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Czechoslovak Mathematical Journal, Vol. 34 (1984), No. 3, 411–416

Persistent URL: <http://dml.cz/dmlcz/101966>

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RECOGNIZABILITY IN THE LATTICE OF CONVEX ℓ -SUBGROUPS
OF A LATTICE-ORDERED GROUP

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(Received December 20, 1982)

The intent of this note is to show that several types of lattice-ordered groups and convex ℓ -subgroups are impossible to recognize from the lattice structure of the lattice of convex ℓ -subgroups of a lattice-ordered group. In particular we show that one cannot recognize if a lattice-ordered group belongs to any nontrivial proper variety, if it is archimedean, or if it is completely distributive, and also that one cannot tell if a given convex ℓ -subgroup is closed. It is already known that no nontrivial proper variety other than \mathcal{N} (the variety of normal-valued ℓ -groups) is recognizable, but our proof that \mathcal{N} is not recognizable yields the fact for all nontrivial proper varieties at once. Our method is to consider two specific ℓ -groups which we label G and G' . We show that the lattices $\mathcal{C}(G)$ and $\mathcal{C}(G')$ of convex ℓ -subgroups are isomorphic as lattices, and then consider the recognizability questions.

Let $A(\mathbb{R})$ represent the lattice-ordered group of order-preserving permutations of the real numbers with pointwise meet and join as lattice operations, and composition for the group operation. Let $C(\mathbb{R})$ be the ℓ -group of continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$, with pointwise meet, join and addition. We define a continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ to be *finitely piecewise linear* if there is a finite set of real numbers $a_1 < a_2 < \dots < a_n$ such that for each interval I of \mathbb{R} which has no a_k in its interior, there are real numbers m and b such that $f(x) = mx + b$ for each $x \in I$. We will say that such a function f has *finitely many pieces* over any interval $J \subseteq \mathbb{R}$, where a *piece of f over J* is defined to be f restricted to a subinterval J' of J that is maximal with respect to the property that there are real numbers m and b with $f(x) = mx + b$ for all x in J' . We also define the 0-support of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ to be $\{x \in \mathbb{R} \mid f(x) \neq 0\}$ and the 1-support of $f = \{x \in \mathbb{R} \mid f(x) \neq x\}$. We will use $\text{supp}(f)$ to represent both the 0-support of a function in $C(\mathbb{R})$ and the 1-support of a function in $A(\mathbb{R})$, since it will be clear from the context just which is intended.

We let $G = \{f \in C(\mathbb{R}) \mid f \text{ is finitely piecewise linear and } \text{supp}(f) \text{ is bounded}\}$ and let $G' = \{f \in A(\mathbb{R}) \mid f \text{ is finitely piecewise linear and } \text{supp}(f) \text{ is bounded}\}$. Since G is an ℓ -subgroup of $C(\mathbb{R})$ and G' is an ℓ -subgroup of $A(\mathbb{R})$, we know that G and G' are ℓ -groups. Ball invented the group G' to show that an ℓ -group can have the DCC on regular subgroups and still fail to be normal-valued. In fact, G' has no nontrivial

proper normal subgroups at all. A variation of this example due to Conrad can be used to show that a locally flat ℓ -group need not be archimedean (take $G'' = G' \cap A(\emptyset)$; neither G' nor G'' is archimedean).

Now we show that the lattices $\mathcal{C}(G)$ and $\mathcal{C}(G')$ are isomorphic. We begin by identifying all of the prime subgroups of G . For each $r \in \mathbb{R}$, let

$$\begin{cases} G_r &= \{f \in G \mid f(r) = 0\} \\ G_{r,+} &= \{f \in G \mid \text{for some } \varepsilon > 0 \text{ and for all } x \text{ in } [r, r + \varepsilon], f(x) = 0\} \\ G_{r,-} &= \{f \in G \mid \text{for some } \varepsilon > 0 \text{ and for all } x \text{ in } [r - \varepsilon, r], f(x) = 0\}. \end{cases}$$

It is easily seen that these are prime subgroups, using the finiteness in the cases of $G_{r,+}$ and $G_{r,-}$. The next three lemmas together show that these are the only prime subgroups of G .

1. Lemma. *Every proper convex ℓ -subgroup of G is contained in some G_r . Hence each prime subgroup lies in a unique G_r .*

Proof. Let $C \in \mathcal{C}(G)$ and suppose that for all $r \in \mathbb{R}$, $C \not\subseteq G_r$. Let $0 < g \in G$, and let $K = \overline{\text{supp}(g)}$. K is compact, and so $g(K)$ is bounded. Let $M = \sup \{g(x) \mid x \in \mathbb{R}\}$. For each $r \in K$, $C \not\subseteq G_r$, so we can find $0 < f_r \in C \setminus G_r$. Since $f_r(r) > 0$, we have $\frac{1}{2}f_r(r) > 0$. Let $U_r = \{x \in \mathbb{R} \mid f_r(x) > \frac{1}{2}f_r(r)\}$. Since each f_r is continuous, each U_r is an open neighbourhood of r , and so $\{U_r \mid r \in K\}$ is an open cover of K . Let $\{U_{r_i} \mid 1 \leq i \leq n\}$ be a finite subcover, and let $m = \min \{\frac{1}{2}f_{r_i}(r_i) \mid 1 \leq i \leq n\}$. We have $m > 0$, so there must be a positive integer N with $mN > M$. Let $f = \bigvee_{i=1}^n Nf_{r_i}$.

Since $0 < g \leq f \in C$, we have $g \in C$, so $C = G$, and every proper convex ℓ -subgroup lies in some G_r . Now, since G_r and G_s are incomparable if $r \neq s$, and since for any prime subgroup the set of convex ℓ -subgroups containing it is totally ordered, each prime subgroup lies in a unique G_r . \square

2. Lemma. *If P is a prime subgroup of G with $P \not\subseteq G_r$, then either $P \subseteq G_{r,+}$ or $P \subseteq G_{r,-}$.*

Proof. Suppose P is a prime subgroup of G with $P \not\subseteq G_{r,+}$ and $P \not\subseteq G_{r,-}$, but $P \subseteq G_r$. Let $0 < h_1 \in P \setminus G_{r,+}$ and $0 < h_2 \in P \setminus G_{r,-}$. Let $0 < g \in G_r$. As in the proof of Lemma 1, let $M = \sup \{g(x) \mid x \in \mathbb{R}\}$. Since $0 < h_1 \in P \subseteq G_r$ and $h_1 \notin G_{r,+}$, there must exist $\varepsilon_1 > 0$ and $m_1 > 0$ with $h_1(x) = m_1(x - r)$ for all x in $[r, r + \varepsilon_1]$. Similarly, there exist $\varepsilon_2 > 0$ and $m_2 < 0$ with $h_2(x) = m_2(x - r)$ for all x in $[r - \varepsilon_2, r]$. Since g consists of finitely many pieces over $[r, r + \varepsilon_1]$, we let $m_+ = \max \{\text{slopes of pieces of } g \text{ over } [r, r + \varepsilon_1]\}$. Similarly, we set $m_- = \min \{\text{slopes of pieces of } g \text{ over } [r - \varepsilon_2, r]\}$. We let N be a positive integer such that $m_1N \geq m_+$ and $m_2N \leq m_-$. Then for all x in $[r - \varepsilon_2, r + \varepsilon_1]$ we have $0 \leq g(x) \leq [N(h_1 \vee h_2)](x)$, with $N(h_1 \vee h_2) \in P$. Now let $K_0 = \overline{\text{supp}(g)} \setminus (r - \varepsilon_2, r + \varepsilon_1)$. K_0 is compact, and $r \notin K_0$. Since P is prime, we know from Lemma 1 that if $s \in K_0$, then $P \not\subseteq G_s$. As in

the proof of Lemma 1, we find $f \in P$ with $f > 0$ and $f(x) \geq g(x)$ for all $x \in K_0$. Then $0 < g \leq f \vee N(h_1 \vee h_2)$, with $f \vee N(h_1 \vee h_2) \in P$, so $g \in P$, and $P = G_r$. \square

3. Lemma. *If P is a prime subgroup of G with $P \subseteq G_{r+}$, then $P = G_{r+}$. If P is a prime subgroup of G with $P \subseteq G_{r-}$ then $P = G_{r-}$. Thus $\{C \in \mathcal{C}(G) \mid \text{for some } r \in \mathbb{R}, C = G_r \text{ or } G_{r+} \text{ or } G_{r-}\}$ is a complete list of the prime subgroups of G .*

Proof. Let $P \in \mathcal{C}(G)$, with $P \not\subseteq G_{r+}$. Let $0 < g \in G_{r+} \setminus P$. Then there exists $\varepsilon > 0$ with $g(x) = 0$ for all x in $[r, r + \varepsilon]$. Define f by:

$$f(x) = \begin{cases} x - r, & \text{if } x \in [r, r + \frac{1}{2}\varepsilon] \\ r + \varepsilon - x, & \text{if } x \in [r + \frac{1}{2}\varepsilon, r + \varepsilon] \\ 0, & \text{if } x \notin [r, r + \varepsilon]. \end{cases}$$

Now, $f \notin G_{r+}$, so $f \notin P$, and $g \notin P$, but $f \wedge g = 0$, so P is not prime. A similar example shows that if $P \in \mathcal{C}(G)$ and $P \not\subseteq G_{r-}$, then P is not prime. \square

In G' , we let $G'_r = \{f \in G' \mid f(r) = r\}$, $G'_{r+} = \{f \in G' \mid \text{for some } \varepsilon > 0 \text{ and for all } x \in [r, r + \varepsilon], f(x) = x\}$, and $G'_{r-} = \{f \in G' \mid \text{for some } \varepsilon > 0 \text{ and for all } x \in [r - \varepsilon, r], f(x) = x\}$. Then, as part of Ball's example, it is known that $\{C' \in \mathcal{C}(G') \mid \text{for some } r \in \mathbb{R}, C' = G'_r \text{ or } G'_{r+} \text{ or } G'_{r-}\}$ is a complete list of the prime subgroups of G' . (The three lemmas we have just proved can be modified slightly to obtain this result.)

Now, recall (Bigard, et al, [1], 2.5.5) that a convex ℓ -subgroup is the intersection of the prime subgroups containing it. We use this to set up a lattice isomorphism between $\mathcal{C}(G)$ and $\mathcal{C}(G')$ via a lattice that describes the collections of prime subgroups above elements of $\mathcal{C}(G)$ and $\mathcal{C}(G')$. This isomorphism will send G_r to G'_r , G_{r+} to G'_{r+} , and G_{r-} to G'_{r-} , as one would expect.

Let $\mathcal{S} = \{(S, S_+, S_-) \mid S_+ \cup S_- \subseteq S \subseteq \mathbb{R}, S \text{ is closed in } \mathbb{R}, \text{ and for each sequence } \{s_n\}_{n=1}^\infty \text{ in } S \text{ with } \lim_{n \rightarrow \infty} s_n = s, \text{ we have that if } \{s_n\}_{n=1}^\infty \text{ is strictly increasing, then } s \in S_-, \text{ while if } \{s_n\}_{n=1}^\infty \text{ is strictly decreasing, then } s \in S_+\}$. Order \mathcal{S} by setting $(S, S_+, S_-) \leq (T, T_+, T_-)$ if and only if $S \supseteq T$ and $S_+ \supseteq T_+$ and $S_- \supseteq T_-$. This is a lattice order on \mathcal{S} , with $(S, S_+, S_-) \vee (T, T_+, T_-) = (S \cap T, S_+ \cap T_+, S_- \cap T_-)$ and $(S, S_+, S_-) \wedge (T, T_+, T_-) = (S \cup T, S_+ \cup T_+, S_- \cup T_-)$.

4. Proposition. *There is a lattice isomorphism between $\mathcal{C}(G)$ and \mathcal{S} .*

Proof. Let $C \in \mathcal{C}(G)$, and set $S_C = \{r \in \mathbb{R} \mid C \subseteq G_r\}$, $S_{C+} = \{r \in \mathbb{R} \mid C \subseteq G_{r+}\}$, and $S_{C-} = \{r \in \mathbb{R} \mid C \subseteq G_{r-}\}$. We want $(S_C, S_{C+}, S_{C-}) \in \mathcal{S}$. Obviously, $S_C \subseteq \mathbb{R}$, and since $G_{r+} \subseteq G_r$ and $G_{r-} \subseteq G_r$ for each $r \in \mathbb{R}$, $S_{C+} \cup S_{C-} \subseteq S_C$. Let $\{s_n\}_{n=1}^\infty$ be a sequence in S_C , with $\lim_{n \rightarrow \infty} s_n = s \in \mathbb{R}$. Let $f \in C$. Then $f(s_n) = 0$ for all positive integers n , and by continuity, $f(s) = 0$, so $f \in G_s$, and $C \subseteq G_s$. Then $s \in S_C$, and S_C is closed. If, in addition, $\{s_n\}_{n=1}^\infty$ is strictly increasing, then because $f \in G$, there must be $\varepsilon > 0$ such that $f(x) = mx + b$ on $[s - \varepsilon, s]$. Since $\lim_{n \rightarrow \infty} s_n = s$, there must be a positive integer N such that if $n \geq N$, then $|s_n - s| < \varepsilon$. Therefore, if $n_1 > n_2 \geq N$

$s_{n_1} \in [s - \varepsilon, s]$ and $s_{n_2} \in [s - \varepsilon, s]$ with $s_{n_1} \neq s_{n_2}$. Hence $f(s_{n_1}) = f(s_{n_2}) = 0$, so $f(x) \equiv 0$ on $[s - \varepsilon, s]$, and $s \in S_{C^-}$. Similarly if $\{s_n\}_{n=1}^\infty$ is strictly decreasing, then $s \in S_{C^+}$, and so $(S_C, S_{C^+}, S_{C^-}) \in \mathcal{S}$ for each $C \in \mathcal{C}(G)$. We let $\theta : \mathcal{C}(G) \rightarrow \mathcal{S}$ by setting $\theta(C) = (S_C, S_{C^+}, S_{C^-})$. If $C, D \in \mathcal{C}(G)$ with $C \neq D$, then the sets of primes exceeding C is not equal to the set of primes containing D , and so $\theta(C) \neq \theta(D)$. Thus θ is one-to-one. Let $(S, S_+, S_-) \in \mathcal{S}$, and set $C = \left(\bigcap_{s \in S} G_s \right) \cap \left(\bigcap_{s \in S_+} G_{s^+} \right) \cap \left(\bigcap_{s \in S_-} G_{s^-} \right)$.

Obviously $C \in \mathcal{C}(G)$, and $\theta(C) \leq (S, S_+, S_-)$. If $s \notin S$, then since S is closed, there is $\varepsilon > 0$ such that if $x \in (s - 2\varepsilon, s + 2\varepsilon)$, then $x \notin S$. Define $f \in G$ by

$$f(x) = \begin{cases} x - s + \varepsilon, & \text{if } s - \varepsilon \leq x \leq s \\ -x + s + \varepsilon, & \text{if } s \leq x \leq s + \varepsilon \\ 0, & \text{if } x \notin [s - \varepsilon, s + \varepsilon]. \end{cases}$$

If $x \in S$, then $x \notin [s - \varepsilon, s + \varepsilon]$, so $f(x) = 0$. If $x \in S_+ \cup S_-$, then $x \in S$ and $f(x) = 0$, and in addition, for each y with $|y - x| < \varepsilon$, we have $y \notin [s - \varepsilon, s + \varepsilon]$, since $x \notin [s - 2\varepsilon, s + 2\varepsilon]$. Thus $f(y) = 0$, so $f \in G_r$ for all $r \in S$, $f \in G_{r^+}$ for all $r \in S_+$, and $f \in G_{r^-}$ for all $r \in S_-$; hence $f \in C$. But $f(s) = \varepsilon > 0$, so $f \notin G_s$, and $C \not\subseteq G_s$. Thus $S_C = S$. Suppose $s \notin S_+$. By definition there is $\varepsilon > 0$ such that if $x \in (s, s + 2\varepsilon)$, then $x \notin S$. Define $g \in G$ by

$$g(x) = \begin{cases} 2x - 2s, & \text{if } x \in [s, s + \frac{1}{2}\varepsilon] \\ -2x + 2s + 2\varepsilon, & \text{if } x \in [s + \frac{1}{2}\varepsilon, s + \varepsilon] \\ 0, & \text{if } x \notin [s, s + \varepsilon]. \end{cases}$$

If $x \in S$, then $x \notin (s, s + \varepsilon)$, so $g(x) = 0$. If $x \in S_-$, then $x \notin (s, s + 2\varepsilon)$, so that for each y in $[x - \varepsilon, x]$, $y \notin (s, s + \varepsilon)$ and $g(y) = 0$. If $x \in S_+$, then $x \neq s$ and $x \notin (s, s + \varepsilon)$ so if $x \geq s + \varepsilon$, then for all y in $[x, x + \varepsilon]$, $y \geq s + \varepsilon$ and $g(y) = 0$, while if $x < s$, then set $\delta = \frac{1}{2}(s - x)$. We have $\delta > 0$, and for each y in $[x, x + \delta]$, $y < s$, so $g(y) = 0$. Thus $g \in C$, but $g \notin G_{s^+}$ since there is no $\alpha > 0$ with $g(x) = 0$ for all x in $[s, s + \alpha]$. Hence $S_+ = S_{C^+}$. A similar argument shows that $S_- = S_{C^-}$, so that $\theta(C) = (S, S_+, S_-)$, and θ is onto. If $C, D \in \mathcal{C}(G)$, then since G_r is a prime subgroup of G for each $r \in \mathbb{R}$, we have that if $C \wedge D \subseteq G_r$, then $C \subseteq G_r$ or $D \subseteq G_r$. Thus $r \in S_{C \wedge D} \Leftrightarrow C \wedge D \subseteq G_r \Leftrightarrow C \subseteq G_r$ or $D \subseteq G_r \Leftrightarrow r \in S_C \cup S_D$, so $S_{C \wedge D} = S_C \cup S_D$. Similarly, $S_{C \wedge D^+} = S_{C^+} \cup S_{D^+}$ and $S_{C \wedge D^-} = S_C \cup S_{D^-}$, so that $\theta(C \wedge D) = \theta(C) \wedge \theta(D)$. Also, $r \in S_{C \vee D} \Leftrightarrow C \vee D \subseteq G_r \Leftrightarrow C \subseteq G_r$ and $D \subseteq G_r \Leftrightarrow r \in S_C \cap S_D$, so that $S_{C \vee D} = S_C \cap S_D$. Similarly, $S_{C \vee D^+} = S_{C^+} \cap S_{D^+}$ and $S_{C \vee D^-} = S_{C^-} \cap S_{D^-}$, so that $\theta(C \vee D) = \theta(C) \vee \theta(D)$, and θ is a lattice isomorphism of $\mathcal{C}(G)$ onto \mathcal{S} . \square

Essentially the same proof shows that if $\theta' : \mathcal{C}(G') \rightarrow \mathcal{S}$. By the rule $\theta'(C') = (S_{C'}, S_{C'^+}, S_{C'^-})$, where $S_{C'} = \{r \in \mathbb{R} \mid C' \subseteq G'_r\}$, $S_{C'^+} = \{r \in \mathbb{R} \mid C' \subseteq G'_{r^+}\}$, and $S_{C'^-} = \{r \in \mathbb{R} \mid C' \subseteq G'_{r^-}\}$, then θ' is a lattice isomorphism of $\mathcal{C}(G')$ onto \mathcal{S} . Thus, the function $(\theta')^{-1} \theta$ is a lattice isomorphism of $\mathcal{C}(G)$ onto $\mathcal{C}(G')$, and it is easily

checked that this map sends G_r to G'_r , G_{r+} to G'_{r+} , and G_{r-} to G'_{r-} for each $r \in \mathbb{R}$.

Now, we consider the questions of recognizability. First, G is a member of the variety of abelian lattice-ordered groups, which is a subvariety of all nontrivial varieties, while G' lies outside the variety of normal-valued ℓ -groups, since each G'_r is a regular subgroup of G' but is not normal (however, $G'_{r+} \triangleleft G'_r$ and $G'_{r-} \triangleleft G'_r$, for each $r \in \mathbb{R}$). But \mathcal{N} contains all proper varieties, so G' cannot lie in any proper variety, while G is in all nontrivial varieties, and it is impossible to tell if a lattice-ordered group belongs to any nontrivial proper variety by considering only its lattice of convex ℓ -subgroups.

Next, G is archimedean, while G' is not, and so it is impossible to recognize from the lattice of convex ℓ -subgroups if a given ℓ -group is archimedean.

Martinez [6] introduced the idea of a torsion class, which generalizes the notion of a variety. The class of archimedean ℓ -groups is not a torsion class, but if we take only those archimedean ℓ -groups with the additional property that every ℓ -homomorphic image is archimedean, then we get the torsion class of hyperarchimedean ℓ -groups. Several characterizations of hyperarchimedean ℓ -groups are known (for a history, see Conrad [3]), including the fact that an ℓ -group is hyperarchimedean if and only if each prime subgroup is a minimal prime subgroup, and so each prime subgroup is a maximal. Since a convex ℓ -subgroup P is prime if and only if the set of convex ℓ -subgroups containing P is totally ordered, we can recognize primes, and certainly maximals are recognizable, so it is possible to recognize if an ℓ -group is hyperarchimedean. Hence some torsion classes are recognizable, while varieties are not. Conrad [4] has considered several other important torsion classes, and determined which are recognizable by looking at torsion radicals. He has shown that the torsion class of divisible abelian ℓ -groups is not recognizable, while the following torsion classes are recognizable:

- \mathcal{A} = all hyperarchimedean ℓ -groups
- \mathcal{F} = all ℓ -groups such that each bounded disjoint subset is finite
- \mathcal{F}_v = all finite-valued ℓ -groups
- \mathcal{D} = all ℓ -groups with DCC on the set of regular subgroups
- \mathcal{O} = all cardinal sums of o-groups
- \mathcal{A} = all cardinal sums of archimedean o-groups
- \mathcal{B} = all ℓ -groups such that each prime exceeds a unique minimal prime.

Recall that in an ℓ -group A , a convex ℓ -subgroup C is closed if for each family $\{x_i \mid i \in I\}$ with $e \leq x_i \in C$ such that $\bigvee_{i \in I} x_i = x$ exists in A , then $x \in C$. In Bigard, et al. [1] (11.1.10), it is shown that in an archimedean ℓ -group, closed convex ℓ -subgroups are polars, so that in G , no G_r , G_{r+} , or G_{r-} can be closed, since they are not polars (for $G_{r+}^\perp = G_{r-}^\perp = \{0\}$). On the other hand, for each $r \in \mathbb{R}$, G'_r is closed in G' . This is a result of McCleary [7], which generalized the work of Lloyd [5]. Hence it is impossible to recognize closed convex ℓ -subgroups in the lattice of convex ℓ -subgroups of a lattice-ordered group.

Finally, Byrd and Lloyd [2] define the distributive radical of an ℓ -group, and show that it is the intersection of all closed prime subgroups. They also show that an ℓ -group is completely distributive if and only if it has a trivial distributive radical. In our examples, no prime subgroups of G are closed, so that G is its own radical, while all of the G'_i are closed in G' , so that the distributive radical of G' is $\{1\}$. Hence it is impossible to recognize the distributive radical, and one cannot tell if an ℓ -group is completely distributive, from the lattice of convex ℓ -subgroups.

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