JOINT ESSENTIAL SPECTRA

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Introduction. The essential spectrum of a bounded linear operator $A$ on a Hilbert space is the spectrum of the canonical image of $A$ in the Calkin algebra. This has been discussed by Fillmore, Stampfli and Williams [3]. Dash [1] has discussed the joint essential spectrum of an $n$-tuple of bounded operators and has extended some of the results of [3]. A bounded operator $A$ on a Hilbert space is said to be Fredholm if the null spaces of $A$ and $A^*$ are finite dimensional and the range of $A$ is closed. By Atkinson’s theorem [4, problem 142], a bounded operator $A$ is Fredholm if and only if zero does not belong to the essential spectrum of $A$. In this note we study the generalization of the notion of a Fredholm operator to an $n$-tuple of closed operators with the same domain which is dense in a Hilbert space. The result analogous to Atkinson’s theorem will be proved and some other characterizations for an $n$-tuple of operators in a Hilbert space to be joint Fredholm will be discussed. Also Weyl’s theorem for an $n$-tuple of commuting normal operators will be proved.

In what follows, $H$ denotes a complex separable infinite dimensional Hilbert space, $\mathcal{B}(H)$ denotes the algebra of all bounded linear operators on $H$. Let $\mathcal{K}$ be the ideal of compact operators on $H$, $\mathcal{Q}$ the quotient (or Calkin) algebra $\mathcal{B}(H)/\mathcal{K}$ and $\pi$ the canonical quotient map of $\mathcal{B}(H)$ onto $\mathcal{Q}$. Let $H^{(n)} = \bigoplus_{i=1}^n H_i$, $(H_i = H)$ and let $T = (T_1, \ldots, T_n)$ be an $n$-tuple of closed linear operators $T_1, \ldots, T_n$ with the same domain $\mathcal{D}(T)$, dense in $H$. We define an operator $T^{(n)} : \mathcal{D}(T) \rightarrow H^{(n)}$ by $T^{(n)} x = (T_1 x, \ldots, T_n x), (x \in \mathcal{D}(T))$. Further, if $T_1^*, \ldots, T_n^*$ have the same domain, then we shall denote $(T_1^*, \ldots, T_n^*)$ by $T^*$. Let $T^{(n)*}$ be the usual Hilbert space adjoint of $T^{(n)}$. Then $T^{(n)*} T^{(n)}$ is a positive self-adjoint operator. If $G = (T^{(n)*} T^{(n)})^{1/2}$, then $\mathcal{D}(G) = \mathcal{D}(T)$ and $\sum_{i=1}^n (T_ix, T_iy) = (T^{(n)}x, T^{(n)}y) = (Gx, Gy); x, y \in \mathcal{D}(G) = \mathcal{D}(T)$ [5, p. 334]. The null space, the range and the closure of an operator $A$ from $H$ to a Hilbert space $K$ will be denoted by $N(A), R(A)$ and $\bar{A}$, respectively.

Definition 1. Let $T_1, \ldots, T_n$ be closed linear operators in $H$ defined on the same dense domain $\mathcal{D}$. Suppose that $T_1^*, \ldots, T_n^*$ also have the same domain $\mathcal{D}^*$.

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(1) The joint left spectrum $\text{Sp}_l(T)$ of $T = (T_1, \ldots, T_n)$ is the set of $(z_1, \ldots, z_n) \in \mathbb{C}^n$ 
($n$-fold Cartesian product of the complex plane $\mathbb{C}$) such that for no $n$-tuple 
$(B_1, \ldots, B_n)$ of operators in $\mathcal{B}(H)$, $\sum_{i=1}^n B_i(T_i - z_i I) \subset I$ holds.

(2) The joint right spectrum $\text{Sp}_r(T)$ of $T = (T_1, \ldots, T_n)$ is the set $(\text{Sp}_l(T^*))^*$, where 
$T^* = (T_1^*, \ldots, T_n^*)$ and for $K \subset \mathbb{C}^n$, $K^* = \{(\bar{z}_1, \ldots, \bar{z}_n) : (z_1, \ldots, z_n) \in K\}$.

(3) The joint spectrum $\text{Sp}(T)$ is the set $\text{Sp}_l(T) \cup \text{Sp}_r(T)$ \cite[Definition 1.1]{6}.

**Definition 2.** The joint left (right) spectrum $\text{Sp}_{l(\mathcal{A})}(a)$ ($\text{Sp}_{r(\mathcal{A})}(a)$) of an $n$-tuple $a = 
(a_1, \ldots, a_n)$ of elements $a_1, \ldots, a_n$ of a unital Banach algebra $\mathcal{A}$ is the set of all 
$z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ such that the left (right) ideal generated by 
$\{a_1 - z_1 e, \ldots, a_n - z_n e\}$ is proper in $\mathcal{A}$. The joint spectrum $\text{Sp}_{l(\mathcal{A})}(a)$ of $a$ is the set $\text{Sp}_{l(\mathcal{A})}(a) \cup \text{Sp}_{r(\mathcal{A})}(a)$.

It is obvious that if $\mathcal{A}$ is a Banach*-algebra, then $\text{Sp}_{l(\mathcal{A})}(a) = \{(\bar{z}_1, \ldots, \bar{z}) : 
(z_1, \ldots, z_n) \in \text{Sp}_{l(\mathcal{A})}(a_1^*, \ldots, a_n^*)\}$.

**Definition 3.** An $n$-tuple $T = (T_1, \ldots, T_n)$ of closed operators with the same domain 
which is dense in $H$, whose adjoints also have the same domain in $H$, is called joint 
upper Fredholm (in short j.u.F.) if $N(T^{(n)})$ is finite dimensional and $R(T^{(n)})$ is a closed 
subspace of $H^{(n)}$. $T$ is called joint lower Fredholm (in short j.l.F.) if $T^* \in j.u.F..$ $T$ is called 
joint Fredholm if $T$ is both j.u.F. and j.l.F..

**Definition 4.** The joint left (right) essential spectrum $\text{Sp}_{le}(T)$ ($\text{Sp}_{re}(T)$) of $T$ is the 
set of all $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ such that $(T_1 - z_1 I, \ldots, T_n - z_n I)$ is not j.u.F. (j.l.F.). 
The essential spectrum $\text{Sp}_e(T)$ is the set $\text{Sp}_{le}(T) \cup \text{Sp}_{re}(T)$.

**Characterizations of an $n$-tuple to be j.u.F.** The following theorem is a result analogous 
to Atkinson's theorem.

**Theorem 5.** Let $T = (T_1, \ldots, T_n)$ be an $n$-tuple of closed operators with the same 
domain $\mathbb{D}(T)$ dense in $H$. Then zero does not belong to $\text{Sp}_{le}(T)$ if and only if there 
exist $B_1, \ldots, B_n$ in $\mathcal{B}(H)$ such that $\sum_{i=1}^n B_i T_i - I$ is a compact operator.

**Proof.** Suppose zero does not belong to $\text{Sp}_{le}(T)$. Then $T$ is j.u.F. So $\mathbb{R}(T^{(n)})$ is closed 
and $N(T^{(n)})$ is of finite dimension. Also $T^{(n)}$ maps $N(T^{(n)})$ onto $\mathbb{D}(T)$ and $\mathbb{R}(T^{(n)})$. It is not difficult to see that $T^{(n)}$ is a closed operator. Hence there exists 
a bounded operator $B$: $H^{(n)} \to H$ such that $BT^{(n)} \subset I_{N(T^{(n)})}$. Define $B_ix = B(0, \ldots, 0, 
\chi, 0, \ldots, 0)$ (where $x$ is at the $i$th place on the right hand side and $\chi \in H$). Then $B_i \in 
\mathbb{B}(H)$ and $\sum_{i=1}^n B_i T_i \subset I_{N(T^{(n)})}$. So $I - \sum_{i=1}^n B_i T_i$ is a projection on $N(T^{(n)})$ and since 
$N(T^{(n)})$ is finite dimensional, $I - \sum_{i=1}^n B_i T_i$ is a compact operator.
Conversely, suppose that there exist bounded operators $B_1, \ldots, B_n$ in $\mathcal{B}(H)$ such that $I - \sum_{i=1}^{n} B_i T_i (= C)$ is a compact operator. Define $B: H^{(n)} \to H$ by $B(x_1, \ldots, x_n) = \sum_{i=1}^{n} B_i x_i$ \((x_1, \ldots, x_n) \in H^{(n)}\). Then $B$ is bounded and $\overline{BT^{(n)}} = I - C$. Hence $N(\overline{BT^{(n)}}) (= N(BT^{(n)})$ is of finite dimension. Since $C$ is a compact operator, $BT^{(n)}$ is bounded below on $N(BT^{(n)})$ [4, Solution 140]. As $\|BT^{(n)}x\| \leq \|B\| \|T^{(n)}x\|$ for $x \in \mathcal{D}(T) \cap N(BT^{(n)})$, $T^{(n)}$ is bounded below on $\mathcal{D}(T) \cap N(BT^{(n)})$. Therefore $T^{(n)}(N(BT^{(n)}) \cap \mathcal{D}(T))$ is closed. Since $N(BT^{(n)})$ is of finite dimension and $T^{(n)}(N(BT^{(n)}) \cap \mathcal{D}(T)) \in T^{(n)}(\mathcal{D}(T))$, $\mathcal{H}(T^{(n)})$ is closed. Thus $T$ is j.u.F.\

\textbf{Remark 6.} Dash [1] has defined the joint left (right) essential spectrum of an $n$-tuple $T = (T_1, \ldots, T_n)$ of bounded operators on $H$ as

$$
\sigma_{le}(T) = \text{Sp}_{\delta}(\pi(T_1), \ldots, \pi(T_n)) \quad \sigma_{re}(T) = \text{Sp}_{\delta}(\pi(T_1), \ldots, \pi(T_n))
$$

(see Definition 2) and the joint essential spectrum as $\sigma_{le}(T) \cup \sigma_{re}(T)$. The last theorem shows that for an $n$-tuple $T$ of operators in $\mathcal{B}(H)$, $\sigma_{le}(T) = \text{Sp}_{le}(T)$ and $\sigma_{re}(T) = \text{Sp}_{re}(T)$; hence $\sigma_{le}(T) = \text{Sp}_{le}(T)$.

Next we give other characterizations for an $n$-tuple to be j.u.F.

\textbf{Theorem 7.} Let $T = (T_1, \ldots, T_n)$ be an $n$-tuple of closed operators with the same domain $\mathcal{D}(T)$ which is dense in $H$. Then the following assertions are equivalent.

(a) $T$ is not j.u.F..

(b) There exists a sequence $\{x_k\}$ of unit vectors in $\mathcal{D}(T)$ such that $x_k \to 0$ (weakly) and $T_i x_k \to 0$ (strongly) as $k \to \infty$ for $i = 1, \ldots, n$.

(c) There exists an orthonormal sequence $\{e_k\}$ in $\mathcal{D}(T)$ such that $T_i e_k \to 0$ (strongly), as $k \to \infty$ for $i = 1, \ldots, n$.

(d) There exists an infinite dimensional projection $P$ such that $PH \subset \mathcal{D}(T)$ and $T_i P$ is compact for each $i = 1, \ldots, n$.

(e) For every $\delta > 0$, there exists a closed infinite dimensional subspace $M_\delta \subset \mathcal{D}(T)$ such that

$$
\sum_i \|T_i x\|^2 \leq \delta \|x\|^2 \quad \text{for} \quad x \in M_\delta.
$$

(f) $(T^{(n)} - T^{(n)})^{1/2}$ is a Fredholm operator in $H$.

\textbf{Proof.} Proof of (d) $\Rightarrow$ (e). Suppose that (d) holds. Let $\{e_k\}$ be an orthonormal basis for $PH$. Since $T_i P$ is compact and $e_k \to 0$ (weakly), $T_i e_k = T_i P e_k \to 0$ (strongly) [2] for $i = 1, \ldots, n$.

(e) $\Rightarrow$ (b) is clear.

Proof of (b) $\Rightarrow$ (a). Let $\{x_k\}$ be a sequence of unit vectors in $\mathcal{D}(T)$ such that $x_k \to 0$ (weakly) and $T_i x_k \to 0$ (strongly) as $k \to \infty$, for $i = 1, \ldots, n$. If there exists $B_1, \ldots, B_n$ in $\mathcal{B}(H)$ such that $I - \sum_{i=1}^{n} B_i T_i (= C)$ is a compact operator. Define $B: H^{(n)} \to H$ by $B(x_1, \ldots, x_n) = \sum_{i=1}^{n} B_i x_i (x_1, \ldots, x_n) \in H^{(n)}$. Then $B$ is bounded and $\overline{BT^{(n)}} = I - C$. Hence $N(\overline{BT^{(n)}}) (= N(BT^{(n)})$ is of finite dimension. Since $C$ is a compact operator, $BT^{(n)}$ is bounded below on $N(BT^{(n)})$ [4, Solution 140]. As $\|BT^{(n)}x\| \leq \|B\| \|T^{(n)}x\|$ for $x \in \mathcal{D}(T) \cap N(BT^{(n)})$, $T^{(n)}$ is bounded below on $\mathcal{D}(T) \cap N(BT^{(n)})$. Therefore $T^{(n)}(N(BT^{(n)}) \cap \mathcal{D}(T))$ is closed. Since $N(BT^{(n)})$ is of finite dimension and $T^{(n)}(N(BT^{(n)}) \cap \mathcal{D}(T)) \in T^{(n)}(\mathcal{D}(T))$, $\mathcal{H}(T^{(n)})$ is closed. Thus $T$ is j.u.F...
in $\mathcal{B}(H)$ such that $I - \sum_{i=1}^{n} B_i T_i$ is a compact operator, then

$$1 = \|x_k\| = \|\sum_{i=1}^{n} B_i T_i x_k + x_k - \sum_{i=1}^{n} B_i T_i x_k\| \leq \|(I - \sum_{i=1}^{n} B_i T_i)x_k\| + \sum_{i=1}^{n} \|B_i\| \|T_i x_k\| \to 0 \text{ as } k \to \infty,$$

which is absurd. Hence there exist no $B_1, \ldots, B_n \in \mathcal{B}(H)$ such that $I - \sum_{i=1}^{n} B_i T_i$ is a compact operator. Thus by Theorem 5, $T$ is not j.u.F..

Proof of (a) $\Rightarrow$ (f). Since $N(T^{(a)}) = N((T^{(a)^\#} T^{(a)})^{1/2})$, it is sufficient to show that if $\mathcal{R}(T^{(a)})$ is not closed, then $\mathcal{R}((T^{(a)^\#} T^{(a)})^{1/2})$ is not closed. If $\mathcal{R}((T^{(a)^\#} T^{(a)})^{1/2})$ is closed, then let $\{T^{(a)} x_k\}$ be a Cauchy sequence in $\mathcal{R}(T^{(a)})$. Then, since $\|T^{(a)} x_k\| = \|(T^{(a)^\#} T^{(a)})^{1/2} x_k\|$, $\{(T^{(a)^\#} T^{(a)})^{1/2} x_k\}$ is a Cauchy sequence in $\mathcal{R}((T^{(a)^\#} T^{(a)})^{1/2})$.

But $\mathcal{R}((T^{(a)^\#} T^{(a)})^{1/2})$ is closed, so $(T^{(a)^\#} T^{(a)})^{1/2} x_k \to (T^{(a)^\#} T^{(a)})^{1/2} x$ for some $x \in \mathcal{D}((T^{(a)^\#} T^{(a)})^{1/2}) = \mathcal{D}(T)$. Hence

$$\|T^{(a)}(x_k - x)\| = \|(T^{(a)^\#} T^{(a)})^{1/2} (x_k - x)\| \to 0$$

as $k \to \infty$. Therefore $\mathcal{R}(T^{(a)})$ is closed which is a contradiction.

Proof of (f) $\Rightarrow$ (d). Since $((T^{(a)^\#} T^{(a)})^{1/2})$ is not Fredholm by [3, Theorem 1.1], there exists an infinite dimensional projection $P$ such that $PH \subset \mathcal{D}((T^{(a)^\#} T^{(a)})^{1/2}) = \mathcal{D}(T)$ and $(T^{(a)^\#} T^{(a)})^{1/2} P$ is a compact operator. Let $\{x_k\}$ be a bounded sequence weakly converging to zero. Then $(T^{(a)^\#} T^{(a)})^{1/2} PX_k \to 0$ (strongly) as $k \to \infty$. Therefore,

$$\|T_i P x_k\|^2 \leq \sum_{j=1}^{n} \|T_j P x_k\|^2 = \sum_{j=1}^{n} (T_j P x_k, T_j P x_k) = (T^{(a)^\#} P x_k, T^{(a)^\#} P x_k) = \|(T^{(a)^\#} T^{(a)})^{1/2} P x_k\|^2 \to 0$$

as $k \to \infty$, for $i = 1, \ldots, n$. Thus $T_i P$ is compact for $i = 1, \ldots, n$.

Since $\mathcal{D}((T^{(a)^\#} T^{(a)})^{1/2}) = \mathcal{D}(T)$ and $\|(T^{(a)^\#} T^{(a)})^{1/2} x\|^2 = \|T^{(a)} x\|^2 = \sum_{i=1}^{n} \|T_i x\|^2$ for $x \in \mathcal{D}(T)$, the equivalence of (e) and (f) follows from [3, Theorem 1.1].

**Corollary 8.** If $T = (T_1, \ldots, T_n)$ is an n-tuple of normal operators with the same domain $\mathcal{D}$ in $H$, then $\text{Sp}_c(T) = \text{Sp}_e(T)$.

**Proof.** Since $T_i - z_i I$ is normal, $\|(T_i - z_i I) x\| = \|(T_i - z_i I)^* x\|$ for $x \in \mathcal{D}(T) = \mathcal{D}(T^*)$ for $i = 1, \ldots, n$. The equivalence of (a) and (b) in the last theorem, yields $\text{Sp}_c(T) = \text{Sp}_e(T)$. Hence $\text{Sp}_c(T) = \text{Sp}_e(T)$.

**Weyl's theorem.** In this section we prove a Weyl-type theorem. To this end we shall need some lemmas. Let $T = (T_1, \ldots, T_n)$ be an n-tuple of closed operators with the same dense domain $\mathcal{D}(T)$ in $H$. We say that $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ is a joint eigenvalue of $T$ if there exists a nonzero vector $x$ in $\mathcal{D}(T)$ such that $(T_i - z_i I)x = 0$ for $i = 1, \ldots, n$. The multiplicity of $z$ is the dimension of $\bigcap_{i=1}^{n} N(T_i - z_i I)$. 601
Lemma 9. Let \( T = (T_1, \ldots, T_n) \) be an \( n \)-tuple of pairwise commuting normal operators with the same domain \( \mathcal{D}(T) \) in \( H \). Let \( z = (z_1, \ldots, z_n) \) be an isolated point of the joint spectrum \( \text{Sp}(T) \) of \( T \) (see Definition 1). Then \( z \) is a joint eigenvalue of \( T \).

Proof. Since \( z \) is an isolated point, \( \chi_z \), the characteristic function of \( \{ z \} \), is a non-zero element in \( m(\text{Sp}(T)) \), the algebra of all equivalence classes (with respect to the equality almost everywhere) of Borel functions on \( \text{Sp}(T) \). By the joint spectral theorem [6, Theorem 2.2], \( \chi_z(T) = P_z \) is a non-zero projection \( H \) and \( T_i P_z - z_i P_z = 0 \) for each \( i = 1, \ldots, n \). Hence \( z \) is a joint eigenvalue of \( T \).

Lemma 10. Let \( S = (S_1, \ldots, S_n) \) be an \( n \)-tuple of closed operators with the same dense domain \( \mathcal{D}(S) \) in a Hilbert space \( H_1 \) and let \( T = (T_1, \ldots, T_n) \) be an \( n \)-tuple of closed operators with the same dense domain \( \mathcal{D}(T) \) in a Hilbert space \( H_2 \). Then \( \text{Sp}(S \oplus T) = \text{Sp}(S) \cup \text{Sp}(T) \), where \( S \oplus T = (S_1 \oplus T_1, \ldots, S_n \oplus T_n) \).

Proof. It is sufficient to show that \( \text{Sp}_1(S \oplus T) = \text{Sp}_1(S) \cup \text{Sp}_1(T) \). Let \( z = (z_1, \ldots, z_n) \notin \text{Sp}_1(S) \cup \text{Sp}_1(T) \). Then there exist \( B_1, \ldots, B_n \in \mathcal{B}(H_1) \) and \( C_1, \ldots, C_n \in \mathcal{B}(H_2) \) such that \( \sum_{i=1}^{n} B_i(S_i - z_i I_{H_1}) \subset I_{H_1} \) and \( \sum_{i=1}^{n} C_i(T_i - z_i I_{H_2}) \subset I_{H_2} \). Thus \( \sum_{i=1}^{n} B_i \oplus C_i(S_i \oplus T_i - z_i I) \subset I \). Hence \( z \notin \text{Sp}_1(S \oplus T) \). Conversely, if \( z \in \text{Sp}_1(S) \cup \text{Sp}_1(T) \), then \( z \in \text{Sp}_1(S) \) or \( z \in \text{Sp}_1(T) \). Without loss of generality assume that \( z \in \text{Sp}_1(S) \). Then there exists a sequence \( \{ x_k \} \) of unit vectors in \( \mathcal{D}(S) \) such that \( (S_i - z_i I_{H_1}) x_k \to 0 \) for \( i = 1, \ldots, n \). Let \( y_k = x_k \oplus 0 \). Then \( \| y_k \| = 1 \) and \( (S_1 \oplus T_1 - z_i I) y_k = (S_i - z_i I_{H_2}) x_k \to 0 \) as \( k \to \infty \) for \( i = 1, \ldots, n \). Hence \( z \in \text{Sp}_1(S \oplus T) \). Thus \( \text{Sp}_1(S \oplus T) = \text{Sp}_1(S) \cup \text{Sp}_1(T) \).

Theorem 11. Let \( T = (T_1, \ldots, T_n) \) be an \( n \)-tuple of pairwise commuting normal operators with the same domain \( \mathcal{D}(T) \) in \( H \). Then \( \text{Sp}_n(T) \) consists precisely of all points in \( \text{Sp}(T) \) except the isolated joint eigenvalues of finite multiplicity.

Proof. Since \( \text{Sp}_n(T) = \text{Sp}(T) \) by Corollary 8, it is sufficient to show that \( (0, \ldots, 0) \) is an isolated joint eigenvalue of \( T \) of finite multiplicity if and only if \( T \) is j.u.F. and \( (0, \ldots, 0) \in \text{Sp}(T) \).

As \( T_i \)'s are pairwise commuting and normal, \( N(T^{(n)}) = \bigcap_{i=1}^{n} N(T_i) \) is a reducing subspace for each \( T_i \). For each \( i \) define \( S_i : N(T^{(n)})^\perp \cap \mathcal{D}(T) \to N(T^{(n)})^\perp \cap \mathcal{D}(T) \) by \( S_i x = T_i x \) \( (x \in N(T^{(n)})^\perp \cap \mathcal{D}(T)) \). Then \( T_i = 0 \oplus S_i \), the null space of \( S^{(n)} \) is \( \{ 0 \} \) and \( S_i \)'s are pairwise commuting normal operators in \( N(T^{(n)})^\perp \). Also by Lemma 10, \( \text{Sp}(T) = \{ 0 \} \cup \text{Sp}(S) \) (where \( S = (S_1, \ldots, S_n) \)).

Now assume that \( (0, \ldots, 0) \) is an isolated joint eigenvalue of \( T \) of finite multiplicity. Since \( N(S^{(n)}) = \{ 0 \} \), by Lemma 9, \( (0, \ldots, 0) \notin \text{Sp}(S) \). Hence \( \mathcal{R}(T^{(n)}) = \mathcal{R}(S^{(n)}) \) is a closed subspace of \( H^{(n)} \). As \( N(T^{(n)}) \) is of finite dimensions, \( T \) is j.u.F.\:.
Conversely, assume that $T$ is j.u.F. and $(0, \ldots, 0) \in \text{Sp}(T)$. Then $\mathcal{R}(T^{(n)})$ is closed and $N(T^{(n)})$ is finite dimensional. Since $N(S^{(n)}) = \{0\}$ and $\mathcal{R}(S^{(n)}) = \mathcal{R}(T^{(n)})$, which is closed, $S^{(n)}$ is bounded below. So $(0, \ldots, 0) \notin \text{Sp}(S) = \text{Sp}(S)$. Hence $(0, \ldots, 0)$ is an isolated point of $\text{Sp}(T)$. As the dimension of $N(T^{(n)})$ is finite, by Lemma 9, $(0, \ldots, 0)$ is an isolated joint eigenvalue of $T$ of finite multiplicity.

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**References**


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