 1 \) which contradicts \( xp = 0 \).

Any Akemann-Giles \( q \)-topology is \( T_1 \) complete regular \([3; III.1 and 8; 3.9]\) so Definition 5.1 of compact \( q \)-sets is applicable.

**Theorem 5.3.** Let \((B_A, \tau_A)\) be the \( q \)-spectrum of a \( C^* \)-algebra \( A \). A \( q \)-set \( p \in \text{Pr}B_A \) is \( q \)-compact after Akemann (see 1.3) iff \( p \) is compact in the sense of Definition 5.1.

**Proof.** In § 2 we have mentioned an isomorphism of \( \bar{A}_m \) on the cone \( U^a(S_A) \). With any \( q \)-closed \( q \)-set \( p \in 1 - \tau_A \) this isomorphism correlates the USC affine function \( \bar{p} \) on \( S_A \) and the closed face \( F(p) = \{ \varphi \in S_A \mid \bar{p}(\varphi) = 1 \} \). The map \( p \mapsto F(p) \) is induced by the Effros-Akemann correspondence between the \( q \)-closed \( q \)-sets and the \( \sigma(A^*, A) \)-closed order ideals of \( A^* \) ([7], [2]). So that map is a bijection of \( 1 - \tau_A \) on the set of all closed faces of \( S_A \) and for each \( p \in 1 - \tau_A \) and \( b \in \bar{A}_m \) we have \( \|pbp\| = \max_{\varphi \in F(p)} |\bar{b}(\varphi)| \).

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Notice that a q-closed p is q-compact after Akemann iff \( F(p) \) is a compact subset of \( S_A \). It may be proved in the same way as Akemann-Urysohn's lemma [3; III.1].

Necessity. It is enough to prove that \( p \) is quasicompact. Consider a decreasing net \((b_\alpha) \subset U(\tau) \) with \( b = \bigwedge \alpha b_\alpha \) and put \( \lambda = \inf_\alpha \| p b_\alpha p \| \geq \| p b p \| \). For each \( \alpha \), \( F_\alpha = \{ \varphi \in F(p) \mid \hat{b}_\alpha(x) \geq \lambda \} \neq \emptyset \) is a closed subset of the compact set \( F(p) \) so \( F_\alpha = \bigcap x F_\alpha = \emptyset \). Take \( \varphi \in F_\alpha \), then \( \hat{b}(\varphi) = \inf_\alpha \hat{b}_\alpha(\varphi) \geq \lambda \) whence \( \| p b p \| = \max_\alpha \hat{b}(\varphi) \geq \lambda \).

 Sufficiency. If \( A \ni 1 \), it follows from Proposition 5.2. Let \( A = A + C_1 \) as in 2.3. Denote as \( \overline{F(p)} = S_{\hat{A}} \) the \( \sigma(\hat{A}^*, \hat{A}) \)-closure of \( F(p) \). Since \( S_{\hat{A}} \) is compact, it is enough to show \( F(p) = \overline{F(p)} \). For any \( \varphi \in \overline{F(p)} \) we have \( \varphi = \varphi_0 + \lambda \varphi_\infty \), where \( \varphi_0 \in A^+, \lambda \geq 0, \varphi_\infty \) is the unique pure state of \( \hat{A} \) which vanishes on \( A \) and \( \| \varphi \| = \| \varphi_0 \| + |\lambda| \). We shall show \( \| \varphi \| = \| \varphi_0 \| \), whence \( \lambda = 0 \) and \( \varphi \in F(p) \). Let \((u_\alpha) \subset C(\tau_0)^+ \) be an increasing approximate unit, then \( \| p u_\alpha p - p \| \to 0 \) since \( (1 - u_\alpha) \subset C(\tau)^+, 0 = \bigwedge_\alpha (1 - u_\alpha) \in B^+ \) and \( p \) is compact. Let us choose \( u_n = u_n \), with \( u_{n+1} \geq u_n \) and \( p u_\alpha p \geq p - 2^{-n} p \), \( n \) being natural. Consider the closed subsets \( F_n = \{ \theta \in \hat{A}^* \mid \hat{u}_n(\theta) \geq 1 - 2^{-n} \} \subset \hat{A}^* \); then \( F(p) \subset F_n \), hence \( \overline{F(p)} \subset F_n \). This means that \( \varphi(u_n) = \varphi_0(u_n) \geq 1 - 2^{-n} \). Since \( n \) was arbitrary, \( \| \varphi_0 \| = 1 = \| \varphi \| \) as desired.

Definition 5.4. A completely regular \( T_1 \) q-topological space \((B, \tau)\) is called locally compact if for any q-point \( x \) there exists an open q-set \( e \ni x \) with the compact q-closure \( \hat{e} \).

Theorem 5.5. A q-topological space \((B, \tau)\) is the q-spectrum of a certain C*-algebra iff \( \tau \) is a locally compact C*-topology.

Proof. Necessity. Let \( A \) be a C*-algebra, then \( \tau_A \) is the complete regular \( T_2 \) C*-topology by the above. For any q-point \( x \in B_A \) take \( a \in A^+ \) with \( ax = x \), then \( e = E_a(1, \infty) = E_a([1, 1]) \geq x, e \in \tau \) and \( \hat{e} \leq E_a([1, 1]) \leq 2a \). So \( e \) is q-compact after Akemann, hence \( \tau \) is compact by Theorem 5.3.

Sufficiency. Let \((B, \tau)\) be a locally compact C*-topological space. Then \( C(\tau) \) is the weakly dense C*-subalgebra of \( B \) and \( \tau = \text{Pr}B \cap (C(\tau)^+)^m \) (see 4.3). Put \((\hat{B}, \hat{\tau}) = = q \text{ spec } C(\tau) \). By [8; 3.4] there exists a central projection \( z \in \text{Pr}\hat{B} \) with \( B \ni z\hat{B} \). Since the indentification \( B = z\hat{B} \) agrees with the inclusions \( C(\tau) \subset B \) and \( C(\tau) \subset \hat{B} \), we have \( \tau = \text{Pr}B \cap (C(\tau)^+)^m = \text{Pr}(z\hat{B}) \cap z(C(\tau)^+)^m = z(\text{Pr}\hat{B} \cap (C(\tau)^+)^m) = z\hat{\tau} \). Let us show \( z \in \hat{\tau} \). By hypothesis, if \( x \leq z \) is a q-point in \( \hat{B} \) (hence in \( B \)) there exists \( e \in \tau \) with \( e \ni x \) and \( \hat{e} \) q-compact. Besides, there exists \( a \in C(\tau)^+ \) with \( ax = x \) and \( a \leq e \). Let \( p \) be the support of \( a \) in \( \hat{B} \). Then \( p \in \hat{\tau} \) and \( x \leq p \). If we show \( p \leq z \), the assertion will follow for then \( z = \bigvee \{ p \in \hat{\tau} \mid x \leq p \leq z, x \) is a q-point \} \) which is \( \hat{\tau} \)-open. To prove \( p \leq z \) we consider any q-point \( y \leq 1 - z \) and show \( ay = 0 \). Indeed, by [8; 3.9 and 4.2] there exists a decreasing net \((b_\alpha) \subset C(\tau)^+ \), with \( y = \bigwedge_\alpha b_\alpha \).
in $\overline{B}$. Then $\|ay\| \leq \inf_a \|ab_a\|$ and
\[
\|ab_a\| = \|(b_{a_2}^{1/2}) \ast (b_{a_2}^{1/2})\| = \|b^{1/2}a^2b^{1/2}\| \leq \|b^{1/2}e^{1/2}\| = \|eb\|.
\]
Since $0 = \bigwedge_d (zb_d)$ we have $\|eb\| \to 0$, for $e$ is a compact q-set. Thus $\|ay\| = 0$, i.e. $ay = 0$. It implies that $py = 0$, hence $p \leq z$. So we have that $z$ is $t$-open. Now consider $A = \{a \in C(\tau) \mid az = a\}$. Then by [8; 5.9] $q$ spec $A = (z\overline{B}, z\overline{t}) = (B, \tau)$, which completes the proof.

**Corollary 5.6.** A $C^*$-topological space $(B, \tau)$ is compact iff $(B, \tau) = q$ spec $C(\tau)$.

This last theorem shows that for an arbitrary locally compact $C^*$-topological space $(B, \tau)$ the $q$-space $(\overline{B}, \overline{t}) = q$ spec $C(\tau)$ may be described as "the Stone-Cech compactification of $(B, \tau)$".

**References**


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