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REPRESENTATIVE PROPERTIES OF THE QUASI-ORDERED SET $F(\alpha, M)$

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In [5] V. Novák improved a result of M. Novotný in [4] proving that a set of type $F(\omega_\nu . 2, \aleph_\nu)$ is an $\aleph_\nu$-universal quasi-ordered set. Moreover, he used the quasi-ordered set $F(\alpha, M)$ for the representation of ordered sets and showed that a set of type $F(\omega_\nu, \aleph_\nu)$ is an $\aleph_\nu$- universal quasi-ordered set for every regular cardinal number $\aleph_\nu$. Finally, L. Mišík [6] proved that a set of type $F(\omega_\nu, \aleph_\nu)$ is an $\aleph_\nu$-universal quasi-ordered set for every number $\aleph_\nu$. In this paper the above-mentioned results are improved and supplemented.

A *quasi-ordered* set is a non-empty set G together with a reflexive and transitive binary relation $\leqq$ (see for instance [1]). If, moreover, the relation $\leqq$ is antisymmetric, the set G is said to be *ordered*. A *chain* is defined as an ordered set such that we have either $x \leqq y$ or $y \leqq x$ for each pair of its elements x, y. By an *antichain* we understand an ordered set for which the implication $x \leqq y \Rightarrow x = y$ holds for each pair of its elements x, y. Two quasi-ordered set G, G' are called *isomorphic* if there exists such a one-one mapping f of the set G onto G' that $x, y \in G$, $x \leqq y \Leftrightarrow f(x) \leqq f(y)$. A set H with a binary relation is called an *m-universal set for quasi-ordered sets* (where $m > 0$ is a cardinality) if for every quasi-ordered set G with card $G \leqq m$ there exists a subset $H' \subseteq H$ isomorphic with G. An *m-universal set for ordered sets*, an *m-universal set for chains* and an *m-universal set for antichains* are defined in an analogous way. If an m-universal set for quasi-ordered sets is quasi-ordered, then we call it an *m-universal quasi-ordered set.*

Let us recall one important property of every quasi-ordered set. If G is a quasi-ordered set, $x, y \in G$, then put $x \equiv y$ if and only if $x \leqq y$, $y \leqq x$. Then the relation $\equiv$ is an equivalence relation, i.e. a reflexive, symmetric and transitive binary relation, which defines a decomposition $\bar{G}$ of G. Let $X, Y \in \bar{G}$ and put $X \leqq Y$ if and only if $x \leqq y$ for any $x \in X$, $y \in Y$. Then the set $\bar{G}$ is an ordered set (see [1]).

Let M be a non-empty set and $\alpha > 0$ an ordinal number. Denote by $F(\alpha, M)$ the set of all sequences of type α consisting of elements of the set M together with the relation $\leqq$ defined as follows: $\{a_\lambda \mid \lambda < \alpha\} \leqq \{b_\lambda \mid \lambda < \alpha\}$ if and only if there exists a strictly increasing sequence $\{\beta_\lambda \mid \lambda < \alpha\}$ of type α of ordinal numbers less than α

such that $a_\lambda = b_{\beta_\lambda}$ for every $\lambda < \alpha$. It is easy to prove that the relation $\leqq$ is reflexive and transitive so that $F(\alpha, M)$ is a quasi-ordered set. This relation, however, is in general not antisymmetric as is shown in [4]. Therefore $F(\alpha, M)$ is generally not an ordered set. If N is a set with card N = card M, then clearly $F(\alpha, N)$ is isomorphic with $F(\alpha, M)$ so that the type of the set $F(\alpha, M)$ depends only on the cardinality m of the set M. We denote this type by $F(\alpha, m)$. Clearly, for $\alpha < \omega_0$ the set of type $F(\alpha, m)$ is an antichain of power $m^{\text{card}\,\alpha}$.

If α is an ordinal number, then we denote the set of all ordinal numbers less than α ordered according to their magnitude by $W(\alpha)$. It is known that $W(\alpha)$ is a chain of type α (see [2]). Let $\{a_\lambda \mid \lambda < \alpha\}$ be a sequence of type α. Let $G = \{x \mid$ there exists an ordinal number $\lambda < \alpha$ such that $a_\lambda = x\}$. For every $x \in G$ put $m_x(\{a_\lambda \mid \lambda < \alpha\}) =$ $=$ card $\{\lambda \mid \lambda \in W(\alpha), a_\lambda = x\}$. We shall need the following two lemmas proved in [5]:

Lemma 1. *Let G be a non-empty set such that* card $G \leqq \aleph_\nu$. *Then the elements of the set G can be written in the form of a sequence of type ω_ν, $\{a_\lambda \mid \lambda < \omega_\nu\}$, such that $m_x(\{a_\lambda \mid \lambda < \omega_\nu\}) = \aleph_\nu$ for every $x \in G$.*

Lemma 2. *Let G be a set with* card $G = m$ *where $2 \leqq m \leqq \aleph_\nu$. Let $\mathscr{S}$ be the set of all sequences of type ω_ν consisting of elements of the set G and such that $m_x(\{a_\lambda \mid \lambda < \omega_\nu\}) = \aleph_\nu$ for any sequence $\{a_\lambda \mid \lambda < \omega_\nu\} \in \mathscr{S}$ and any element $x \in G$. Then* card $\mathscr{S} = 2^{\aleph_\nu}$.

Let α denote a given ordinal number. If α_1 and α_2 are ordinal numbers such that $\alpha = \alpha_1 + \alpha_2$, then the number α_2 is called the *remainder of number α corresponding to the segment α_1* (see [3]). Now we shall prove the following important theorem:

Theorem 1. *Let α, β be ordinal numbers, $0 < \alpha \leqq \beta$, and let m, n be cardinal numbers, $0 < m \leqq n$. Let at least one of the following three assumptions hold:*

(I) $m < n$,

(II) $m \geqq \aleph_0$,

(III) $\alpha_2 + (\beta - \alpha) > \beta - \alpha$ *for every remainder $\alpha_2 > 0$ of number α.*

Then for every quasi-ordered set $F(\alpha, M)$ of type $F(\alpha, m)$ there exists a subset of a quasi-ordered set of type $F(\beta, n)$ isomorphic with $F(\alpha, M)$.

Proof. Let $F(\alpha, M)$, $F(\beta, N)$ be quasi-ordered sets of types $F(\alpha, m)$, $F(\beta, n)$ where $0 < \alpha \leqq \beta$, $0 < m \leqq n$, i.e. card $M = m$, card $N = n$. We can suppose $M \subseteq N$ without loss of generality.

Let the assumption (I) hold. Then the set $N - M$ is non-empty. Let $x \in N - M$ be an element and for every sequence $a = \{a_\lambda \mid \lambda < \alpha\} \in F(\alpha, M)$ put $\varphi(a) = b =$ $= \{b_\lambda \mid \lambda < \beta\}$ where $\{b_\lambda \mid \lambda < \beta\}$ is a sequence defined in the following way:

$$b_\lambda = \begin{cases} a_\lambda & \text{for} \quad \lambda < \alpha\,, \\ x & \text{for} \quad \alpha \leqq \lambda < \beta\,. \end{cases}$$

Then clearly $b \in F(\beta, N)$ and φ is a one-one mapping of $F(\alpha, M)$ onto $\Sigma = \{\varphi(a) \mid a \in$ $\in F(\alpha, M)\} \subseteq F(\beta, N)$. We shall show that φ is an isomorphism of $F(\alpha, M)$ onto Σ.

Let $a = \{a_\lambda \mid \lambda < \alpha\}$, $a' = \{a'_\lambda \mid \lambda < \alpha\} \in F(\alpha, M)$, $a \leqq a'$ and $\varphi(a) = b = \{b_\lambda \mid \lambda < \beta\}$, $\varphi(a') = b' = \{b'_\lambda \mid \lambda < \beta\}$. Then there exists a strictly increasing sequence $\{\gamma_\lambda \mid \lambda < \alpha\}$ of type α of ordinal numbers less than α such that $a_\lambda = a'_{\gamma_\lambda}$ for every $\lambda < \alpha$. Let us define a sequence $\{\delta_\lambda \mid \lambda < \beta\}$ of type β of ordinal numbers less than β in the following way:

$$\delta_\lambda = \begin{cases} \gamma_\lambda & \text{for } \lambda < \alpha\,, \\ \lambda & \text{for } \alpha \leqq \lambda < \beta\,. \end{cases}$$

The sequence $\{\delta_\lambda \mid \lambda < \beta\}$ is strictly increasing and $b_\lambda = a_\lambda = a'_{\gamma_\lambda} = a'_{\delta_\lambda} = b'_{\delta_\lambda}$ for every $\lambda < \alpha$ and $b_\lambda = x = b'_\lambda = b = b'_{\delta_\lambda}$ for every $\alpha \leqq \lambda < \beta$. Therefore $b_\lambda = b'_{\delta_\lambda}$ for every $\lambda < \beta$, i.e. $b \leqq b'$. Suppose, on the contrary, that $b = \varphi(a) = \{b_\lambda \mid \lambda < \beta\}$, $b' = \varphi(a') = \{b'_\lambda \mid \lambda < \beta\} \in \Sigma$, $b \leqq b'$. Then there exists a strictly increasing sequence $\{\delta_\lambda \mid \lambda < \beta\}$ of type β of ordinal numbers less than β such that $b_\lambda = b'_{\delta_\lambda}$ for every $\lambda < \beta$. If $\lambda < \alpha$, then $\delta_\lambda < \alpha$, for, if $\delta_{\lambda_0} \geqq \alpha$ for some $\lambda_0 < \alpha$, then $b_{\lambda_0} = b'_{\delta_{\lambda_0}} = x$ which contradicts $b_{\lambda_0} = a_{\lambda_0} \in M$. Let us define the sequence $\{\gamma_\lambda \mid \lambda < \alpha\}$ such that $\gamma_\lambda = \delta_\lambda$ for every $\lambda < \alpha$. Then $\{\gamma_\lambda \mid \lambda < \alpha\}$ is a strictly increasing sequence of type α of ordinal numbers less than α and such that $a_\lambda = b_\lambda = b'_{\delta_\lambda} = b'_{\gamma_\lambda} = a'_{\gamma_\lambda}$ for every $\lambda < \alpha$, i.e. $a \leqq a'$. Thus φ is an isomorphism.

Let the assumption (II) hold. Then we can suppose that the set $N - M$ is non-empty and the proof coincides with the previous one.

Let the assumption (III) hold. Let $x \in N$ be an element and let us define the mapping φ of $F(\alpha, M)$ into $F(\beta, N)$ in the same way as in the first part of the proof. Then φ is a one-one mapping of $F(\alpha, M)$ onto $\Sigma = \{\varphi(a) \mid a \in F(\alpha, M)\} \subseteq F(\beta, N)$ and we shall show that φ is an isomorphism of $F(\alpha, M)$ onto Σ. Let $a, a' \in F(\alpha, M)$, $a \leqq a'$, $b = \varphi(a)$, $b' = \varphi(a')$. We are able to prove that $b \leqq b'$ in the same way as in the first part of the proof. Suppose, on the contrary, that $b = \varphi(a) = \{b_\lambda \mid \lambda < \beta\}$, $b' = \varphi(a') = \{b'_\lambda \mid \lambda < \beta\} \in \Sigma$, $b \leqq b'$. Then there exists a strictly incteasing sequence $\{\delta_\lambda \mid \lambda < \beta\}$ of type β of ordinal numbers less than β such that $b_\lambda = b'_{\delta_\lambda}$ for every $\lambda < \beta$. We shall prove that $\delta_\lambda < \alpha$ for every $\lambda < \alpha$. Suppose that there exists $\lambda_0 < \alpha$ such that $\delta_{\lambda_0} \geqq \alpha$. Then $\delta_{\lambda_0} \leqq \delta_\lambda < \beta$ for every $\lambda_0 \leqq \lambda < \beta$, i.e. the sequence $\{b_\lambda \mid \lambda_0 \leqq \lambda < \beta\}$ results by omitting a set (empty or non-empty) of members of the sequence $\{b'_\lambda \mid \delta_{\lambda_0} \leqq \lambda < \beta\}$. Let α_2 denote the remainder of the number α corresponding to the segment λ_0, i.e. $\alpha = \lambda_0 + \alpha_2$. As the type of the sequence $\{b_\lambda \mid \lambda_0 \leqq \lambda < \beta\}$ is $\alpha_2 + (\beta - \alpha)$ and the type of the sequence $\{b'_\lambda \mid \delta_{\lambda_0} \leqq \lambda < \beta\}$ is $\leqq \beta - \alpha$ we have $\alpha_2 + (\beta - \alpha) \leqq \beta - \alpha$, which is a contradiction. Therefore $\delta_\lambda < \alpha$ for every $\lambda < \alpha$ and this implies, similarly as in the first part of the proof, that $a \leqq a'$. Thus φ is an isomorphism and the theorem is proved.

Now we shall investigate the set $F(\alpha, M)$ as an m-universal set.

Theorem 2. *Let m be a cardinal number such that $0 < m \leqq \aleph_\nu$. Then a quasi-ordered set of type $F(\omega_\nu, m)$ is an m-universal set for ordered sets.*

P r o o f. Let the assumptions of Theorem be true and let G be an ordered set such that card $G \leqq m$. Then there exists a one-one mapping f of G into M where M is a set with card $M = m$. Denote by $\mathscr{S}$ the set of all subsets of M, i.e. $\mathscr{S} = \{N \mid N \subseteq M\}$, ordered by the set inclusion. If we assign to every element $x \in G$ a subset $\psi(x) = \{f(t) \mid t \leqq x\} \subseteq M$, then clearly ψ is an isomorphism of G onto a certain cubset $\mathscr{S}' \subseteq \mathscr{S}$ and card $N' \geqq 1$ for every $N' \in \mathscr{S}'$. Since card $M = m \leqq \aleph_\nu$, according to Lemma 1 it is possible to write the elements of the set M in the form of a sequence $\{b_\lambda \mid \lambda < \omega_\nu\}$ of type ω_ν such that $m_x(\{b_\lambda \mid \lambda < \omega_\nu\}) = \aleph_\nu$ for every $x \in M$. Now let us define a mapping φ of $\mathscr{S}'$ into $F(\omega_\nu, M)$ in the same way as in the proof of Theorem 1 of [5], i.e., let us assign to every set $N' \in \mathscr{S}'$ a sequence $\varphi(N') = \{a_\lambda \mid \lambda < \omega_\nu\}$ of type ω_ν in the following way: $a_0 = b_{\mu_0}$ where μ_0 is the smallest ordinal number such that $b_{\mu_0} \in N'$; suppose that we have defined a_λ for every $\lambda < \lambda_0$ $(\lambda_0 < \omega_\nu)$ and put $a_{\lambda_0} = b_{\mu_{\lambda_0}}$ where μ_{λ_0} is the smallest ordinal number with the following properties: $\mu_{\lambda_0} > \mu_\lambda$ for every $\lambda < \lambda_0$, $\mu_{\lambda_0} < \omega_\nu$, $b_{\mu_{\lambda_0}} \in N'$. In the above mentioned proof [5] it is shown that such an ordinal number always exists and that φ is an isomorphism of $\mathscr{S}'$ onto $\Sigma = \{\varphi(N') \mid N' \in \mathscr{S}'\} \subseteq F(\omega_\nu, M)$. Hence it follows that the composite mapping $\varphi\psi$ is an isomorphism of G onto $\Sigma \subseteq F(\omega_\nu, M)$. Because the type of the set $F(\omega_\nu, M)$ is $F(\omega_\nu, m)$, the theorem is proved.

Theorem 3. *Let $\aleph_\nu$ be a regular cardinal number and let m be a cardinal number such that $0 < m \leqq \aleph_\nu$. Then a quasi-ordered set of type $F(\omega_\nu, m + 1)$ is an m-universal quasi-ordered set.*

Proof. Let the assumptions of Theorem 3 be fulfilled and let G be a quasi-ordered set such that card $G \leqq m$. Then card $\bar{G} \leqq m$ and similarly as in the proof of Theorem 2 there exists an isomorphism ψ of the ordered set $\bar{G}$ onto a certain subset $\mathscr{S}' \subseteq \mathscr{S}$ where $\mathscr{S}$ is the set of all subsets of a set M with card $M = m$ ordered by the set inclusion. The definition of the mapping ψ yields that card $N' \geqq 1$ for every $N' \in \mathscr{S}'$. Let $a \bar{\in} M$ be an element and for every $N' \in \mathscr{S}'$ put $N'' = N' \cup \{a\}$. Then the system $\mathscr{S}'' = \{N'' \mid N' \in \mathscr{S}'\}$ is a system of sets such that $2 \leqq$ card $N'' \leqq \aleph_\nu$ for every $N'' \in \mathscr{S}''$ which – ordered by the set inclusion – is isomorphic with $\bar{G}$. Denote by χ an isomorphism of $\bar{G}$ onto $\mathscr{S}''$. Let $\Sigma(N'')$ be the set of all sequences $\{a_\lambda \mid \lambda < \omega_\nu\}$ of type ω_ν consisting of elements of the set N'' and such that $m_x(\{a_\lambda \mid \lambda < \omega_\nu\}) = \aleph_\nu$ for every $x \in N''$. According to Lemma 2 we have card $\Sigma(N'') = 2^{\aleph_\nu}$ for every $N'' \in \mathscr{S}''$. As card $X \leqq \aleph_\nu$ for every $X \in \bar{G}$ it is possible to define a one-one mapping φ_X of the set X into $\Sigma[\chi(X)]$. Finally, let us define a mapping φ of G into $F(\omega_\nu, M \cup \{a\})$ in the same way as in the proof of Theorem 3 of [5], i.e. let $\varphi(x) = \varphi_X(x)$ where $x \in X \in \bar{G}$. In [5] it is shown that φ is an isomorphism of G onto a certain subset of $F(\omega_\nu, M \cup \{a\})$. Because the type of the set $F(\omega_\nu, M \cup \{a\})$ is $F(\omega_\nu, m + 1)$, the theorem is proved.

Theorem 4. *Let m be a cardinal number such that $0 < m \leqq \aleph_\nu$. Then a quasi-ordered set of type $F(\omega_\nu, m + 2)$ is an m-universal quasi-ordered set.*

Proof. Let the assumptions of Theorem 4 be fulfilled. If $m \leqq \aleph_0$, then the statement follows from Theorem 3 and Theorem 1. If $m > \aleph_0$, then we obtain the statement in the following way:

Let G be a quasi-ordered set such that card $G \leqq m$. Then card $\bar{G} \leqq m$ and according to Theorem 2 the ordered set $\bar{G}$ is isomorphic with a certain subset $H \subseteq \subseteq F(\omega_\nu, M)$, where $\nu > 0$ and M is a set with card $M = m$. Denote by ψ an isomorphism of $\bar{G}$ onto H. Let $a \bar{\in} M$, $b \bar{\in} M$, $a \neq b$, be two elements. Let us construct the class $\Sigma[\psi(X)]$ for every element $\psi(X) = \{a_\lambda \mid \lambda < \omega_\nu\} \in H (X \in \bar{G})$ where $\Sigma[\psi(X)]$ is the set of all sequences which we obtain by inserting the sequence $\{a, b, a, b, \ldots\}$ or $\{b, a, b, a, \ldots\}$ of type ω_0 after every element a_λ, $\lambda < \omega_\nu$. Every element $\xi \in \in \Sigma[\psi(X)]$ belongs to the set $F(\omega_\nu, M \cup \{a, b\})$ for every $X \in \bar{G}$. For $\psi(X) \leqq \psi(Y)$ and $\xi \in \Sigma[\psi(X)]$, $\eta \in \Sigma[\psi(Y)]$ $\xi \leqq \eta$ holds. Because card $\Sigma[\psi(X)] = 2^{\aleph_\nu}$ for every $X \in \bar{G}$ there exists a one-one mapping φ_X of X into $\Sigma[\psi(X)]$ for every $X \in \bar{G}$. If we define a mapping φ of G into $F(\omega_\nu, M \cup \{a, b\})$ in the same way as in the proof of Theorem of [6], i.e. $\varphi(x) = \varphi_X(x)$ for $x \in X \in \bar{G}$, then φ is an isomorphism of G onto a certain subset of $F(\omega_\nu, M \cup \{a, b\})$. Because the type of the set $F(\omega_\nu, M \cup \cup \{a, b\})$ is $F(\omega_\nu, m + 2)$, the theorem is proved.

Now we shall deal with representations of finite chains and finite antichains by the set $F(\alpha, M)$.

Theorem 5. *If B is a chain of type ω_ν, then a quasi-ordered set of type $F(\omega_\nu, 2)$ contains a subset isomorphic with B.*

Proof. If B is a chain of type ω_ν, then we can suppose $B = W(\omega_\nu)$ without loss of generality. Let $F(\omega_\nu, M)$ be a quasi-ordered set of type $F(\omega_\nu, 2)$, where $M = = \{a, b\}$. To every ordinal number $\mu \in W(\omega_\nu)$ let us assign a sequence $f(\mu) = = \{c_\lambda^\mu \mid \lambda < \omega_\nu\}$ defined in the following way:

$$c_\lambda^\mu = \begin{cases} a & \text{for } \lambda < \mu\,, \\ b & \text{for } \mu \leqq \lambda < \omega_\nu\,. \end{cases}$$

It is clear that $f(\mu) \in F(\omega_\nu, M)$ for every $\mu \in W(\omega_\nu)$ and that f is a one-one mapping of the chain $W(\omega_\nu)$ onto a certain subset $K \subseteq F(\omega_\nu, M)$. We shall show that f is an isomorphism of $W(\omega_\nu)$ onto K. Hence let $\mu_1, \mu_2 \in W(\omega_\nu)$, $\mu_1 \leqq \mu_2$. Then $f(\mu_1) = = \{c_\lambda^{\mu_1} \mid \lambda < \omega_\nu\}$, $f(\mu_2) = \{c_\lambda^{\mu_2} \mid \lambda < \omega_\nu\}$ and put

$$\gamma_\lambda = \begin{cases} \lambda & \text{for } \lambda < \mu_1\,, \\ \mu_2 + (\lambda - \mu_1) & \text{for } \mu_1 \leqq \lambda < \omega_\nu\,. \end{cases}$$

Then $\{\gamma_\lambda \mid \lambda < \omega_\nu\}$ is a strictly increasing sequence of ordinal numbers of type ω_ν and because $\mu_2 + (\lambda - \mu_1) < \mu_2 + (\omega_\nu - \mu_1) = \omega_\nu$ for $\mu_1 \leqq \lambda < \omega_\nu$ we have $\gamma_\lambda < \omega_\nu$ for every $\lambda < \omega_\nu$. Now if $c_\lambda^{\mu_1} = a$, then $\lambda < \mu_1$ and therefore $\gamma_\lambda = \lambda < \mu_1 \leqq \mu_2$. This implies $c_{\gamma_\lambda}^{\mu_2} = a$. If $c_\lambda^{\mu_1} = b$, then $\mu_1 \leqq \lambda < \omega_\nu$ and therefore $\gamma_\lambda = = \mu_2 + (\lambda - \mu_1) \geqq \mu_2$. This implies $c_{\gamma_\lambda}^{\mu_2} = b$. Thus, $c_\lambda^{\mu_1} = c_{\gamma_\lambda}^{\mu_2}$ for every $\lambda < \omega_\nu$, .e. $f(\mu_1) \leqq f(\mu_2)$. Suppose, on the contrary, that $f(\mu_1) = \{c_\lambda^{\mu_1} \mid \lambda < \omega_\nu\} \leqq \leqq \{c_\lambda^{\mu_2} \mid \lambda < \omega_\nu\} = f(\mu_2)$. Then there exists a strictly increasing sequence

$\{\gamma_\lambda \mid \lambda < \omega_\nu\}$ of type ω_ν of ordinal numbers less then ω_ν such that $c_\lambda^{\mu_1} = c_{\gamma_\lambda}^{\mu_2}$ for every $\lambda < \omega_\nu$. If $\lambda < \mu_1$, then $c_\lambda^{\mu_1} = a$ and therefore $c_{\gamma_\lambda}^{\mu_2} = a$ which implies $\gamma_\lambda < \mu_2$. Because $\lambda \leqq \gamma_\lambda$ for every $\lambda < \omega_\nu$ we obtain $\lambda < \mu_1 \Rightarrow \lambda < \mu_2$. This implies $\mu_1 \leqq \mu_2$ and the proof is complete.

Corollary. *Let m be a cardinal number such that $0 < m < \aleph_0$. Then a quasi-ordered set of type $F(\omega_0, 2)$ is an m-universal set for chains.*

Proof. Every finite chain is isomorphic with a certain subset of a chain of type ω_0. Now the statement follows from Theorem 5 for $\nu = 0$.

Theorem 6. *Let m be a cardinal number such that $0 < m < \aleph_0$. Then a quasi-ordered set of type $F(\omega_0, 3)$ is an m-universal set for antichains.*

Proof. Let the assumptions of Theorem 6 be fulfilled. Let P be an antichain such that card $P \leqq m$. Let α be an ordinal number with card $\alpha = m$. Then a set of type $F(\alpha, 2)$ is an antichain of power 2^m and thus it contains a certain subset isomorphic with P. According to Theorem 1 every set of type $F(\alpha, 2)$ is isomorphic with a certain subset of a set of type $F(\omega_0, 3)$. Thus Theorem 6 is proved.

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