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GLOBALS OF UNARY ALGEBRAS

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The *global* $P(X, f)$ of a universal algebra (X, F) is the family of all subsets of X with operations given by

$$f(X_1, \ldots, X_n) = \{f(x_1, \ldots, x_n);\; x_i \in X_i\}$$

whenever f is an n-ary operation in F and $X_1, \ldots, X_n$ are subsets of X.

We extend this definition to partial algebras as well. Unlike some other authors we do not exclude the void subset from the definition. (The difference is only formal, but useful when dealing with partial algebras.)

A class K of (partial) algebras is said to be *globally determined*, if any two members of K having isomorphic globals are themselves isomorphic.

E.g., some classes of finite semigroups have been shown to be globally determined (see [2], [3], [5–9], [11–14]); the problem concerning the class of all finite semigroups seems to be open, though.

In this paper we will show that all finite (partial) monounary algebras (i.e. unary algebras with only one operation) constitute a globally determined class. The proof is constructive, and could be used as a basis for an algorithm deciding if a finite (partial) monounary algebra is the global of any (partial) monounary algebra and determining that algebra up to isomorphism.

On the other hand, an example of two non-isomorphic infinite monounary algebras is given.

Since we shall deal only with monounary algebras, the prefix mono- will be almost always omitted.

A monounary algebra (X, f) is called *connected* if for each pair $x, y \in X$ there are natural numbers m, n with $f^n(x) = f^m(y)$. Each unary algebra is a disjoint union of its maximal connected subalgebras – its *components*. If (X, f) and (Y, g) are unary algebras and $\{X_i;\; i \in I\}$, $\{Y_j;\; j \in J\}$ are the sets of all components of (X, f) and (Y, g), then (X, f) is isomorphic to (Y, g) if and only if there is a bijection $h: I \to J$ such that X_i (as a subalgebra of (X, f)) is isomorphic to $Y_{h(i)}$ (as a subalgebra of (Y, g)). An element $x \in X$ is a *cyclic point* of (X, f) if $f^n(x) = x$ for some $n > 0$; denote $r(x) = \min\{n > 0;\; f^n(x) = x\}$. For a non-cyclic point x, $r(x)$ is not defined. If (X, f) is a finite non-empty unary algebra, then for each component C of (X, f)

there is a cyclic point in C. If C is a component of a unary algebra and $x, y \in C$ are cyclic points, then $r(x) = r(y)$; therefore we can put $r(C) = r(x)$. For a unary algebra $\mathscr{X} = (X, f)$ denote by $\mathscr{A}(\mathscr{X})$ the sequence $(a_n; n \in \mathbb{N})$ where a_n is the number of components C of (X, f) with $r(C) = n$ and $\mathbb{N} = 1, 2, 3, \ldots$ is the set of all positive integers. It is called the *cyclic type of* (X, f). The following result is evident:

Proposition 1. *Let (X, f), (Y, g) be unary algebras such that f and g are bijections and each component of them contains a cyclic point. Then (X, f) and (Y, g) are isomorphic if and only if they have the same cyclic type.*

If (X, f) is a finite unary algebra, then for every $x \in X$, $h(x) = \min\{n \geqq 0; f^n(x)$ is cyclic$\}$ is the *height* of x; it is defined for each $x \in X$.

By a routine calculation we get

Proposition 2. *Let (X, f) be a finite unary algebra, then $U \subset X$ is a cyclic point of $P(X, f)$ if and only if each $x \in U$ is a cyclic point of (X, f). For $\emptyset \neq U \subset X$, $h(U) = \max\{h(x); x \in X\}$. If $U \subset X$ is a cyclic point of $P(X, f)$, then $r(U)$ equals the least common multiple of $p_U(x)$, $x \in U$, where $p_U(x) = \min\{n \in \mathbb{N};$ for every $c \in \mathbb{N}, f^{cn}(x) \in U\}$.*

We easily get

Proposition 3. *Let (X, f) be a finite unary algebra, then the cyclic type of (X, f) can be evaluated from the cyclic type of $P(X, f)$.*

Proof. Let (a_n) be the cyclic type of (X, f) and (b_n) the cyclic type of $P(X, f)$. By Proposition 2, $f(U) = U$ holds for any $U \subset X$ if and only if for each component C of (X, f) the set $C \cap U$ is either empty or contains all cyclic points of C. Hence $b_1 = 2^{\Sigma a_n}$, i.e. the number of components of (X, f) is determined by the number of fixed points of $P(X, f)$. Since the cyclic type of $P(X, f^p)$, $p \in \mathbb{N}$ is derived from (b_n), we can get the number c_n of components of (X, f^n) for $n \in \mathbb{N}$. We have $c_n = t_1 + (n - 1)\, t_n$, where t_n is the number of components C of (X, f) with n dividing $r(C)$. Furthermore $t_n = \sum a_{kn}$, $k \in \mathbb{N}$. Therefore (b_n) determines (c_n), which determines (t_n), which determines (a_n).

Example 4. There are two non-isomorphic infinite algebras (X, f) and (Y, g) such that

a) f and g are bijections,
b) all points of (X, f) and (Y, g) are cyclic,
and $P(X, f)$ and $P(Y, g)$ are isomorphic.

Proof. Let X be the set of all non-negative integers, Y the set of all positive integers. For a positive integer i put $f(2i - 1) = g(2i - 1) = 2i$, $f(2i) = g(2i) = 2i - 1$ and $f(0) = 0$. If (a_n) is the cyclic type of (X, f) and (b_n) is the cyclic type of (Y, g), then $a_1 = 1$, $a_2 = \aleph_0 = b_2$ and $a_i = b_i = 0 = b_1$ for each $i > 2$. Clearly (X, f) and (Y, g) fulfil a) and b). On the other hand $P(X, f)$ and $P(Y, g)$ contain $2^{\aleph_0}$ fixed points,

$2^{\aleph_0}$ cycles of length 2 and no cycle of length higher than 2 – thus $P(X,f)$ and $P(Y,g)$ are isomorphic.

Let (X,f) be a partial unary algebra. For $x \in X$, the *nest* of x is the partial unary algebra (Y,g) where $Y = \{z \in X$; there is $n \geqq 0$ such that $f^n(z) = x\}$ and $g(z) = f(z)$ whenever $f(z) \in Y$. Thus $g(x)$ is undefined for non-cyclic x and the nest of a cyclic point coincides with its component. Further, for $x_1, x_2 \in X$ define $x_1 \leqq x_2$ if and only if the nest of x_1 can be embedded into the nest of x_2.

Lemma 5. *Let (X,f) be a (partial) unary algebra and let Z be a subset of X which contains at least one non-cyclic point. Denote by (Y,g) the nest of Z in $P(X,f)$ and by (Y_z, g_z) the nest of $\{z\}$ in $P(X,f)$, for every $z \in Z$. Then (Y,g) is isomorphic to the component of $\Pi\{(Y_z, g_z);\ z \in Z\}$ containing the element $(x_z;\ z \in Z)$, where $x_z = \{z\}$ for each $z \in Z$.*

Proof. Denote by C the described component. If $(y_z;\ z \in Z) \in \Pi(Y_z, g_z)$ then $(y_z;\ z \in Z) \in C$ if and only if for some $n \geqq 0, f^n(y_z) = x_z$ for each $z \in Z$. On the other hand, if $V \in P(X,f)$ then $V \in (Y,g)$ if and only if there exist $m \geqq 0$ and a decomposition $\{V_z;\ z \in Z\}$ of the set V such that $f^m(V_z) = x_z$ for each $z \in Z$. Because Y is non-cyclic in $P(X,f)$, m is uniquely determined by the height of V, $m = h(V) - h(Y)$, and therefore the decomposition $\{V_z;\ z \in Z\}$ is unique, too. Hence by mapping $(y_z;\ z \in C)$ onto $\bigcup_{z \in Z} y_z$ we get the required isomorphism.

If (X,f) is a (partial) unary algebra, $x \in X$, then $s(x) = \max\{n$; for some $y \in X$, $f^n(y) = x\}$; thus $s(x) = \infty$ for x cyclic.

Lemma 6. *If (X,f) is a (partial) unary algebra, $V \subset Z \subset X$ are not cyclic and $s(V) \leqq s(Z)$ in $P(X,f)$, then the nest of V in $P(X,f)$ can be embedded into the nest of Z in $P(X,f)$.*

Proof. For U in the nest of V in $P(X,f)$ with $f^n(U) = V$ we map U onto $f^{-n}(Z \setminus V) \cup U$ – we directly verify that this is the required embedding.

Proposition 7. *Let (X,f) be a finite unary algebra, $x \in X$ and (Y,g) the nest of a unary algebra. If (Y,g) is isomorphic to the nest of $\{x\}$ in $P(X,f)$, then we can determine the nest of x in (X,f) by (Y,g).*

Proof. Let $y_1, \ldots, y_n$ be all different points $y \in X$ such that $f(y) = x$. The idea of the proof lies in finding n different points $v_1, \ldots, v_n \in Y$ such that $g(v_i) = u$ for $1 \leqq i \leqq n$ and the nest of v_i (in (Y,g)) is isomorphic to the nest of $\{y_i\}$ (in $P(X,f)$). Then we get our statement for x non-cyclic by induction over $s(x)$ because for $s(x) = 0$ it is clearly true. If x is cyclic, we denote by y_1 and v_1 those cylic points for which $f(y_1) = x$, $g(v_1) = u$. The nest of a cyclic point coincides with its component, therefore the nest of v_1 is isomorphic to the nest of $\{y_1\}$. The points $y_2, \ldots, y_n$ are non-cyclic and hence we can determine their nests. By doing so with all cyclic points we determine the whole component.

To find the points $v_1, \ldots, v_n$ we again use induction. Let us suppose that points

$y_1, \dots, y_k$ and $v_1, \dots, v_k$ $(0 \leqq k \leqq n)$ have already been determined in such a way that $s(w_1) \geqq s(w_2)$ for any $w_1, w_2 \in Y$ with $g(w_1) = u = g(w_2)$, $w_1 \in V_k$, $w_2 \notin V_k$, where for every non-void subset $A \subseteq \{1, \dots, k\}$, V_k contains a different element $v_A \in Y$ such that

a) $f(v_A) = u$,

b) the nest of v_A in (Y, g) is isomorphic to the nest of $(v_i;\ i \in A)$ in $(Y, g)^A$.

The existence of the set V_k follows from Lemma 5. We can assume $v_i \in V_k$ for $1 \leqq \leqq i \leqq k$, of course. Denote by W_k the set of $w \in Y$ with $f(w) = u$, $w \notin V_k$. If x is cyclic, we choose y_1 and v_1 as above. Hence we can assume that W_k contains only non-cyclic points. If $W_k = \emptyset$, then $k = n$. Let $k < n$ and let $p_k = \max\{s(w);\ w \in W_k\}$, $P_k = \{w \in W_k;\ s(w) = p_k\}$. We choose v_{k+1} as a minimal element of P_k with respect to $\leqq$. By Lemma 6 there is $y_{k+1} \in X$ with $f(y_{k+1}) = x$, such that the nest of v_{k+1} is isomorphic to the nest of $\{y_{k+1}\}$. The proof is complete.

If (X, f) is a finite unary algebra and C a component of (X, f), let $p(C)$ be the number of non-cyclic points of the component C.

Lemma 8. *Let (X, f) be a connected finite unary algebra, $x \in X$ a cyclic point and A the component of $P(X, f)$ containing $\{x\}$. If B is a component of $P(X, f)$, B distinct from the component of the void subset, then $p(A) \leqq p(B)$ and $r(A) \geqq r(B)$. If $p(A) = p(B)$ and $r(A) = r(B)$, then A and B are isomorphic.*

Proof. Let $Z \in B$ be cyclic with $x \in Z$. Denote by C the set of all cyclic points of B. For $U \in A$ with $f^n(U) = \{x\}$ we map U onto $U \cup (f^{-n}(Z) \cap C)$ – this mapping is injective on non-cyclic points and defines an embedding if $r(A) = r(B)$. The inequality $r(A) \geqq r(B)$ comes from Proposition 2.

Combining this lemma with Proposition 7 we get

Corollary 9. *If (X, f), (Y, g) are connected finite unary algebras, then (X, f) and (Y, g) are isomorphic if and only if $P(X, f)$ and $P(Y, g)$ are isomorphic.*

Lemma 10. *If (X_1, f_1) and (X_2, f_2) are partial unary algebras, $X_1 \cap X_2 = \emptyset$ and (X, f) is their joint* (i.e. $X = X_1 \cup X_2, f = f_1 \cup f_2$), *then $P(X, f)$ is isomorphic to $P(X_1, f_1) \times P(X_2, f_2)$.*

Proof. Easy.

Lemma 11. *Let (X_1, f_1) and (X_2, f_2) be connected finite unary algebras and Y a component of $(X_1, f_1) \times (X_2, f_2)$. For $j = 1, 2$ let $p_j = p(X_j)$, $r_j = r(X_j)$, $r = r_j s_j$, where r is the least common multiple of r_1, r_2. Then $p(Y) \geqq p_1 s_1 + p_2 s_2$.*

Proof. Easy.

Let (X, f) be a unary algebra, C one of its components with $x \in C$ cyclic and A a component of $P(X, f)$. We shall say that C *participates* in A if there is $x \in Z$ for some $Z \in A$.

Lemma 12. *Let (X,f) be a finite unary algebra, C its component with $x \in C$ cyclic and A the component of $P(X,f)$ containing $\{x\}$. Let C participate in a component B of $P(X,f)$. Then $p(A) \leqq p(B)$ and if $p(A) = p(B)$, then $r(A) \geqq r(B)$. If $p(A) = p(B)$ and $r(A) = r(B)$, then A and B are isomorphic.*

Proof. Use Lemmas 8, 10 and 11.

Theorem 13. *Let (X,f) be a finite unary algebra, then it can be determined by $P(X,f)$.*

Proof. Let (Y, g) be a subalgebra of $P(X,f)$ consisting of all components A with $p(A) = 0$. By Proposition 3 we can determine all components C of (X,f) with $p(C) = 0$. Let $\mathscr{P}$ be a class of all components $A \not\subset Y$ of $P(X,f)$ with $p(A)$ minimal. Choose $B \in \mathscr{P}$ with maximal $r(B)$. By Lemma 12 there is a cyclic $x \in X$ such that B is isomorphic to the nest of $\{x\}$ in $P(X,f)$ and thus, by Proposition 7, we can determine a certain component of (X,f). Now we can enlarge the subalgebra (Y,f) and repeat the process until (Y,f) coincides with $P(X,f)$. Note that Proposition 3 is not needed if $p(C) > 0$ for all components C of (X,f).

We new proceed to generalize our result for partial unary algebras.

First, let (X,f) be a unary algebra and let $x \in X$. The *prolonged nest* of x is derived from the nest (Y, g) of x by adding the element $f(x)$; more exactly, it is the partial subalgebra $(Y \cup \{f(x)\}, h)$ of (X,f) where $h(z) = g(z)$ for any $z \in Y \setminus \{x\}$, $h(x) = f(x)$, and $h(f(x))$ is defined if and only if $f(x) \in Y$. Thus $h(f(x))$ is undefined for non-cyclic x and the prolonged nest of a cyclic point coincides with its nest.

Now, let (X,f) be a partial unary algebra. Fot any $A \subset X$ we define $o(A) \subset A$ to be the set of all points $x \in A$ where f vanishes, i.e. where $f(x)$ is not defined. Note that $o(A) = \emptyset$ whenever A is cyclic in $P(X,f)$. Let A be non-cyclic. The *envelope* of A is a partial subalgebra (Y, g) of $P(X,f)$ where $f(A) \in Y$ and for $f(A) \neq C \subset X$ the inclusion $C \in Y$ holds if and only if $o(f^{n-1}(C)) \supset o(A)$ and $f^{n-1}(C) \setminus o(f^{n-1}(C)) = A \setminus o(A)$ for some $n > 0$. (Then obviously $f^n(C) = f(A)$.) Further, $g(C) = f(C)$ for $f(A) \neq C \in Y$ and $g(f(A))$ is undefined. (Loosely speaking, the envelope of A is the greatest subalgebra of the nest of $f(A)$, whose members of the height 1 contain in addition to A only vanishing elements. In other words, it is the union of prolonged nests of $A \cup W$, with $\emptyset \subset W \subset o(X)$.)

If (X,f) is finite and contains no cyclic points, then for any $A \subset X$ there evidently exists $n \geqq 0$ such that $f^n(A) = \emptyset$, i.e. $P(X,f)$ is connected and the empty set is its only cyclic point. First we shall discuss such partial unary algebras.

Lemma 14. *Let (X,f) be a finite partial unary algebra with no cyclic point and let A be a subset of X with $o(A) = \emptyset$, $f(A) = \{x\}$ for some $x \in X$. Denote by (Y, g) the prolonged nest of A. Then the envelope of A is isomorphic to a component of $(Y, g) \times P(X,f)$ containing the element $(\{x\}, \emptyset)$.*

Proof. Denote by Z the above described component of $(Y, g) \times P(X,f)$. For

$B, C \subset X$ we have $(B, C) \in Z \setminus \{(x_,), \emptyset)\}$ if and only if $f^n(B) = A$ and $f^{n+1}(C) = \emptyset$ for some $n \geqq 0$. On the other hand, if $D \neq \{x\}$ belongs to the envelope of A, there must exist a decomposition $D = B \cup C$ and $n \geqq 0$ with $f^n(B) = A$ and $f^n(C) \subset o(X)$. Because x is not cyclic and $A \cap o(X) = o(A) = \emptyset$, the number n and the decomposition are determined uniquely. Hence by mapping $(B, C) \in Z$ onto $B \cup C$ we get the required isomorphism.

The lemma enables us to determine the envelope of A from the knowledge of $P(X, f)$ and the nest of A. This is used in the proof of the following proposition.

Proposition 15. *Let (X, f) be a finite partial unary algebra with no cyclic point, $x \in X$ and let (Y, g) be the nest of an element $u \in Y$ of a partial algebra. If (Y, g) is isomorphic to the nest of $\{x\}$ in $P(X, f)$, then we can determine the nest of x in (X, f) by (Y, g) and $P(X, f)$.*

Proof. We shall proceed by induction over $s(x)$. For $s(x) = 0$ the statement clearly holds. Let $s = s(x) = s(u) > 0$. Let $y_1, \ldots, y_n$ be all distinct points $y \in X$ such that $f(y) = x$. The idea of the proof consists in finding n distinct points $v_1, \ldots, v_n \in Y$ such that $g(v_i) = u$ for $1 \leqq i \leqq n$ and the nest of v_i (in (Y, g)) is isomorphic to the nest of $\{y_i\}$ (in $P(X, f)$).

Denote by P the set of all $w \in Y$ with $s(w) = s - 1$ and choose $v \in P$ minimal with respect to $\leqq$. By Lemma 6 there is $y \in X$ with $f(y) = x$ such that the nest of v is isomorphic to the nest of $\{y\}$. Put $v_1 = v$, $y_1 = y$. Using Lemma 14 we can determine a subalgebra of (Y, g) isomorphic to the envelope of $\{y_1\}$. Choosing $v \in Y$ (if there is any) outside this subalgebra with maximal $s(v)$ and minimal with respect to $\leqq$ we can determine v_2. Combining Lemma 6 and Lemma 14 we can use the same technique as that introduced in the proof of Proposition 7, and thus determine the nest of every element $y \in X$ with $f(y) = x$.

Corollary 16. *If (X, f) is a finite partial unary algebra with no cyclic point, then it can be determined by $P(X, f)$.*

Proof. (X, f) has no cyclic point if and only if $P(X, f)$ has exactly one cyclic point, namely, the empty set. For $A \subset X$ we have $f(A) = \emptyset$ if and only if $A \subset o(X)$. Using Lemma 6 we can recognize the nests of $\{x\}$ for all $x \in o(X)$. The rest of the proof comes from the preceding proposition.

We could join the techniques used in the proofs of Theorem 13 and Proposition 15 to get a self-contained, constructive proof of Theorem 18. However, since no new ideas would appear, we choose a shorter way, using the following lemma.

Lemma 17. *Let (X_1, f_1), (X_2, f_2), (Y, g) be finite unary algebras and let $g(y) = y$ for some $y \in Y$. Then, if $(X_1, f_1) \times (Y, g)$ and $(X_2, f_2) \times (Y, g)$ are isomorphic, (X_1, f_1) and (X_2, f_2) are isomorphic as well.*

Proof can be easily deduced from more general theorems of [1] or [4] or [10].

Theorem 18. *Let (X, f) be a finite partial unary algebra. Then it can be determined by $P(X, f)$.*

Proof. The partial algebra (X, f) can be evidently decomposed in a unique way into two partial algebras (Y, g), (X_1, f_1), $X = X_1 \cup Y$, $f = g \cup f_1$, such that (Y, g) has no cyclic point and (X_1, f_1) is a unary algebra. For any $A \subset X$, $B \subset Y$ the sets A and $A \cup B$ belong to the same component of $P(X, f)$. Hence $P(Y, g)$ can be determined from $P(X, f)$ as isomorphic to (any) component with the smallest number of points. By Lemma 10, $P(X, f)$ is isomorphic to $P(Y, g) \times P(X_1, f_1)$ and therefore we can determine $P(X_1, f_1)$ from $P(X, f)$ by Lemma 17. By applying Proposition 16 to $P(Y, g)$ and Theorem 13 to $P(X_1, f_1)$ we conclude the proof.

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