C. T. Tucker
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POINTWISE AND ORDER CONVERGENCE FOR SPACES OF CONTINUOUS FUNCTIONS AND SPACES OF BAIRE FUNCTIONS

C. T. Tucker, Houston

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Suppose $L$ is a Riesz space (a lattice ordered vector space). For notation and basic terminology concerning Riesz spaces, the reader is referred to Luxemburg and Zaanen [6]. A Riesz homomorphism between two Riesz spaces is a positive linear map that preserves the finite lattice operations. The Riesz space $L$ is almost $\sigma$-complete if it is Riesz isomorphic to a subspace $L^\sim$ of a $\sigma$-complete space $M$ with the property that if $m \in M^+$, there is a sequence $(u_n)$ such that $u_n \in L^\sim$ $(n = 1, 2, \ldots)$, $0 \leq u_1 \leq u_2 \leq \ldots$, and $\bigvee_{n=1}^\infty u_n = m$. See Aliprantis and Langford [1] or Quinn [10] for some properties of almost $\sigma$-complete spaces.

A sequence $f_1, f_2, f_3, \ldots$ is order Cauchy if there is a sequence $y_1 \geq y_2 \geq y_3 \geq \ldots \geq 0$, $\bigwedge_{n=1}^\infty y_n = 0$, such that, for $m \geq n$, $|f_m - f_n| \leq y_n$. If every order Cauchy sequence converges then $L$ is order Cauchy complete. By Corollary 8.5 of Quinn [12], $L$ is $\sigma$-complete if and only if it is almost $\sigma$-complete and order Cauchy complete.

Suppose $Q$ is a set and $\Omega$ is a Riesz space of real valued functions defined on $Q$ containing the constants. Then $B_1(\Omega)$ (the first Baire class of $\Omega$) is the set of all pointwise limits of sequences of $\Omega$. Also, $USB_1(\Omega)$ is the set of all pointwise limits of non-increasing sequences of $B_1(\Omega)$. For a discussion of Baire spaces see Mauldin [7] or [8].

If $X$ is a topological space, then $X$ is said to be a $P$-space, if $C(X)$, the set of all continuous functions on $X$, is closed with respect to pointwise convergence. In Gilman and Henriksen [4], $P$-spaces were introduced. See also Regoli [13] for a discussion of some properties of $P$-spaces.

It is the purpose of this paper to compare monotone pointwise convergence with monotone order convergence and bounded pointwise convergence with order convergence in $B_1(\Omega)$ and in $C(X)$. All convergence referred to here is sequential convergence.

A first easy observation is that monotone pointwise convergence implies monotone order convergence in both $B_1(\Omega)$ and $C(X)$.
Theorem 1. Suppose $X$ is either a completely regular Baire space with the countable chain condition or a perfectly normal Baire space. Then bounded pointwise convergence implies order convergence in $C(X)$.

Proof. Suppose $X$ is a completely regular Baire space with the countable chain condition. Suppose $f_1, f_2, f_3, \ldots$ is a sequence of functions in $C(X)$ pointwise converging to 0 and $0 \leq f_i \leq 1$ for each positive integer $i$. Let $f_{ij} = ((f_i \wedge (1/2)^j) \vee (1/2)^j) - (1/2)^j$ for each positive integer $i$ and $j$. Note that $f_i = \sum_{j=1}^{\infty} f_{ij}$ and $0 \leq f_{ij} \leq (1/2)^j$. Also for every point $x$ of $X$ and positive integer $j$ there exists a positive integer $k$ such that if $i > k$, then $f_{ij}(x) = 0$.

For each positive integer $i$ and $j$ let $Z_{ij}$ be the set to which $x$ belongs if $f_{ij}(x) = 0$ for $p \geq i$ and let $U_{ij}$ be the interior of $Z_{ij}$. Note that $Z_{ij}$ is closed. For each positive integer $j$, $Z_{1j} \subset Z_{2j} \subset Z_{3j} \subset \ldots$ and $\bigcup_{i=1}^{\infty} Z_{ij} = X$. Let $M_j$ be the complement of $\bigcup_{i=1}^{\infty} U_{ij}$. Since $\bigcup_{i=1}^{\infty} U_{ij}$ is open, $M_j$ is closed. The interior of $M_j$ is an open set which is the countable union of a collection of sets each of which is the subset of the boundary of $Z_{ij}$, for some $i$. As the boundary of each $Z_{ij}$ is nowhere dense and $X$ is a Baire space, $M_j$ does not contain an interior point and $M_j$ is nowhere dense. Therefore $\bigcup_{i=1}^{\infty} U_{ij}$ is an open dense set.

As $X$ is completely regular, for each point $x$ of $U_{ij}$ there is a continuous function $h_x$ such that $0 \leq h_x \leq 1$, $h_x(x) = 1$ and $h_x(y) = 0$ for each $y$ not in $U_{ij}$. Let $V_x$ be the cozero set of $h_x$. Let $\omega$ be a maximal disjoint collection of $V_x$ and $B_{ij}$ be the union of the sets in $\omega$. As $X$ has the countable chain condition, $\omega = \{V_1, V_2, \ldots\}$ is countable. Suppose $y$ belongs to $U_{ij}$ and $N$ is an open set containing $y$. If $N$ is disjoint from the sets in $\omega$, a $V_p$ containing $y$ and no point of $B_{ij}$ can be constructed as $X$ is completely regular. Then $\omega$ is not maximal. Thus $y$ is a limit point of $B_{ij}$ and $B_{ij}$ is dense in $U_{ij}$.

It follows that $\bigcup_{i=1}^{\infty} B_{ij}$ is an open dense set.

Let $\{k_1, k_2, \ldots\}$ be a collection of continuous functions such that for each positive integer $p$, $0 \leq k_p \leq 1$ and $V_p$ is the cozero set of $k_p$ and let $t_{ij} = \sum_{n=1}^{\infty} (1/2)^n k_n$. Then $t_{ij}$ is continuous and its cozero set is $B_{ij}$. Let $r_{ij} = \bigwedge_{p \geq i} (1 - ((p \cdot t_{ij}) \wedge 1))$. Then $r_{1j} \geq r_{2j} \geq r_{3j} \geq \ldots$ is a sequence of continuous functions converging to zero on an open dense set $W_j = \bigcup_{i=1}^{\infty} B_{ij}$ such that $r_{ij} \geq f_{ij}$. Let $g_i = \sum_{j=1}^{\infty} r_{ij}$. Thus $g_1 \geq g_2 \geq \ldots \geq g_3 \geq \ldots$ is a sequence of continuous functions converging to zero on $\bigcap_{i=1}^{\infty} W_j$. As $X$ is a Baire space $\bigcap_{j=1}^{\infty} W_j$ is dense. Further $g_i \geq f_i$. Therefore $f_1, f_2, f_3, \ldots$ order converges to zero.
Suppose $X$ is a perfectly normal Baire space. Define $f_{ij}$ as before. Let $q_{ij} = \bigvee_{p \geq i} f_{pj}$.

Then $q_{ij}$ is lower semi-continuous, $q_{ij}$ converges pointwise to zero as $i \to \infty$, and for every point $x$ in $X$ there exists a positive integer $k$ such that $q_{k}(x) = 0$. Let $m_{ij}$ be the pointwise limit of $\{(nq_{ij}) \land (1/2)^j\}$, $n = 1, 2, 3, \ldots$. Then $m_{ij}$ is lower semi-continuous, is equal to zero wherever $q_{ij}$ is equal to zero, and is equal to $(1/2)^j$ wherever $q_{ij}$ is not equal to zero. Let $r_{ij}(x) = 0$ if $x$ belongs to the interior of $m_{ij}^{-1}(0)$ and equal $(1/2)^j$ otherwise. Since $m_{ij}^{-1}(0)$ is closed, its boundary is nowhere dense. Thus $r_{ij}$ is a non-increasing sequence which converges pointwise to zero as $i \to \infty$ on a set of the second category in $X$. Also $r_{ij}$ is upper semi-continuous. Since $X$ is perfectly normal there exists a non-increasing sequence $h_{ijk}$, $k = 1, 2, 3, \ldots$, of continuous functions converging pointwise to $r_{ij}$. Let $g_{1j} = h_{1j1} \land (1/2)^j$ and $g_{pj} = \bigwedge_{i+k=p} h_{ijk} \land (1/2)^j$ if $p > 1$. Then $g_{pj}$, $p = 1, 2, 3, \ldots$, is a non-increasing sequence of continuous functions such that $g_{pj} \geq r_{pj} \geq m_{pj} \geq q_{pj} \geq f_{pj}$ and $g_{pj}$ converges pointwise to zero on a set of the second category in $X$.

Let $g_{p} = \sum_{j=1}^{\infty} g_{pj}$. Then $g_{p}$ is continuous, $g_{p} \geq f_{p}$, and $g_{p}$ converges to zero except possibly on a set of the first category in $X$. In particular $g_{p}$ converges to zero on a dense subset of $X$. Thus $f_{p}$ order converges to zero.

Example 2. Let $X$ be the square of the interval $[0, 1]$ ordered lexicographically. Take as a basis for the topology on $X$ the sets of the form

\[
\{(a, x) \mid b > x > c\},
\]

\[
\{(a, x) \mid 1 \geq x > b\}, \text{ and}
\]

\[
\{(a, x) \mid c > x \geq 0\}.
\]

This is a perfectly normal Baire space without the countable chain condition.

The following example is due to H. Cook.

Let $M$ be the subset of the plane to which $x$ belongs if the first coordinate of $x$ is either 0 or 1 and the second coordinate of $x$ is in the closed interval $[0, 1]$. Order $M$ lexicographically. Let a basis for the topology on $M$ be the collection of sets

\[
\{(x, y) \mid a < x < b, \; c \leq y < d\}
\]

and

\[
\{(x, y) \mid a < x \leq b, \; c < y < d\}.
\]

Let $L = M \times M$. Then $L$ is compact and has the countable chain condition. On the other hand, let $R$ be the subset of $L$ consisting of all pairs of points of $M$ each of whose first coordinate is 1. Then $R$ is the square of an interval of the Sorgenfrey line and is not normal. This implies that $L$ is not completely normal (and thus not perfectly normal).

Theorem 3. Suppose $L$ is a linear lattice of real valued functions and bounded pointwise convergence implies order convergence in $L$. Then $L$ is almost $\sigma$-complete.
Proof. Let \( \{f_1, f_2, f_3, \ldots\} \) be a bounded disjoint sequence in \( L \). As \( f_1, f_2, f_3, \ldots \), converges pointwise to 0, it must order converge to 0. By Theorem 2.2 of Fremlin [3], \( L \) is almost \( \sigma \)-complete.

In [10] Meyer gives a necessary and sufficient condition on a normal topological space \( X \) for \( C(X) \) to be almost \( \sigma \)-complete. Compare Theorems 1 and 3 here with his Corollary 8. Also see section 9 of [5].

**Example 4.** There exists a compact Hausdorff space \( X \) such that \( C(X) \) is not almost \( \sigma \)-complete. Let \( H \) denote the set of all continuous functions on the interval \([0, 1]\) and \( K \) denote the set of bounded functions in \( B_1(H) \). Since \( B_1(H) \) is not closed with respect to pointwise convergence it is not \( \sigma \)-complete (Tucker [18], Lemma 4). As it is order Cauchy complete (Tucker [18], Proposition 6), \( B_1(H) \) can not be almost \( \sigma \)-complete. If \( \omega \) denotes the functions it is \( \sigma \)-complete.

Suppose \( f_1 \geq f_2 \geq f_3 \geq \ldots \) is a sequence of functions in \( C(X) \) converging pointwise to a function \( f \) and, further, suppose \( x \) is a point of \( X \) such that \( f \) is not continuous at \( x \). Let \( k = f(x) - 1 \). Replace each of \( f_i \) with \( f_i \lor k \) and \( f \) with \( f \lor k \).

Since \( \{f_i \lor k\} \) is bounded below and \( C(X) \) is \( \sigma \)-complete, \( \bigwedge (f_i \lor k) \) is in \( C(X) \) and since monotone order convergence implies pointwise convergence \( \bigwedge (f_i \lor k) = f \lor k \).

By Theorem 1 of Tucker [14], any function which is the pointwise limit of a sequence of functions in \( C(X) \) can be uniformly approximated by the difference of two functions each of which is the pointwise limit of a non-increasing sequence of functions in \( C(X) \) and is therefore continuous.

Since \( C(X) \) is closed under pointwise convergence, \( X \) is a \( P \)-space.

(3) implies (2). If \( X \) is a \( P \)-space, \( C(X) \) is closed with respect to pointwise convergence. If \( f_1, f_2, f_3, \ldots \) is a bounded sequence converging pointwise to \( f \), let \( y_n(x) = \max f_i(x) \). Then \( y_1, y_2, y_3, \ldots \) is a nonincreasing sequence of functions converging pointwise to \( f \) and \( y_n \geq f_n \). Thus \( f_1, f_2, f_3, \ldots \) order converges to \( f \). Also any monotone order convergent sequence is pointwise convergent (Tucker [16], Lemma 3), so that any order convergent sequence is pointwise convergent.

(2) implies (1). By Theorem 3, \( C(X) \) is almost \( \sigma \)-complete.

A Riesz space \( L \) is said to have the **sequential mapping continuity property** (abbreviated s.m.c. property) if every positive linear map from \( L \) into an Archimedean, directed, partially ordered vector space is sequentially order continuous. This property was defined in [16] and was shown in \( K \) which take on only the values 0 and 1, every function in \( K \) is the uniform limit of a sequence of functions each of which is a linear combination of the functions in \( \omega \). (Tucker [17], Theorem 7). The functions in \( \omega \) form a Boolean algebra, so by the Stone representation theorem they are isomorphic to the open and closed sets of a totally disconnected compact Hausdorff space \( X \). The natural mapping between \( K \) and \( C(X) \) is a Riesz isomorphism. As one
can note from the definition of almost $\sigma$-complete, it is a property which is preserved by Riesz isomorphisms. Thus $C(X)$ is not almost $\sigma$-complete.

**Theorem 5.** The following three statements are equivalent:

1. $C(X)$ is almost $\sigma$-complete and monotone order convergence implies pointwise convergence.
2. Order convergence in $C(X)$ is equivalent to bounded pointwise convergence, and
3. $X$ is a $P$-space.

**Proof.** (1) implies (3). By Lemma 2.10 of Papangelou [11], $C(X)$ is order Cauchy complete if and only if when each of $x_1 \geq x_2 \geq x_3 \geq \ldots$ and $w_1 \leq w_2 \leq w_3 \leq \ldots$ is a sequence of functions in $C(X)$ with the property that $x_n \geq w_n$ for each $n$ and $\{x_n - w_n\}$ order converges to 0 then there is a function $u$ in $C(X)$ such that $\bigwedge x_n = u$ and $\bigvee w_n = u$. Since monotone order convergence is assumed to imply pointwise convergence in $C(X)$, then if $x_n$ and $w_n$ are as described above then $\{x_n - w_n\}$ pointwise converges to 0. Thus the pointwise infimum of $x_n$ is the pointwise supremum of $w_n$ and since it is both upper semi-continuous and lower semi-continuous, it is continuous. So that $C(X)$ is order Cauchy complete and since it is assumed to be almost $\sigma$-complete, to apply to a large class of Riesz spaces. Also see [3]. Huijmsans and de Pagter in [5] prove a theorem (Theorem 9.3) for general Riesz spaces which when specialized to $C(X)$ yields that $X$ is a $P$-space if and only if $C(X)$ is almost $\sigma$-complete and has the s.m.c. property. Combining this with Theorem 5 gives that if $C(X)$ is almost $\sigma$-complete, then $C(X)$ has the s.m.c. property if and only if order convergence implies pointwise convergence. This is not true for Riesz spaces in general, e.g. let $L$ be the space of bounded sequences. Then $L$ is almost $\sigma$-complete and order convergence implies pointwise convergence, but $L$ does not have the s.m.c. property.

**Corollary 6.** If $C(X)$ is almost $\sigma$-complete then every positive linear functional on $C(X)$ is an integral if and only if $X$ is a $P$-space.

**Proof.** If $X$ is a $P$-space, then $C(X)$ is closed with respect to pointwise convergence and every positive linear functional on $C(X)$ is sequentially continuous. (Tucker [16], Proposition 4). If every positive linear functional on $C(X)$ is sequentially continuous, then $\varphi(x) = f(x)$ is sequentially continuous and order convergence implies pointwise convergence. By Theorem 5, $X$ is a $P$-space.

**Example 7.** There exists a topological space $X$ such that $C(X)$ is almost $\sigma$-complete but bounded pointwise convergence does not imply order convergence for sequences. Let $X$ be the set of rational numbers in $[0, 1]$ with the topology induced by the ordinary topology on $[0, 1]$. Suppose $H$ is a subset of $C(X)$ and $f = \bigvee h$. For each $x_i$ in $X$ let $g(x_i) = 1$.u.b. $\{h(x_i)\}$. There is a countable subset $\{h_{ij}\}, j = 1, 2, \ldots, of H$.
such that l.u.b. \( \{h_{ij}(x_i)\} = g(x_i) \). Thus \( \{h_{ij}\}, i = 1, 2, \ldots, j = 1, 2, \ldots, \) is a countable subset of \( H \) such that l.u.b. \( \{h_{ij}\} = g \wedge \bigvee_{i,j} h_{ij} = f \). So that, \( C(X) \) is order separable and thus almost \( \sigma \)-complete. (Aliprantis and Langford [1]). For each positive integer \( j \) let \( N_{ij}, i = 1, 2, \ldots, j \), be a collection of circles such that \( N_{ij} \) has center \( x_j \), the radius of \( N_{ij} \to 0 \) as \( j \to \infty \), and \( N_{pj} \cap N_{qj} = \emptyset \), \( p < q \leq j \). Let \( f_j(x) = 1 \) if \( x \) is exterior to each of \( N_{1j}, N_{2j}, \ldots, N_{jj} \) and \( f_j \) decreases linearly to zero between each circle and its center. If \( k_j \geq f_p \), for all \( p \geq j \), then \( k_j \geq 1 \) and thus \( \{f_j\} \) does not order converge to 0 even though it converges pointwise.

The situation of \( B_1(\Omega) \) is much less complicated. It has been shown previously that order convergence implies pointwise convergence. (See Tucker [16], Lemma 3). For the converse we have the following theorem.

**Theorem 8.** Bounded pointwise convergence implies order convergence in \( B_1(\Omega) \) if and only if \( B_1(\Omega) \) is closed with respect to pointwise convergence which implies that it is the set of \( \Lambda \) measurable functions for some \( \sigma \)-algebra \( \Lambda \).

**Proof.** If bounded pointwise convergence implies order convergence, by Theorem 3, \( B_1(\Omega) \) is almost \( \sigma \)-complete. By Proposition 6 of Tucker [18], \( B_1(\Omega) \) is order Cauchy complete. Thus \( B_1(\Omega) \) is \( \sigma \)-complete. By Lemma 4 of Tucker [18], it is closed under pointwise convergence. Therefore the conclusion follows. (See Regoli [13] or Bogdan [2].)

If \( B_1(\Omega) \) is closed with respect to pointwise convergence and \( f_1, f_2, f_3, \ldots \) is a bounded sequence converging pointwise to \( f \), let \( y_n(x) = \max_{i \geq n} f_i(x) \). Then \( y_1, y_2, y_3, \ldots \) is a nonincreasing sequence of functions in \( B_1(\Omega) \) converging pointwise to \( f \) and \( y_n \geq f_n \). Thus \( f_1, f_2, f_3, \ldots \) order converges to \( f \).

In [9], Meyer shows that if \( X \) is an infinite dispersed compact Hausdorff space then \( C(X) \) is not closed with respect to pointwise convergence but \( B_1(C(X)) \) is.

The following theorems expand on the relationship between \( P \)-spaces and spaces of Baire functions. These theorems also supplement the results in Tucker [17].

In the following, \( B_2(\Omega) = B_1(B_1(\Omega)) \), and in general if \( \alpha \) is an ordinal, \( \alpha > 0 \), \( B_\alpha(\Omega) \) is the family of pointwise limits of sequences from \( \bigcup_{\gamma \geq \alpha} B_\gamma(\Omega) \). If \( \omega_1 \) is the first uncountable ordinal then \( B_{\omega_1}(\Omega) = B_{\omega_1+1}(\Omega) \) which will be denoted as \( B(\Omega) \).

**Theorem 9.** If \( \varrho \) is a real valued Riesz homomorphism defined on \( B_1(\Omega) \), then \( \varrho \) can be extended to \( B_2(\Omega) \).

**Proof.** By Corollary 3 of Tucker [18], \( \varrho \) can be extended as a positive linear functional. To show that the extension of \( \varrho \) is a Riesz homomorphism consider \( f \) and \( g \) in \( USB_1(\Omega) \) such that \( f \wedge g = 0 \). There exists a sequence \( \{f_i\} \) of \( B_1(\Omega) \) such that \( f = \bigwedge f_i \) and there exists a sequence \( \{g_i\} \) of \( B_1(\Omega) \) such that \( g = \bigvee g_i \). Since \( \{f_i \wedge g_i\} \) converges pointwise to zero, \( \varrho(f_i \wedge g_i) = \varrho(f_i) \wedge \varrho(g_i) \) converges point-
wise to zero by Theorem 3 of Tucker [17]. As \( \varrho(f) = \wedge \varrho(f_i) \) and \( \varrho(g) = \wedge \varrho(g_i) \) also by Theorem 3, \( \varrho(f) \wedge \varrho(g) = 0 \). Now suppose \( h \) and \( k \) are in \( B_2(\Omega) \) and \( h \wedge k = 0 \). There exists a sequence \( \{h_i\} \) of points in \( (USB_1(\Omega))^+ \) and a sequence \( \{k_i\} \) of points in \( (USB_1(\Omega))^+ \) such that \( h = \vee h_i \) and \( k = \vee k_i \). Since \( h \wedge k = 0 \), \( h_i \wedge k_i = 0 \), which implies \( \varrho(h_i) \wedge \varrho(k_i) = 0 \), which in turn implies \( \varrho(h) \wedge \varrho(k) = 0 \). Thus the extension is a Riesz homomorphism.

**Theorem 10.** If \( B_1(\Omega) \) is mapped by a Riesz homomorphism \( \varrho \) into \( C(X) \) so that for each \( f \) in \( C^+(X) \) there is a subset \( \omega \) of \( \varrho(B_1(\Omega)) \) such that \( f \) is the pointwise suprema of the functions in \( \omega \), then \( X \) is a \( P \)-space.

**Proof.** By Theorem 2 of Tucker [17], \( \varrho = \alpha \beta \) where \( \beta \) is a Riesz homomorphism from \( B_1(\Omega) \) to \( C(X) \), \( \beta(1) = 1 \), and \( \alpha \) is multiplication by a function \( h \) in \( C(X) \). Suppose there exists a point \( x \) in \( X \) such that \( h(x) = 0 \). Then \( f(x) = 0 \) for each \( f \) in \( \varrho(B_1(\Omega)) \) and the constant function 1 is not the pointwise suprema of any subset of \( \varrho(B_1(\Omega)) \). Therefore \( h(x) \neq 0 \) for each \( x \) in \( X \) and \( 1/h \) is a continuous function. Thus \( \alpha^{-1} \alpha \beta = \beta \) is a Riesz homomorphism of \( B_1(\Omega) \) into \( C(X) \) with the property that \( \beta(1) = 1 \) and for each \( f \) in \( C^+(X) \) there is a subset \( \omega \) of \( \beta(B_1(\Omega)) \) such that \( f \) is the pointwise supremum of the functions in \( \omega \).

Let \( Z \) be a zero set of \( C(X) \). There exists a function \( f \) in \( C(X) \) such that \( f(x) = 1 \) for each \( x \) in \( Z \) and \( 0 \leq f(x) < 1 \) for each \( x \) not in \( Z \). There exists a subset \( \omega \) of \( \beta(B_1(\Omega)) \) such that \( f \) is the pointwise supremum of the functions in \( \omega \).

By Theorem 7 of Tucker [17], each bounded function in \( B_1(\Omega) \) can be uniformly approximated by a function in \( B_1(\Omega) \) with a finite range. Let \( g \) be a function in \( B_1^{-1}(\omega) \) such that \( 0 \leq g \leq 1 \) and \( g_r \) be a function in \( B_1(\Omega) \) with a finite range that uniformly approximates \( g - 1/2^{r+1} \) within \( 1/2^{r+1} \). Note that \( g_r \leq g \) and \( \{g_r\} \) converges pointwise to \( g \). Thus \( \beta(g_r) \leq \beta(g) \) and by Theorem 3 of Tucker [17], \( \{\beta(g_r)\} \) converges pointwise to \( \beta(g) \). The function \( g \) is the sum of a finite number of disjoint functions in \( B_1(\Omega) \) each of which takes on only one non-zero value. Denote the set of all such functions for all \( r \) and all \( g \) as \( \Gamma \). Then \( f \) is the pointwise supremum of the functions in \( \beta(\Gamma) \). Now if \( h \) is a function in \( B_1(\Omega) \) that takes on only the values 0 and 1, \( h \leq 1 \), \( (1 - h) \leq 1 \), and \( h \wedge (1 - h) = 0 \). This implies that \( \beta(h) \) takes on only the values 0 and 1. Thus each function in \( \beta(\Gamma) \) takes on only one non-zero value.

Pick \( x \) in \( Z \). There exists a subsequence \( f_1, f_2, f_3, \ldots \) of \( \beta(\Gamma) \) such that \( 0 < f_1(x) \leq f_2(x) \leq f_3(x) \ldots \) and \( \{f_i(x)\} \) converges to 1. For each positive integer \( i \) let \( c_i \) be a positive number and \( t_i \) be a function that takes on only the values 0 and 1 such that \( \beta(c_i t_i) = f_i \). Let \( s_i = \beta(c_i (t_1 \wedge t_2 \wedge \ldots \wedge t_i)) \). Then \( s_i(x) = f_i(x) \) so that \( \{s_i(x)\} \) converges to 1. The sequence \( \{c_i (t_1 \wedge t_2 \wedge \ldots \wedge t_i)\} \) converges pointwise to a function \( k \) in \( B_2(\Omega) \). By Theorem 9, \( \beta \) can be extended to \( B_2(\Omega) \). Let \( \beta(k) = v \). By Theorem 4 of Tucker [17], \( v \) is continuous. By Theorem 3 of Tucker [17], \( v \) is the pointwise limit of \( \{s_i\} \). Therefore \( v(x) = 1 \), \( v^{-1}(1) \) is open, and \( Z \) is open. This implies that \( x \) is a \( P \)-space by Theorem 5.3 of Gillman and Henriksen [4].

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References


Author’s address: University of Houston, Central Campus, Houston, Texas 77004, U.S.A.