ON BASIC CONCEPTS OF NON-COMMUTATIVE TOPOLOGY

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A non-commutative generalization of locally compact Hausdorff spaces was independently offered by Ch. A. Akemann [1 – 3], R. Giles and H. Kummer [8; 9]. They used the description of such topological spaces in terms of the bounded continuous functions algebra as the matter to be extended to the non-abelian situation. In that way a non-commutative or "quantum" topology associated with a C*-algebra was defined as a certain family of projections in its atomic W*-envelope, and the C*-algebra was interpreted as an algebra of "continuous" elements in accordance with this topology.

In the present paper we give an intrinsic axiomatic definition of a general non-commutative topology in terms of the lattice of all projections in an arbitrary atomic W*-algebra B. The system of axioms connects the properties of non-commutative topology with order, Jordan and C*-structures on B. For all that two key ideas are pursued: firstly, to generalize to non-abelian case the description of any topology by means of the bounded lower semicontinuous functions cone and, secondly, to provide that the set of "continuous" elements in B be a C*-algebra. Moreover, we give an effective characterization of compactness and show that a non-commutative topology is locally compact iff it is the Akemann-Giles topology associated with a certain C*-algebra.

1. Preliminaries. Let us consider, together with any set X, the commutative W*-algebra B(X) of all complex-valued bounded functions on X. The field of all subsets of X may be naturally identified with the atomic boolean algebra PrB(X) of all projections in B(X) (each subset of X is assigned its characteristic function); the points of X are in one to one correspondence with the atoms in the lattice PrB(X). Now, if there is a topology τ on X, it may be regarded as some family of projections in B(X). Namely, τ = PrB(X) ∩ L(X), where L(X) is the convex cone of all lower semi-continuous bounded functions on X. In particular, if τ is completely regular, let C(τ) denote the C*-algebra of all continuous bounded functions on X and let C(τ) denote the set of suprema in B(X) (i.e. pointwise suprema) of all bounded increasing nets of real-valued elements in C(τ), then τ = PrB ∩ C(τ).

It easily follows from the spectral theory that an arbitrary commutative atomic
$W^*$-algebra $M$ is isomorphic to the $W^*$-algebra $B(X)$, where $X$ is the set of all minimal projections in $M$ (i.e., atoms in the atomic boolean algebra $\text{Pr}M$). In the light of these facts the lattice $\text{Pr}B$ of all projections in an arbitrary non-commutative atomic $W^*$-algebra $B$ can be regarded as a non-commutative analogue of the concept of set; minimal projections in $B$ play the role of points, any projection in $B$, being the supremum of minimal ones, plays the role of a subset. Finally, a “non-commutative topology” arises as the appropriate family of projections in a non-commutative atomic $W^*$-algebra.

This outlook at a non-commutative generalization of topology was the basis for the Akemann-Giles analogue of a locally compact Hausdorff space. In what follows we construct a non-commutative analogue of a general topological space and show that the Akemann-Giles construction coincides with ours in the locally compact case.

For the general theory of $C^*$- and $W^*$-algebras we shall make systematic use of books [6] and [12].

1.1. $q$-sets. An atomic $W^*$-algebra $B$ together with the set of all minimal projections therein will be called a $q$-set, the elements of $\text{Pr}B$ $q$-subsets and the atoms in $\text{Pr}B$ $q$-points. Union and intersection of $q$-sets are to be taken in the lattice $\text{Pr}B$. Two $q$-sets $e$ and $f$ will be called disjoint iff $ef = 0$. (The terminology is lifted from [8]).

Any $C^*$-algebra $A$ is associated with a $q$-set as follows. The second conjugate space $A^{**}$ is a $W^*$-algebra; let $Z_A$ be the supremum of all minimal projections in $A^{**}$, then $Z_A$ belongs to the center of the algebra $A^{**}$ [1; p. 278]. Set $B_A = Z_A A^{**}$, then $B_A$ is the atomic $W^*$-algebra and we can consider $A$ as the weakly dense sub-algebra of $B_A$, since $A \subset A^{**}$ and $A \to Z_A A$ is an isomorphism (see [3], p. 1). If $A$ is abelian with the spectrum $X$, then the points of $X$ are in one to one correspondence with the minimal non-zero projections of $A^{**}$ so that $B_A \approx B(X)$ and the Gelfand isomorphism give $A \approx C_0(X) \subset B(X)$.

1.2. Notation. For any subset $E \subset B$ put $E^+ = \{a \in E \mid a \geq 0\}$, $E^s = \{a \in E \mid a = a^*\}$, $E_1 = \{a \in E \mid \|a\| \leq 1\}$. Recall that the self-adjoint part $B^s$ of a $W^*$-algebra $B$ is an ordered space, in which every norm bounded increasing net has a supremum. Let $E^m$ denote the set of suprema in $B^s$ of all norm-bounded increasing nets of elements of $E^*$; put $E_m = -(-E)^m$.

1.3. The Akeman-Giles $Q$-topology. For any $C^*$-algebra $A$ consider $A \subset B_A$ as above. The family of $q$-sets $\tau_A = \text{Pr}B_A \cap (A^+)^m$ will be called the Akemman-Giles $q$-topology on $B_A$. The elements of $\tau_A$ are called $q$-open $q$-sets, the elements of $1 - \tau_A = \{1 - e \mid e \in \tau_A\}$ are called $q$-closed. The pair $(B_A, \tau_A)$ will be called the $q$-spectrum of the $C^*$-algebra $A$ and denote as $q$ spec $A$. If $A$ is commutative, then $\tau_A$ is the usual topology on the spectrum of $A$ and therefore is Hausdorff and locally compact. In the non-abelian case $\tau_A$ has similar properties. Namely, a $q$-topology $\tau_A$
is Hausdorff: i.e., given disjoint q-points x and y there exist disjoint open q-sets e and f with x ≤ e and y ≤ f [8; III.6]. After Akemann a q-set p ∈ PrB_A is called q-compact if p is closed and there exists a ∈ A^+_1 with p ≤ a [3; II.I and II.5]. The q-topology τ_A is locally compact: i.e., for any q-point x there exist an open q-set e and a compact q-set p with x ≤ e ≤ p (this follows from [3; III.I]). In the abelian case, when A ≈ C_0(X) ⊂ B(X), these concepts of q-compactness are equivalent to the usual definitions.

1.4. Gelfand-Akemann-Giles theorem. An element a ∈ B_A^+ is called τ_A-continuous if each spectral projection of a which corresponds to an open subset of the real numbers is also an open q-set. A τ_A-continuous element a is called vanishing at ∞ if each spectral projection of a corresponding to a closed subset of the real numbers which do not contain 0 is q-compact [3; I.I and III.3].

Denote by C(τ_A)^+ the set of all τ_A-continuous elements of B_A^+ and by C_0(τ_A)^+ the set of all elements vanishing at ∞. Set C(τ_A) = C(τ_A)^+ + iC(τ_A)^+ and C_0(τ_A) = C(τ_A)^+ + iC_0(τ_A)^+. The elements of C(τ_A) and C_0(τ_A) will also be called τ_A-continuous and vanishing at ∞, respectively.

Theorem [3; 5; 8]. A C*-algebra A is exactly the algebra of all τ_A-continuous elements of B_A vanishing at ∞, i.e. A = C_0(τ_A); the set of all continuous elements of B_A coincides with the C*-algebra of all multipliers of A, i.e. C(τ_A) = M(A) ≡ ≡ {b ∈ B_A | Ab + bA ⊂ A}.

Throughout the whole paper B will always be used to denote an arbitrary atomic W*-algebra, and τ ⊂ PrB a family of projections (i.e. q-sets) in B. We start to discuss the individual axioms which will connect the properties of τ with various structures on B.

2. Order axioms. It is easy to see that any Akemann-Giles q-topology contains 0 and 1 and that a union of open q-sets is also open. Akemann has shown that in contradistinction to the usual topology, the intersection of two open q-sets is not necessarily open (see a counterexample in [1]). The first three axioms describe the corresponding properties of the general q-topology.

AXIOM A1. 0, 1 ∈ τ;

AXIOM A2. (e_x) ⊂ τ ⇒ V_x e_x ∈ τ, i.e. “union of open q-sets is open”.

AXIOM A3. e, f ∈ τ, [e, f] = 0 ⇒ e ∧ f ∈ τ, i.e. “intersection of two commuting open q-sets is open”.

Remark that actually axiom A3 is an order condition since [e, f] = 0 ⇐⇒ e = (e ∨ f) ∨ (e ∧ (1 − f)). Any Akemann-Giles q-topology satisfies all these axioms [1].

We have noticed in § 1 that in the commutative case a topology of a topological space X may be described algebraically by the equality τ = PrB(X) ∩ L(X). Our fourth axiom will be a non-commutative version of this description, therefore we need an appropriate definition of the class of lower semicontinuous (LSC) elements in B.
Definition 2.1. The set $E \subset B^*$ is lower monotone closed (LMC) if $E = E^m$; $E$ is upper monotone closed (UMC) if $E = E^m$. The minimal LMC set in $B^*$ containing $E$ is called the lower monotone closure of $E$ and is denoted by $L(E)$. The upper monotone closure of $E$ is similarly defined and denoted by $U(E)$.

Lemma 2.2. If $E$ is a convex cone in $B^*$, then $L(E)$ is a convex cone, too. If, besides, $E \supseteq \mathbb{R} \cdot 1$, then the convex cone $L(E)$ is norm-closed in $B^*$.

Proof. Take $\lambda, \mu \in \mathbb{R}^+$ and $a \in E$, then

\[ (*) \quad E \subset M(a) \equiv \{ b \in L(E) \mid \lambda a + \mu b \in L(E) \} \subset L(E) \]

and the set $M(a)$ is LMC. So $M(a) = L(E)$ and for each $a \in E$ and $b \in L(E)$ we have $\lambda a + \mu b \in L(E)$. This implies that for each $a \in L(E)$ $(*)$ is correct and we get similarly $M(a) = L(E)$. The last equality shows that $L(E)$ is a convex cone. Let $E \supseteq \mathbb{R} \cdot 1$. If a sequence $(a_n) \subset L(E)$ converges to an element $b \in B^*$ we may suppose that $\|a_n - b\| \leq 2^{-n}$. Then the increasing sequence $\frac{a_n}{n} = a_n - 2^{-n+1} \cdot 1$ is contained in $L(E)$ and also converges to $b$. Hence $b = \bigvee_n \frac{a_n}{n} \in L(E)$ and $L(E)$ is norm-closed.

Put $A^+(\tau) = \{ \sum_{i=1}^n \lambda_i e_i \mid \lambda_i \geq 0, e_i \in \tau, n \in \mathbb{N} \}$ and consider $A(\tau) = A^+(\tau) + \mathbb{R} \cdot 1$, the minimal convex cone containing $\tau$ and "constants". We now define the class of lower semicontinuous (LSC) elements in $B^*$ as $L(\tau) = L(A(\tau))$. By Lemma 2.2, $L(\tau)$ is a norm-closed convex cone in $B^*$. If $\tau$ were the usual topology on a set $X$, $L(\tau)$ would be the class of all lower semicontinuous real-valued functions on $X$.

Similarly we may define the class $U(\tau)$ of upper semicontinuous (USC) elements in $B^*$ by setting

\[ A^+(1 - \tau) = \{ \sum_{i=1}^n \lambda_i f_i \mid \lambda_i \geq 0, f_i \in 1 - \tau \}, \]

\[ A(1 - \tau) = A^+(1 - \tau) + \mathbb{R} \cdot 1 \quad \text{and} \quad U(\tau) = U(A(1 - \tau)). \]

For any subset $E \subset B^*$ we shall denote by $\bar{E}$ the norm-closure of $E$ in $B^*$.

Lemma 2.3. If $A$ is a $C^*$-algebra and $\tau_A$ is the Akemann-Giles topology associated with $A$, then

\[ L(\tau_A) = \bar{A}^m, \]

where $\bar{A} = A + \mathbb{C} \cdot 1 \subset B_A$ is the $C^*$-algebra obtained by adjoining the unit 1 of $B_A$ to $A$.

Proof. We have $A(\tau_A) \subset \bar{A}^m \subset \bar{A}^m$ since $\tau_A \subset (A^+)^m$. By [4; 3.3] the convex cone $\bar{A}^m$ is LMC, therefore we get $L(\tau_A) \subset \bar{A}^m$. To show the converse inclusion notice that by virtue of the Gelfand-Akemann-Giles theorem (1.4) we have $\bar{A} \subset C(\tau_A)$ and so by the spectral theory $\bar{A}^m \subset A(\tau_A)^m$. This implies $\bar{A}^m \subset L(\tau_A)$, which means $\bar{A}^m \subset L(\tau_A)$ as desired.

The proof of the last lemma was based on the assertion in [4], which was stated
there in terms of $A^{**}$, but applying theorems [9; 3.8 and 4; 2.6] we get the result in terms of $B_{A}$.

From the papers [9; 4] by Pedersen and Akemann we know that there is an isometric map of $\tilde{A}^{*}$ onto the set $L^{c}(S(A))$ of all bounded LSC affine functions on the state space $S_{A} = \{[\varphi] \in A^{**} | \|\varphi\| = 1\}$ provided with the weak* topology of $A^{*}$. Through Lemma 2.3 we may therefore identify the convex cone $L(\tau_{A})$ with the cone $L^{c}(S(A))$ (see also § 5 below).

The next definition coincides in essence with that in 1.1, and in the commutative case it is the usual definition of continuous functions.

**Definition 2.4.** An element $a \in B^{*}$ is called $q$-continuous, if for each open set $I \subset \mathbb{R}$ the spectral projection $E_{a}(I)$ belongs to $\tau$, $a = \int_{\lambda} dE_{\lambda}$ being the spectral representation of $a$. Let $C^{0}(\tau)$ denote the set of all $q$-continuous elements of $B^{*}$ and $C(\tau) = C(\tau)^{+} + iC(\tau)^{-}$. The elements of $C(\tau)$ will also be called $q$-continuous.

**Proposition 2.5.** If a family $\tau \subset \mathfrak{P}B$ satisfies axioms A1 - A3, then

$$C^{0}(\tau)^{+} = \bigcup\{C^{\tau} \mid C \subset C(\tau), C \text{ is commutative } C^{*}-\text{subalgebra}\}.$$  

**Proof.** Let $a \in C^{0}(\tau)^{+}$ and let $C^{0}(1, a)$ be the commutative $C^{*}$-subalgebra of $B$, generated by $a$ and the unit 1. By the Gelfand representation theorem $C^{0}(1, a)$ is isomorphic to the $C^{*}$-algebra $C(\text{Sp}a)$ of all continuous functions on the spectrum of $a$, and any $b \in C^{0}(1, a)^{+}$ is associated with the continuous real-valued function $f_{b}$ on $\text{Sp}a$ with $E_{b}(I) = E_{a}(f_{b}^{-1}(I))$ for each $I \subset \mathbb{R}$. This shows that for any $b \in C^{0}(1, a)^{+}$ and any open set $I \subset \mathbb{R}$ the $q$-set $E_{b}(I) \in \tau$, which means that $C^{0}(1, a)^{+} \subset C^{0}(\tau)^{+}$ and completes the proof.

It follows from the spectral theory that $C^{0}(\tau)^{+} \subset L^{c}(\tau) \cap U^{c}(\tau)$, i.e. that any continuous element in $B^{*}$ is both lower and upper semicontinuous. Set

$$Q(\tau)^{+} = L^{c}(\tau) \cap U^{c}(\tau) \quad \text{and} \quad Q(\tau) = Q(\tau)^{+} + iQ(\tau)^{\cdot}.$$  

The elements of $Q(\tau)$ will be called $q$-quasicontinuous. By Lemma 2.2, $Q(\tau)$ is a norm-closed linear subspace of $B$. In the commutative case, when $\tau$ is the usual topology, $Q(\tau)$ certainly coincides with $C(\tau)$. But it is not so for Akemann-Giles topologies.

**Proposition 2.6.** Let $A$ be a $C^{*}$-algebra and $(B_{A}, \tau_{A}) = q \text{ spec } A$. Then

$$Q(\tau_{A}) = \{x \in B_{A} \mid a \times b \in A \forall a, b \in A\} \equiv Q(A),$$  

i.e. $Q(\tau_{A})$ is the space of all quasimultiples of $A$.

**Proof.** It follows from Lemma 2.3 that $Q(\tau_{A}) = \tilde{A}^{*} \cap \tilde{A}_{m}^{*}$; by [4; 4.1] this intersection is exactly the space $Q(A)$ of all quasimultiples of $A$ in $B_{A}$.

By the Gelfand-Akemann-Giles theorem, $C(\tau_{A}) = M(A)$ where $M(A)$ is the $C^{*}$-algebra of all multiples of $A$ in $B_{A}$; in the paper [4] an example is given of a $C^{*}$-algebra $A$ for which $Q(A) \neq M(A)$ and $Q(A)$ is not a Jordan algebra.
AXIOM A4. \( \tau = \Pr B \cap L(\tau) \).

**Definition 2.7.** A family \( \tau \subseteq \Pr B \), which obeys the axioms A1−A4, is called a q-topology. The elements of \( \tau \) are q-open q-sets the elements of \( 1 - \tau \) are q-closed q-sets. Given any q-set \( e \in \Pr B \), its q-closure is \( \bar{e} = \bigwedge \{ f \mid f \text{ is q-closed and } e \subseteq f \} \); similarly its q-interior is \( ^o e = \bigvee \{ g \mid g \text{ is open and } e \not\subseteq g \} \). The pair \((B, \tau)\) is called the q-topological space.

It follows from Lemma 2.3 and [4; 36] that any Akemann-Giles q-topology satisfies axiom A4. In the commutative case that axiom follows from A1−A3, but if \( B \) is non-abelian, A4 is independent of them. Axioms A1−A3 imply neither that \( C(\tau) \) is a linear subspace of \( B \) nor that it is norm-closed.

**Theorem 2.8.** If \( \tau \) is a q-topology, then

\[
C(\tau)^o = \{ a \in B^o \mid a^n \in Q(\tau)^o \text{ for all } n \in \mathbb{N} \}.
\]

**Proof.** If \( a \in C(\tau)^o \) then by Proposition 2.5 \( a^n \in C(\tau)^o \) for all \( n \in \mathbb{N} \) and so \( C(\tau)^o \subseteq \{ a \in B^o \mid a^n \in Q(\tau)^o \text{ for all } n \in \mathbb{N} \} \). Conversely, let \( b \in B^o \), \( b^n \in Q(\tau)^o \) for all \( n \in \mathbb{N} \) and denote by \( C^*(1, b) \) the \( C^* \)-subalgebra of \( B \) generated by \( b \) and the unit 1. Then through the Stone-Weierstrass theorem we get \( C^*(1, b)^o \subseteq Q(\tau)^o \). Since for any open \( l \in \mathbb{R} \) there exists an increasing sequence \( (b_n) \subseteq C^*(1, b)^o \) with \( E_b(I) = \bigvee_n b_n \), it implies that \( E_b(I) \in (Q(\tau))^o \subseteq L(\tau) \). Thus in virtue of axiom A4 \( E_b(I) \) is open, whence \( b \in C(\tau)^o \).

**Corollary 2.9.** If \( \tau \) is a q-topology then \( C(\tau)^o \) is a norm-closed subset of \( B^o \).

Recall that \( B^o \) is a real Jordan algebra and \( B \) is a Jordan \( C^* \)-algebra (\( JC^* \)-algebra) with multiplication \( a \circ b = \frac{1}{2}(ab + ba) \).

**Proposition 2.10.** Let \( \tau \) be a q-topology. If \( C(\tau)^o \) is a convex cone in \( B^o \) then \( C(\tau)^o \) is a norm-closed real Jordan subalgebra of \( B^o \) and \( C(\tau) \) is a \( JC^* \)-subalgebra of \( B \).

**Proof.** By Proposition 2.5, given \( a, b \in C(\tau)^o \) we have \( a + \|a\|, 1, b + \|b\| \in C(\tau)^o \) so \( (a + b) + (\|a\| + \|b\|) \cdot 1 \in C(\tau)^o \) whence \( a + b \in C(\tau)^o \). Moreover we get again by Proposition 2.5 \( a^2 \in C(\tau)^o \) whenever \( a \in C(\tau)^o \) and thus \( a \cdot b = \frac{1}{2}((a + b)^2 - (a - b)^2) \in C(\tau)^o \) whenever \( a, b \in C(\tau)^o \). Together with Corollary 2.9 this proves the first assertion. Take a sequence \( (a_n) \subseteq C(\tau) \) which uniformly converges to \( b \in B \). Setting \( \text{Re} \, x = = \frac{1}{2}(x + x^*) \), \( \text{Im} \, x = \frac{1}{2}(x - x^*) \in B^* \) for any \( x \in B \) we see that \( \|\text{Re} \, a_n - \text{Re} \, b\| \to 0 \) and \( \|\text{Im} \, a_n - \text{Im} \, b\| \to 0 \). Since \( C(\tau)^o \) is closed, we obtain that \( \text{Re} \, b, \text{Im} \, b \in C(\tau)^o \) which implies \( b \in C(\tau) \).

**Lemma 2.11.** Let \( \tau \) be a q-topology. Suppose \( \rangle \subseteq Q(\tau)^o \) is a convex cone satisfying

(i) \( \langle \tau \rangle \supseteq \mathbb{R}^+ \cdot 1 \); (ii) \( a^{1/2} \in Q(\tau)^o \) for any \( a \in X \).

Then \( X \subseteq C(\tau)^o \).

**Proof.** Take \( a \in X \). By Theorem 2.8 it suffices to prove that \( a^n \in Q(\tau)^o \) whenever \( n \in \mathbb{N} \). This assertion is trivial for \( n = 1 \) and we assume that it has been proved for all values of \( n < m \). Consider \( x > \|a\| \); since by (i) \( 1 + x^{-1}a \in X \) we see from (ii)
that the element
\[(1 + \alpha^{-1} a)^{1/2} = 1 + \frac{1}{2} \alpha^{-1} a + \frac{1}{4} (1 - 1) \alpha^{-2} a^2 + \frac{1}{3!} (1 - 2) \alpha^{-3} a^3 + \ldots\]
belongs to \(Q(\tau)^+\). So \(a^m\) is the uniform limit
\[a^m = \lim_{\alpha \to +\infty} \alpha^m \left( \frac{1}{m} \right)^{-1} \left[ (1 + \alpha^{-1} a)^{1/2} - \sum_{k=0}^{m-1} \left( \frac{1}{k} \right) \alpha^{-k} a^k \right]\]
of elements in \(Q(\tau)^+\). Because \(Q(\tau)^+\) is closed, we get the lemma.

3. Algebraic regularity of \(q\)-topology. Next we introduce a condition connecting \(\tau\) with the Jordan algebra structure on \(B\).

**Axiom A5.** If \(a \in A^+(\tau)\), then \(a^{1/2} \in L(\tau)\); if \(a \in A^+(1 - \tau)\), then \(a^{1/2} \in U(\tau)\).

An arbitrary Akemann-Giles \(q\)-topology satisfies this axiom. Indeed, if \(\tau = \tau_A\) for a \(C^*\)-algebra \(A\), then \(\tau \subset (A^+)^m\) whence \(A^+(\tau) \subset (A^+)^m\) and \(A^+(\tau)^{1/2} \subset (A^+)^m\) Since \((A^+)^{1/2} \subset A^+\) and \(t^{1/2}\) is an operator monotone function on \(B^+\) we have \(((A^+)^m)^{1/2} \subset (A^+)^m\); applying Lemma 2.3 we see that \((A^+)^m \subset L(\tau)\). From all that we obtain \((A^+(\tau))^{1/2} \subset L(\tau)\). Similarly, we can conclude that \((A^+(1 - \tau))^{1/2} \subset U(\tau)\).

**Theorem 3.1.** If a \(q\)-topology \(\tau\) satisfies axiom A5, then \(C(\tau)^+\) is a norm-closed real Jordan subalgebra of \(B\) and \(C(\tau)^+\) is a \(JC^*\)-subalgebra of \(B\).

**Proof.** Let \(X\) denote the convex cone \(A^+(\tau) \cap A^+(1 - \tau)\). In view of Proposition 2.10 it suffices to show that \(C(\tau)^+ = X\). The inclusion \(C(\tau)^+ \subset X\) follows from the spectral theory. The inverse inclusion is valid since the cone \(X\) satisfies all the conditions of Lemma 2.11 (we have \(a^{1/2} \in Q(\tau)^+\) whenever \(a \in X\) in virtue of axiom A5 and norm-continuity of the operator function \(t^{1/2}\)).

Axiom A5 may be weakened to get a necessary and sufficient condition for the set \(C(\tau)^+\) to be a \(JC^*\)-subalgebra of \(B\). Such a weak variant, being equivalent to the original axioms for completely regular \(q\)-topologies (see Definition 3.4 below), will concern only the part of \(q\)-topology \(\tau\), which can be reproduced by means of \(C(\tau)^+\).

Let us define the regularized \(\tau^{reg}\) of a \(q\)-topology \(\tau\) by setting \(\tau^{reg} = \text{Pr} B \cap C(\tau)^+\) (in general the projections family \(\tau^{reg}\) need not be a \(q\)-topology).

**Lemma 3.2.** If \(\tau \in \text{Pr} B\) satisfies axioms A1 – A3, then \(C(\tau) = C(\tau^{reg})\).

**Proof.** Clearly, \(C(\tau^{reg}) \subset C(\tau)^+\). Conversely, by Proposition 2.5, given \(a \in C(\tau)^+\) the \(C^*\)-subalgebra \(C^*(1, a) \subset C(\tau)\), so for any open \(I \subset R\) the spectral projection \(E_d(I)\), being a supremum of an increasing sequence of elements in \(C^*(1, a)\), belongs to \(\tau^{reg}\). Thus we get \(C(\tau)^+ \subset C(\tau^{reg})\) and, consequently, \(C(\tau) \subset C(\tau^{reg})\).

**Axiom A5.** If \(a \in A^+(\tau^{reg})\), then \(a^{1/2} \in L(\tau)\), if \(a \in A^+(1 - \tau^{reg})\) then \(a^{1/2} \in U(\tau)\).

**Theorem 3.3.** A \(q\)-topology \(\tau\) satisfies axiom A5° iff \(C(\tau)^+\) is a \(JC^*\)-subalgebra of \(B\).
Proof. Necessity follows from Proposition 2.9 and the equality
\[ C(\tau)^+ = A^+(\tau_{\text{reg}}) \cap A^+(1 - \tau_{\text{reg}}), \]
which can be easily deduced from Lemma 3.2 and axiom A5\degree in a manner similar to the proof of Theorem 3.1.

Sufficiency. If \( C(\tau) \) is a Jordan algebra, then \( C(\tau)^+ \) is a convex cone and \( A^+(\tau_{\text{reg}}) \subseteq (C(\tau)^+)_m \). Indeed, whenever \( a = \sum_{i=1}^{n} \lambda_i e_i, \lambda_i \geq 0, e_i \in \tau_{\text{reg}}, \) there exist increasing sequences \( (b_{ik})_{k=1}^{\infty} \subseteq C(\tau)^+ \), \( i = 1, 2, \ldots, n, \) with \( e_i = \bigvee_k b_{ik} \) and \( a = \bigvee_k c_k \), where \( c_k = \sum_{i=1}^{n} \lambda_i b_{ik} \) is an increasing sequence in \( C(\tau)^+ \). Now by Proposition 2.5 we have \( (C(\tau)^+)^{1/2} \subseteq C(\tau)^+ \) and the operator monotonicity of the function \( t^{1/2} \) gives \( (C(\tau)^+)^{1/2} \subseteq (C(\tau)^+)^m \subseteq L(\tau) \). Similarly we can conclude that \( A^+(1 - \tau_{\text{reg}})^{1/2} \subseteq U(\tau) \). Hence axiom A5\degree holds.

Definition 3.4. A q-topology \( \tau \) is called completely regular, if for any q-point \( x \) disjoint from a closed q-set \( f \) there exists an element \( a \in C(\tau)^+ \) with \( ax = x \) and \( af = 0 \) (this means that \( a \) takes the value 1 at \( x \) and the value 0 on \( f \)).

Any Akemann-Giles q-topology is completely regular [8; 47].

Theorem 3.5. If a completely regular q-topology \( \tau \) satisfies axiom A5\degree, then \( \tau = \tau_{\text{reg}} \).

Proof. Given any open q-set \( e \in \tau \) put
\[ I(e) = \{(1 + a)^{-1} \cdot a \mid a \in C(\tau)^+, a \leq \lambda e \text{ for some } \lambda > 0 \}. \]

Whenever \((1 + a_i)^{-1} \cdot a_i \in I(e), i = 1, 2,\) we have \( \bar{a} = (1 + (a_1 + a_2))^{-1} \cdot (a_1 + a_2) \in I(e) \) and \((1 + a_i)^{-1} \cdot a_i \leq \bar{a} \) for \( i = 1, 2 \) (since by [6; 16.8] the function \((1 + t)^{-1} \) is antimonotone and so the function \((1 + t)^{-1} \cdot t = 1 - (1 + t)^{-1} \) is operator monotone). This means that \( I(e) \) is a directed set and there is a supremum \( e_1 \) of \( I(e) \) in \( B^* \). To complete the proof we shall show that \( e = e_1 \). The implication \( a \leq \lambda e \Rightarrow (1 + a)^{-1} \cdot a \leq \lambda e \) gives \( e_1 \leq e \). Inasmuch \( \tau \) is completely regular, for any q-point \( x \leq e \) there exists \( a_x \in C(\tau)^+ \) with \( a_x x = x \) and \( a_x \leq e \). So for each natural \( n \) we have \((1 + na_x)^{-1} na_x \in I(e) \) whence \( e_1 \geq (1 + na_x)^{-1} na_x \geq (1 + nx)^{-1} \cdot nx \) and \( e_1 \geq x \). Finally, we have \( e_1 \geq \bigvee_{x \leq e} x = e \) and \( e_1 = e \).

Corollary 3.6. A complete regular q-topology \( \tau \) satisfies axiom A5 iff the set \( C(\tau) \) is a JC*-subalgebra of \( B \).

4. Symmetry: the sixth axiom. Every unitary \( u \in B \) (i.e., such that \( uu^* = 1 = u^*u \)) yields an * automorphism \( \varphi_u : a \rightarrow u^*au \) of the W*-algebra \( B \), which induces an automorphism of the lattice \( \text{Pr}B \) onto itself. If \( \tau \) is a q-topology in \( B \) and a unitary element \( u \) is \( \tau \)-continuous, it is very natural to require \( \varphi_u \) to be a "homeomorphism" of \( \tau \). Such requirement seems to be independent of axioms A1 - A5 and so it becomes our last axiom.

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AXIOM A6. $u^*u \subseteq \tau$ for any unitary $u \in C(\tau)$.
As a matter of fact we need only the weakened variant of A6 like that in § 3.
AXIOM A6°. $u^*\tau^{\text{reg}}u \subseteq \tau$ for any unitary $u \in C(\tau)$.

**Theorem 4.1.** Let $\tau$ be a q-topology. Then $C(\tau)$ is a $C^*$-subalgebra of $B$ iff $\tau$ satisfies axioms A5° and A6°.

**Proof. Necessity.** In view of Theorem 3.3 we need to check axiom A6° only. Take a unitary $u \in C(\tau)$, then $u^* C(\tau)^+ = C(\tau)^+$ since $\varphi_u$ is an * automorphism and $C(\tau)$ is a $C^*$-algebra. Axiom A6° holds because

$$u^*\tau^{\text{reg}}u = u^*((C(\tau)^+)^n \cap \text{Pr}B)u = u^*(C(\tau)^+)^n u \cap \text{Pr}B = (C(\tau)^+)^n \cap \text{Pr}B = \tau^{\text{reg}}.$$

**Sufficiency.** By Theorem 3.3, $C(\tau)$ is a JC*-subalgebra of $B$ and we have to prove that $i[a, b] = i(ab - ba) = C(\tau)^+$ whenever $a, b \in C(\tau)^+$ for this implies that $ab = ab + \frac{1}{2}[a, b] \in C(\tau)$. For any $t > 0$ consider the unitary element.

$$u_t = \exp(itb) \equiv \sum_{n=0}^{\infty} \frac{(it)^n}{n!} b^n \in C(\tau)^+.$$

Then $(1/t)(u_t^*au_t - a) = a$, uniformly converges to $i[a, b]$ as $t$ tends to 0. Since for any unitary $u \in C(\tau)$ we have by the spectral theory, Lemma 3.2 and by virtue of axiom A6°

$$u^* C(\tau)^+ u = u^* C(\tau)^+ u = C(u^*\tau^{\text{reg}}u)^+ \subseteq C(\tau)^+,$$

the elements $a, b$ belong to $C(\tau)^+$ and thus $i[a, b] \subseteq C(\tau)^+$.

**Definition 4.2.** A q-topology $\tau$ is a $T_1$ q-topology, if for any two disjoint q-points $x$ and $y$ there exists an open q-set $e \in \tau$ with $e \supseteq x$ and $ey = 0$. It also means that any q-point is $\tau$-closed.

**Proposition 4.3.** Let a $T_1$ q-topology $\tau$ satisfy axioms A5° and A6°, then $\tau$ is completely regular iff $\tau = \tau^{\text{reg}}$.

**Proof.** By virtue of Theorem 3.5 it is enough to prove sufficiency. Since $\tau$ is $T_1$, elements of $C(\tau)$ distinguish normal pure states of $B$ and so the $C^*$-algebra $C(\tau)$ is weakly dense in $B$. For any $e \in \tau$ consider the $C^*$-subalgebra $A(e) = \{a \in C(\tau) \mid eae = a\}$. By [8; 4.2 and 4.5] $A(e)$ is weakly dense in the $W^*$-algebra $eBe$, so by the transitivity theorem [8; 2.7] for any q-point $x \leq e$ there exists $a \in A(e)^+$ (i.e. $a \leq e$) with $ax = x$.

We now define a $C^*$-topology as a q-topology which obeys axioms A5 and A6. Any Akemann-Giles q-topology is a $C^*$-topology.

5. **Compactness.** Ch. Akemann introduced the notion of q-compactness in terms of the $C^*$-algebra $A$ (see 1.3), but an intrinsic q-topological description of q-compact q-sets has not been given. Nevertheless, Akemann showed in [3] that the intersection
condition for a \( q \)-set \( p \) (for any decreasing net \( (q_n) \) of \( \tau_A \)-closed \( q \)-sets \( \forall p \wedge (\bigwedge_{i=1}^n q_n) \neq 0 \) implies \( p \wedge (\bigwedge q_n) \neq 0 \)) and the regularity after Effros [7] follow from the \( q \)-compactness of \( p \), it being unknown whether these conditions are sufficient. We make use of a multiplicative version of the intersection condition.

Let us define a \( q \)-topological space to be a pair \((B, \tau)\) where \( \tau \) is a \( q \)-topology. A \( q \)-set \( p \in \Pr B \) is called regular if for any open \( q \)-set \( e \in \tau \), \( \|pe\| = \|\bar{p}e\| \) (\( \bar{p} \) is \( \tau \)-closure of \( p \)).

**Definition 5.1.** Let \((B, \tau)\) be a \( q \)-topological space. A \( q \)-set \( p \in \Pr B \) is called quasi-compact if for any decreasing net \( (b_n) \subseteq U(\tau)^{+} \) with \( b = \bigwedge b_n \in U(\tau)^{+} \), \( \inf \|pb_n p\| = \|pbp\| \). If \((B, \tau)\) is a completely regular \( T_1 \) \( q \)-topological space, then a \( q \)-set \( p \in \Pr B \) is called compact if \( p \) is quasicompact and regular. \((B, \tau)\) is called a compact \( q \)-space if the unit 1 is a compact \( q \)-set.

**Proposition 5.2.** If a completely regular \( T_1 \) \( q \)-topology satisfies axiom A5, then any compact \( q \)-set \( p \) is closed.

**Proof.** Suppose, on the contrary, that \( p \neq \bar{p} \) and consider a \( q \)-point \( x \) with \( x \leq \bar{p} - p \). By Theorem 3.5 there exists an increasing net \( (a_x) \subseteq C(\tau)_1^{+} \) with \( 1 - x = \bigvee a_x \). Put \( b = 1 - a_x \), then \( x = \bigwedge b_x \), \( (b_x) \subseteq C(\tau)^{+} \subseteq U(\tau)^{+} \). For all natural \( n \) and all \( \alpha \) we have \( x \leq E_{a_n}\left((1 - 1/n, \infty)\right) \equiv e_{a_n} \in \tau \) and \( b_x \geq \left((n-1)/n\right) e_{a_n} \). Since \( p \) is regular we see that

\[
1 \geq \|pb_x p\| \geq \frac{n-1}{n} \|pe_{a_n} p\| = \frac{n-1}{n} \|\bar{p}e_{a_n} p\| \geq \frac{n-1}{n} \|\bar{p} x \| =
\]

and finally \( \|pb_x p\| = 1 \). Since \( p \) is compact, this implies that \( \|px p\| = 1 \) which contradicts \( xp = 0 \).

Any Akemann-Giles \( q \)-topology is \( T_1 \) complete regular [3; III.1 and 8; 3.9] so Definition 5.1 of compact \( q \)-sets is applicable.

**Theorem 5.3.** Let \((B_A, \tau_A)\) be the \( q \)-spectrum of a \( C^* \)-algebra \( A \). A \( q \)-set \( p \in \Pr B_A \) is \( q \)-compact after Akemann (see 1.3) iff \( p \) is compact in the sense of Definition 5.1.

**Proof.** In § 2 we have mentioned an isomorphism of \( \widehat{\bigwedge}_m \) on the cone \( U^a(S_A) \). With any \( q \)-closed \( q \)-set \( p \in \tau_A \) this isomorphism correlates the USC affine function \( \hat{p} \) on \( S_A \) and the closed face \( F(p) = \{ \phi \in S_A \mid \hat{p}(\phi) = 1 \} \). The map \( p \mapsto F(p) \) is induced by the Effros-Akemann correspondence between the \( q \)-closed \( q \)-sets and the \( \sigma(A^*, A) \)-closed order ideals of \( A^* \) ([7], [2]). So that map is a bijection of \( 1 - \tau_A \) on the set of all closed faces of \( S_A \) and for each \( p \in 1 - \tau_A \) and \( b \in \bigwedge_m \) we have \( \|pbp\| = \max_{\phi \in F(p)} |\hat{b}(\phi)| \).

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Notice that a q-closed p is q-compact after Akemann iff \( F(p) \) is a compact subset of \( S_{\alpha} \). It may be proved in the same way as Akemann-Urysohn's lemma \[3; \text{III.1}].

Necessity. It is enough to prove that \( p \) is quasicompact. Consider a decreasing net \( (b_n) \subset U(\alpha)^{+} \) with \( b = \bigwedge_n b_n \) and put \( \lambda = \inf_a \| pb_n p \| \geq \| p b p \|. \) For each \( \alpha \), \( F_n = \{ \varphi \in F(p) \mid \hat{b}_n(\varphi) \geq \lambda \} \in \emptyset \) is a closed subset of the compact set \( F(p) \) so \( F_0 = \bigcap F_n = \emptyset \). Take \( \varphi \in F_0 \), then \( \tilde{b}(\varphi) = \inf_a \hat{b}_a(\varphi) \geq \lambda \) whence \( \| p b p \| = \max_{\varphi \in F(p)} \tilde{b}(\varphi) \geq \lambda \).

Sufficiency. If \( A \cong 1 \), it follows from Proposition 5.2. Let \( A \cong 1 \) and \( \tilde{A} = A + C \). Denote as \( \overline{F(p)} = S_{\tilde{A}} \) the \( \sigma(\tilde{A}^*, \tilde{A}) \)-closure of \( F(p) \). Since \( S_{\tilde{A}} \) is compact, it is enough to show \( F(p) = \overline{F(p)} \). For any \( \varphi \in \overline{F(p)} \) we have \( \varphi = \varphi_0 + \lambda \varphi_\infty \), where \( \varphi_0 \in \tilde{A}^{*+}, \lambda \geq 0, \varphi_\infty \) is the unique pure state of \( \tilde{A} \) which vanishes on \( A \) and \( \| \varphi \| = \| \varphi_0 \| + |\lambda| \). We shall show \( \| \varphi \| \geq \| \varphi_0 \| \), whence \( \lambda = 0 \) and \( \varphi \in F(p) \). Let \( (u_n) \subset \subset \tilde{A}^* \) be an increasing approximate unit, then \( \| p u_n p - p \| \to 0 \) since \( (1 - u_n) \subset \subset C(\tilde{A})^{*+} \), \( 0 = \Lambda_\alpha(1 - u_n) \) in \( B^* \) and \( p \) is compact. Let us choose \( u_n = u_n^* \) with \( u_{n+1} \geq u_n \) and \( p u_n p \geq p - 2^{-n} p \), \( n \) being natural. Consider the closed subsets \( F_n = \{ \theta \in \tilde{A}^{*+} \mid \hat{u}_n(\theta) \geq 1 - 2^{-n} \} \subset \tilde{A}^+ + \); then \( F(p) \subset F_n \), hence \( \overline{F(p)} \subset F_n \). This means that \( \varphi(u_n) = \varphi_0(u_n) \geq 1 - 2^{-n} \). Since \( n \) was arbitrary, \( \| \varphi_0 \| = 1 = \| \varphi \| \) as desired.

Definition 5.4. A completely regular \( T_1 \) q-topological space \( (B, \tau) \) is called locally compact if for any q-point \( x \) there exists an open q-set \( e \ni x \) with the compact q-closure \( \tilde{e} \).

Theorem 5.5. A q-topological space \( (B, \tau) \) is the q-spectrum of a certain \( C^* \)-algebra iff \( \tau \) is a locally compact \( C^* \)-topology.

Proof. Necessity. Let \( A \) be a \( C^* \)-algebra, then \( \tau_A \) is the complete regular \( T_2 \) \( C^* \)-topology by the above. For any q-point \( e \subset B \), take \( a \in A^+ \) with \( a x = x \), then \( e = E_a(\frac{1}{2}, \infty]) = E_a([\frac{1}{2}, 1]) \ni x, e \in \tau \) and \( \tilde{e} \subset E_a(\frac{1}{2}, 1) \subset 2a \). So \( e \) is q-compact after Akemann, hence \( e \) is compact by Theorem 5.3.

Sufficiency. Let \( (B, \tau) \) be a locally compact \( C^* \)-topological space. Then \( C(\tau) \) is the weakly dense \( C^* \)-subalgebra of \( B \) and \( \tau = \text{PrB} \cap (C^+(\tau))^{m} \) (see 4.3). Put \( \hat{(\tilde{B}, \tilde{\tau})} = q \text{ spec } C(\tau) \). By \[8; 3.4] there exists a central projection \( z \in \text{PrB} \approx z \hat{B} \).

Since the indentification \( \hat{B} = z \hat{B} \) agrees with the inclusions \( C(\tau) \subset B \) and \( C(\tau) \subset \hat{B} \), we have

\[ \tau = \text{PrB} \cap (C(\tau)^{+})^{m} = \text{Pr}(z \hat{B}) \cap z(C(\tau)^{+})^{m} = z(\text{PrB} \cap (C(\tau)^{+})^{m}) = z \tilde{\tau}. \]

Let us show \( z \in \tilde{\tau} \). By hypothesis, if \( x \subset z \) is a q-point in \( \tilde{B} \) (hence in \( B \)) there exists \( e \in \tau \) with \( e \ni x \) and \( \tilde{e} \) q-compact. Besides, there exists \( a \in C(\tau)^{+} \) with \( a x = x \) and \( a \subset e \). Let \( p \) be the support of \( a \) in \( \tilde{B} \). Then \( p \in \tilde{\tau} \) and \( x \subset p \). If we show \( p \subset z \), the assertion will follow for \( z = \bigvee \{ \hat{p} \in \tilde{\tau} \mid x \subset \hat{p} \subset z, x \text{ is a q-point} \} \) which is \( \tilde{\tau} \)-open. To prove \( p \subset z \) we consider any q-point \( y \subset 1 - z \) and show \( ay = 0 \).

Indeed, by \[8; 3.9 \text{ and } 4.2 \] there exists a decreasing net \( (b_n) \subset C(\tau)^{+}, \) with \( y = \bigwedge_n b_n \).
in \( \mathcal{B} \). Then \( \|a y a\| \leq \inf \|a b \| \) and

\[
\|a b \| = \left\| (b_{a}^{1/2}) \ast (b_{a}^{1/2}) \right\| = \left\| b^{1/2} a^{2} b^{1/2} \right\| \leq \left\| b^{1/2} \bar{e} b^{1/2} \right\| = \left\| \bar{e} \right\|.
\]

Since \( \bigwedge_{a} (z b_{a}) \) we have \( \| \bar{e} b_{a} \| \to 0 \), for \( \bar{e} \) is a compact \( q \)-set. Thus \( \|a y a\| = 0 \), i.e. \( a y = 0 \). It implies that \( p y = 0 \), hence \( p \leq z \). So we have that \( z \) is \( \tau \)-open. Now consider \( A = \{ a \in C(\tau) \mid a z = a \} \). Then by [8; 5.9] \( q \) spec \( A = (z \mathcal{B}, z \bar{\tau}) = (B, \tau) \), which completes the proof.

**Corollary 5.6.** A C*-topological space \((B, \tau)\) is compact iff \((B, \tau) = q \) spec \( C(\tau) \).

This last theorem shows that for an arbitrary locally compact C*-topological space \((B, \tau)\) the \( q \)-space \((\mathcal{B}, \bar{\tau}) = q \) spec \( C(\tau) \) may be described as "the Stone-Čech compactification of \((B, \tau)\)."

**References**


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