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JOINT ESSENTIAL SPECTRA

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Introduction. The essential spectrum of a bounded linear operator A on a Hilbert space is the spectrum of the canonical image of A in the Calkin algebra. This has been discussed by Fillmore, Stampfli and Williams [3]. Dash [1] has discussed the joint essential spectrum of an n-tuple of bounded operators and has extended some of the results of [3]. A bounded operator A on a Hilbert space is said to be *Fredholm* if the null spaces of A and A^* are finite dimensional and the range of A is closed. By Atkinson's theorem [4, problem 142], a bounded operator A is Fredholm if and only if zero does not belong to the essential spectrum of A. In this note we study the generalization of the notion of a Fredholm operator to an n-tuple of closed operators with the same domain which is dense in a Hilbert space. The result analogous to Atkinson's theorem will be proved and some other characterizations for an n-tuple of operators in a Hilbert space to be joint Fredholm will be discussed. Also Weyl's theorem for an n-tuple of commuting normal operators will be proved.

In what follows, H denotes a complex separable infinite dimensional Hilbert space, $\mathbb{B}(H)$ denotes the algebra of all bounded linear operators on H. Let $\mathbb{K}$ be the ideal of compact operators on H, $\mathcal{Q}$ the quotient (or Calkin) algebra $\mathbb{B}(H)/\mathbb{K}$ and π the canonical quotient map of $\mathbb{B}(H)$ onto $\mathcal{Q}$. Let $H^{(n)} = \sum_{i=1}^{n} \oplus H_i$, $(H_i = H)$ and let $T = (T_1, \ldots, T_n)$ be an n-tuple of closed linear operators $T_1, \ldots, T_n$ with the same domain $\mathbb{D}(T)$, dense in H. We define an operator $T^{(n)}: \mathbb{D}(T) \to H^{(n)}$ by $T^{(n)}x = (T_1x, \ldots, T_nx)$, $(x \in \mathbb{D}(T))$. Further, if $T_1^*, \ldots, T_n^*$ have the same domain, then we shall denote $(T_1^*, \ldots, T_n^*)$ by T^*. Let $T^{(n)\#}$ be the usual Hilbert space adjoint of $T^{(n)}$. Then $T^{(n)\#} T^{(n)}$ is a positive self-adjoint operator. If $G = (T^{(n)\#} T^{(n)})^{1/2}$, then $\mathbb{D}(G) = \mathbb{D}(T)$ and $\sum_{i=1}^{n} (T_ix, T_iy) = (T^{(n)}x, T^{(n)}y) = (Gx, Gy)$; $x, y \in \mathbb{D}(G) = \mathbb{D}(T)$ [5, p. 334]. The null space, the range and the closure of an operator A from H to a Hilbert space K will be denoted by $N(A)$, $R(A)$ and $\bar{A}$, respectively.

Definition 1. Let $T_1, \ldots, T_n$ be closed linear operators in H defined on the same dense domain $\mathbb{D}$. Suppose that $T_1^*, \ldots, T_n^*$ also have the same domain $\mathbb{D}^*$.

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(1) The joint left spectrum $\mathrm{Sp}_l(T)$ of $T=(T_1, \ldots, T_n)$ is the set of $(z_1, \ldots, z_n) \in \mathbb{C}^n$ (n-fold Cartesian product of the complex plane $\mathbb{C}$) such that for no n-tuple $(B_1, \ldots, B_n)$ of operators in $\mathbb{B}(H)$, $\sum_{i=1}^{n} B_i(T_i - z_i I) \subset I$ holds.

(2) The joint right spectrum $\mathrm{Sp}_r(T)$ of $T=(T_1, \ldots, T_n)$ is the set $(\mathrm{Sp}_l(T^*))^*$, where $T^* = (T_1^*, \ldots, T_n^*)$ and for $K \subset \mathbb{C}^n$, $K^* = \{(\bar{z}_1, \ldots, \bar{z}_n) : (z_1, \ldots, z_q) \in K\}$.

(3) The joint spectrum $\mathrm{Sp}(T)$ is the set $\mathrm{Sp}_l(T) \cup \mathrm{Sp}_r(T)$ [6, Definition 1.1].

Definition 2. The joint left (right) spectrum $\mathrm{Sp}_{\mathfrak{A}}^{l}(a)$ $(\mathrm{Sp}_{\mathfrak{A}}^{r}(a))$ of an n-tuple $a = (a_1, \ldots, a_n)$ of elements $a_1, \ldots, a_n$ of a unital Banach algebra $\mathfrak{A}$ is the set of all $z = (z_1, \ldots, z_n)$ in $\mathbb{C}^n$ such that the left (right) ideal generated by $\{a_1 - z_1 e, \ldots, a_n - z_n e\}$ is proper in $\mathfrak{A}$. The joint spectrum $\mathrm{Sp}_{\mathfrak{A}}(a)$ of a is the set $\mathrm{Sp}_{\mathfrak{A}}^{l}(a) \cup \mathrm{Sp}_{\mathfrak{A}}^{r}(a)$.

It is obvious that if $\mathfrak{A}$ is a Banach*-algebra, then $\mathrm{Sp}_{\mathfrak{A}}^{r}(a) = \{(\bar{z}_1, \ldots, \bar{z}) : (z_1, \ldots, z_n) \in \mathrm{Sp}_{\mathfrak{A}}^{l}(a_1^*, \ldots, a_n^*)\}$.

Definition 3. An n-tuple $T = (T_1, \ldots, T_n)$ of closed operators with the same domain which is dense in H, whose adjoints also have the same domain in H, is called *joint upper Fredholm* (in short j.u.F.) if $N(T^{(n)})$ is finite dimensional and $R(T^{(n)})$ is a closed subspace of $H^{(n)}$. T is called *joint lower Fredholm* (in short j.l.F.) if T^* is j.u.F.. T is called *joint Fredholm* if T is both j.u.F. and j.l.F..

Definition 4. The joint left (right) essential spectrum $\mathrm{Sp}_{le}(T)$ $(\mathrm{Sp}_{re}(T))$ of T is the set of all $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ such that $(T_1 - z_1 I, \ldots, T_n - z_n I)$ is not j.u.F. (j.l.F.). The essential spectrum $\mathrm{Sp}_e(T)$ is the set $\mathrm{Sp}_{le}(T) \cup \mathrm{Sp}_{re}(T)$.

Characterizations of an n-tuple to be j.u.F. The following theorem is a result analogous to Atkinson's theorem.

Theorem 5. *Let $T = (T_1, \ldots, T_n)$ be an n-tuple of closed operators with the same domain $\mathbb{D}(T)$ dense in H. Then zero does not belong to $\mathrm{Sp}_{le}(T)$ if and only if there exist $B_1, \ldots, B_n$ in $\mathbb{B}(H)$ such that $\sum_{i=1}^{n} B_i T_i - I$ is a compact operator.*

Proof. Suppose zero does not belong $\mathrm{Sp}_{le}(T)$. Then T is j.u.F. So $\mathbb{R}(T^{(n)})$ is closed and $N(T^{(n)})$ is of finite dimension. Also $T^{(n)}$ maps $N(T^{(n)})^{\perp} \cap \mathbb{D}(T)$ one to one onto $\mathbb{R}(T^{(n)})$. It is not difficult to see that $T^{(n)}$ is a closed operator. Hence there exists a bounded operator $B\colon H^{(n)} \to H$ such that $BT^{(n)} \subset I_{N(T^{(n)})^{\perp}}$. Define $B_i x = B(0, \ldots, 0, x, 0, \ldots, 0)$ (where x is at the ith place on the right hand side and $x \in H$). Then $B_i \in \mathbb{B}(H)$ and $\sum_{i=1}^{n} B_i T_i \subset I_{N(T^{(n)})^{\perp}}$. So $I - \sum_{i=1}^{n} B_i T_i$ is a projection on $N(T^{(n)})$ and since $N(T^{(n)})$ is finite dimensional, $I - \sum_{i=1}^{n} B_i T_i$ is a compact operator.

Conversely, suppose that there exist bounded operators $B_1, \ldots, B_n$ in $\mathbb{B}(H)$ such that $I - \sum_{i=1}^{n} B_i T_i$ $(= C)$ is a compact operator. Define $B: H^{(n)} \to H$ by $B(x_1, \ldots, x_n) = \sum_{i=1}^{n} B_i x_i$ $(x_1, \ldots, x_n) \in H^{(n)}$. Then B is bounded and $\overline{BT^{(n)}} = I - C$. Hence $N(\overline{BT^{(n)}})$ $(= N(BT^{(n)}))$ is of finite dimension. Since C is a compact operator, $BT^{(n)}$ is bounded below on $N(BT^{(n)})$ [4, Solution 140]. As $\|BT^{(n)}x\| \leqq \|B\| \, \|T^{(n)}x\|$ for $x \in \mathbb{D}(T) \cap N(BT^{(n)})^{\perp}$, $T^{(n)}$ is bounded below on $\mathbb{D}(T) \cap N(BT^{(n)})^{\perp}$. Therefore $T^{(n)}(N(BT^{(n)})^{\perp} \cap \mathbb{D}(T))$ is closed. Since $N(BT^{(n)})$ is of finite dimension and $T^{(n)}(N(BT^{(n)})^{\perp} \cap \mathbb{D}(T)) + T^{(n)}(N(BT^{(n)})) = T^{(n)}(\mathbb{D}(T))$, $\mathbb{R}(T^{(n)})$ is closed. Thus T is j.u.F.. ∎

Remark 6. Dash [1] has defined the joint left (right) essential spectrum of an n-tuple $T = (T_1, \ldots, T_n)$ of bounded operators on H as

$$\sigma_{le}(T) = \mathrm{Sp}^{l}_{\mathscr{Q}}(\pi(T_1), \ldots, \pi(T_n)) \; (\sigma_{re}(T) = \mathrm{Sp}^{r}_{\mathscr{Q}}(\pi(T_1), \ldots, \pi(T_n)))$$

(see Definition 2) and the joint essential spectrum as $\sigma_{le}(T) \cup \sigma_{re}(T)$. The last theorem shows that for an n-tuple T of operators in $\mathbb{B}(H)$, $\sigma_{le}(T) = \mathrm{Sp}_{le}(T)$ and $\sigma_{re}(T) = \mathrm{Sp}_{re}(T)$; hence $\sigma_e(T) = \mathrm{Sp}_e(T)$.

Next we give other characterizations for an n-tuple to be j.u.F..

Theorem 7. *Let $T = (T_1, \ldots, T_n)$ be an n-tuple of closed operators with the same domain $\mathbb{D}(T)$ which is dense in H. Then the following assertions are equivalent.*

(a) *T is not j.u.F..*
(b) *There exists a sequence $\{x_k\}$ of unit vectors in $\mathbb{D}(T)$ such that $x_k \to 0$ (weakly) and $T_i x_k \to 0$ (strongly) as $k \to \infty$ for $i = 1, \ldots, n$.*
(c) *There exists an orthonormal sequence $\{e_k\}$ in $\mathbb{D}(T)$ such that $T_i e_k \to 0$ (strongly), as $k \to \infty$ for $i = 1, \ldots, n$.*
(d) *There exists an infinite dimensional projection P such that $PH \subset \mathbb{D}(T)$ and $T_i P$ is compact for each $i = 1, \ldots, n$.*
(e) *For every $\delta > 0$, there exists a closed infinite dimensional subspace $M_\delta \subset \mathbb{D}(T)$ such that*

$$\sum \|T_i x\|^2 \leqq \delta \|x\|^2 \quad \text{for} \quad x \in M_\delta .$$

(f) *$(T^{(n)\#} \, T^{(n)})^{1/2}$ is a Fredholm operator in H.*

Proof. Proof of (d) $\Rightarrow$ (c). Suppose that (d) holds. Let $\{e_k\}$ be an orthonormal basis for PH. Since $T_i P$ is compact and $e_k \to 0$ (weakly), $T_i e_k = T_i P e_k \to 0$ (strongly) [2] for $i = 1, \ldots, n$.

(c) $\Rightarrow$ (b) is clear.

Proof of (b) $\Rightarrow$ (a). Let $\{x_k\}$ be a sequence of unit vectors in $\mathbb{D}(T)$ such that $x_k \to 0$ (weakly) and $T_i x_k \to 0$ (strongly) as $k \to \infty$, for $i = 1, \ldots, n$. If there exists $B_1, \ldots, B_n \in$

$\in \mathbb{B}(H)$ such that $I - \sum_{i=1}^{n} B_i T_i$ is a compact operator, then

$$1 = \|x_k\| = \|\sum_{i=1}^{n} B_i T_i x_k + x_k - \sum_{i=1}^{n} B_i T_i x_k\| \leqq$$
$$\leqq \|(I - \sum_{i=1}^{n} B_i T_i)\, x_k\| + \sum_{i=1}^{n} \|B_i\| \, \|T_i x_k\| \to 0 \quad \text{as} \quad k \to \infty\,,$$

which is absurd. Hence there exist no $B_1, \ldots, B_n \in \mathbb{B}(H)$ such that $I - \sum_{i=1}^{n} B_i T_i$ is a compact operator. Thus by Theorem 5, T is not j.u.F..

Proof of (a) $\Rightarrow$ (f). Since $N(T^{(n)}) = N((T^{(n)\#}\, T^{(n)})^{1/2})$, it is sufficient to show that if $\mathbb{R}(T^{(n)})$ is not closed, then $\mathbb{R}((T^{(n)\#}\, T^{(n)})^{1/2})$ is not closed. If $\mathbb{R}((T^{(n)\#}\, T^{(n)})^{1/2})$ is closed, then let $\{T^{(n)} x_k\}$ be a Cauchy sequence in $\mathbb{R}(T^{(n)})$. Then, since $\|T^{(n)} x_k\| = \|(T^{(n)\#}\, T^{(n)})^{1/2}\, x_k\|$, $\{(T^{(n)\#}\, T^{(n)})^{1/2}\, x_k\}$ is a Cauchy sequence in $\mathbb{R}((T^{(n)\#}\, T^{(n)})^{1/2})$. But $\mathbb{R}((T^{(n)\#}\, T^{(n)})^{1/2})$ is closed, so $(T^{(n)\#}\, T^{(n)})^{1/2}\, x_k \to (T^{(n)\#}\, T^{(n)})^{1/2}\, x$ for some $x \in \mathbb{D}((T^{(n)\#}\, T^{(n)})^{1/2}) = \mathbb{D}(T)$. Hence

$$\|T^{(n)}(x_k - x)\| = \|(T^{(n)\#}\, T^{(n)})^{1/2}\, (x_k - x)\| \to 0$$

as $k \to \infty$. Therefore $\mathbb{R}(T^{(n)})$ is closed which is a contradiction.

Proof of (f) $\Rightarrow$ (d). Since $((T^{(n)\#}\, T^{(n)})^{1/2})$ is not Fredholm by [3, Theorem 1.1], there exists an infinite dimensional projection P such that $PH \subset \mathbb{D}((T^{(n)\#}\, T^{(n)})^{1/2}) = \mathbb{D}(T)$ and $(T^{(n)\#}\, T^{(n)})^{1/2}\, P$ is a compact operator. Let $\{x_k\}$ be a bounded sequence weakly converging to zero. Then $(T^{(n)\#}\, T^{(n)})^{1/2}\, Px_k \to 0$ (stıongly) as $k \to \infty$. Therefore,

$$\|T_i P x_k\|^2 \leqq \sum_{j=1}^{n} \|T_j P x_k\|^2 = \sum_{j=1}^{n} (T_j P x_k, T_j P x_k) =$$
$$= (T^{(n)} P x_k, T^{(n)} P x_k) = \|(T^{(n)\#}\, T^{(n)})^{1/2}\, P x_k\|^2 \to 0$$

as $k \to \infty$, for $i = 1, \ldots, n$. Thus $T_i P$ is compact for $i = 1, \ldots, n$.

Since $\mathbb{D}((T^{(n)\#}\, T^{(n)})^{1/2}) = \mathbb{D}(T)$ and $\|(T^{(n)\#}\, T^{(n)})^{1/2}\, x\|^2 = \|T^{(n)} x\|^2 = \sum_{i=1}^{n} \|T_i x\|^2$ for $x \in \mathbb{D}(T)$, the equivalence of (e) and (f) follows from [3, Theorem 1.1]. ∎

Corollary 8. *If $T = (T_1, \ldots, T_n)$ is an n-tuple of normal operators with the same domain $\mathbb{D}$ in H, then* $\mathrm{Sp}_{\mathrm{le}}(T) = \mathrm{Sp}_{\mathrm{e}}(T)$.

Proof. Since $T_i - z_i I$ is normal, $\|(T_i - z_i I)\, x\| = \|(T_i - z_i I)^*\, x\|$ for $x \in \mathbb{D}(T) = \mathbb{D}(T^*)$ for $i = 1, \ldots, n$. The equivalence of (a) and (b) in the last theorem, yields $\mathrm{Sp}_{\mathrm{le}}(T) = \mathrm{Sp}_{\mathrm{re}}(T)$. Hence $\mathrm{Sp}_{\mathrm{le}}(T) = \mathrm{Sp}_{\mathrm{e}}(T)$. ∎

Weyl's theorem. In this section we prove a Weyl-type theorem. To this end we shall need some lemmas. Let $T = (T_1, \ldots, T_n)$ be an n-tuple of closed operators with the same dense domain $\mathbb{D}(T)$ in H. We say that $z = (z_1, \ldots, z_n) \in C^n$ is a joint eigenvalue of T if there exists a nonzero vector x in $\mathbb{D}(T)$ such that $(T_i - z_i I)\, x = 0$ for $i = 1, \ldots, n$. The multiplicity of z is the dimension of $\bigcap_{i=1}^{n} N(T_i - z_i I)$.

Lemma 9. *Let* $T = (T_1, \ldots, T_n)$ *be an n-tuple of pairwise commuting normal operators with the same domain* $\mathbb{D}(T)$ *in H. Let* $z = (z_1, \ldots, z_n)$ *be an isolated point of the joint spectrum* $\mathrm{Sp}(T)$ *of* T (see Definition 1). *Then z is a joint eigenvalue of T.*

Proof. Since z is an isolated point, χ_z, the characteristic function of $\{z\}$, is a non-zero element in $m(\mathrm{Sp}(T))$, the algebra of all equivalence classes (with respect to the equality almost everywhere) of Borel functions on $\mathrm{Sp}(T)$. By the joint spectral theorem [6, Theorem 2.2], $\chi_z(T) = P_z$ is a non zero projection H and $T_iP_z - z_iP_z = 0$ for each $i = 1, \ldots, n$. Hence z is a joint eigenvalue of T. ∎

Lemma 10. *Let* $S = (S_1, \ldots, S_n)$ *be an n-tuple of closed operators with the same dense domain* $\mathbb{D}(S)$ *in a Hilbert space* H_1 *and let* $T = (T_1, \ldots, T_n)$ *be an n-tuple of closed operators with the same dense domain* $\mathbb{D}(T)$ *in a Hilbert space* H_2. *Then* $\mathrm{Sp}(S \oplus T) = \mathrm{Sp}(S) \cup \mathrm{Sp}(T)$, *where* $S \oplus T = (S_1 \oplus T_1, \ldots, S_n \oplus T_n)$.

Proof. It is sufficient to show that $\mathrm{Sp}_1(S \oplus T) = \mathrm{Sp}_1(S) \cup \mathrm{Sp}_1(T)$. Let $z = (z_1, \ldots, z_n) \notin \mathrm{Sp}_1(S) \cup \mathrm{Sp}_1(T)$. Then there exist $B_1, \ldots, B_n \in \mathbb{B}(H_1)$ and $C_1, \ldots, C_n \in \mathbb{B}(H_2)$ such that $\sum_{i=1}^{n} B_i(S_i - z_iI_{H_1}) \subset I_{H_1}$ and $\sum_{i=1}^{n} C_i(T_i - z_iI_{H_2}) \subset I_{H_2}$. Thus $\sum_{i=1}^{n} B_i \oplus C_i(S_i \oplus T_i - z_iI) \subset I$. Hence $z \notin \mathrm{Sp}_1(S \oplus T)$. Conversely, if $z \in \mathrm{Sp}_1(S) \cup \mathrm{Sp}_1(T)$, then $z \in \mathrm{Sp}_1(S)$ or $z \in \mathrm{Sp}_1(T)$. Without loss of generality assume that $z \in \mathrm{Sp}_1(S)$. Then there exists a sequence $\{x_k\}$ of unit vectors in $\mathbb{D}(S)$ such that $(S_i - z_iI_{H_1})\, x_k \to 0$ for $i = 1, \ldots, n$. Let $y_k = x_k \oplus 0$. Then $\|y_k\| = 1$ and $(S_i \oplus T_i - z_iI)\, y_k = (S_i - z_iI_{H_2})\, x_k \to 0$ as $k \to \infty$ for $i = 1, \ldots, n$. Hence $z \in \mathrm{Sp}_1(S \oplus T)$. Thus $\mathrm{Sp}_1(S \oplus T) = \mathrm{Sp}_1(S) \cup \mathrm{Sp}_1(T)$. ∎

Theorem 11. *Let* $T = (T_1, \ldots, T_n)$*be an n-tuple of pairwise commuting normal operators with the same domain* $\mathbb{D}(T)$ *in H. Then* $\mathrm{Sp}_e(T)$ *consists precisely of all points in* $\mathrm{Sp}(T)$ *except the isolated joint eigenvalues of finite multiplicity.*

Proof. Since $\mathrm{Sp}_{le}(T) = \mathrm{Sp}_e(T)$ by Corollary 8, it is sufficient to show that $(0, \ldots, 0)$ is an isolated joint eigenvalue of T of finite multiplicity if and only if T is j.u.F. and $(0, \ldots, 0) \in \mathrm{Sp}(T)$.

As T_i's are pairwise commuting and normal, $N(T^{(n)}) = \bigcap_{i=1}^{n} N(T_i)$ is a reducing subspace for each T_i. For each i define $S_i\colon N(T^{(n)})^\perp \cap \mathbb{D}(T) \to N(T^{(n)})^\perp$ by $S_ix = T_ix$ $(x \in N(T^{(n)})^\perp \cap \mathbb{D}(T))$. Then $T_i = 0 \oplus S_i$, the null space of $S^{(n)}$ is $\{0\}$ and S_i's are pairwise commuting normal operators in $N(T^{(n)})^\perp$. Also by Lemma 10, $\mathrm{Sp}(T) = \{0\} \cup \mathrm{Sp}(S)$ (where $S = (S_1, \ldots, S_n)$).

Now assume that $(0, \ldots, 0)$ is an isolated joint eigenvalue of T of finite multiplicity. Since $N(S^{(n)}) = \{0\}$, by Lemma 9, $(0, \ldots, 0) \notin \mathrm{Sp}(S)$. Hence $\mathbb{R}(T^{(n)}) = \mathbb{R}(S^{(n)})$ is a closed subspace of $H^{(n)}$. As $N(T^{(n)})$ is of finite dimensions, T is j.u.F..

Conversely, assume that T is j.u.F. and $(0, \ldots, 0) \in \mathrm{Sp}(T)$. Then $\mathbb{R}(T^{(n)})$ is closed and $N(T^{(n)})$ is finite dimensional. Since $N(S^{(n)}) = \{0\}$ and $\mathbb{R}(S^{(n)}) = \mathbb{R}(T^{(n)})$, which is closed, $S^{(n)}$ is bounded below. So $(0, \ldots, 0) \notin \mathrm{Sp}_1(S) = \mathrm{Sp}(S)$. Hence $(0, \ldots, 0)$ is an isolated point of $\mathrm{Sp}(T)$. As the dimension of $N(T^{(n)})$ is finite, by Lemma 9, $(0, \ldots, 0)$ is an isolated joint eigenvalue of T of finite multiplicity. ■

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