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A NON ABSOLUTELY CONVERGENT INTEGRAL
WHICH ADMITS TRANSFORMATION AND CAN BE USED
FOR INTEGRATION ON MANIFOLDS

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"Which is the most profound and most difficult mathematical
theorem that admits a concrete and unquestionable physical
interpretation?
For me, Stokes' theorem is the number one candidate ..."


0. INTRODUCTION

J. Mawhin in [1] introduced the notion of the generalized Perron integral (GP-integral) in the n-dimensional Euclidean space $\mathbb{R}^n$, using a Riemann-type definition but restricting the class of "admissible" partitions of the domain of integration. He proved that the divergence theorem holds for the GP-integral for any differentiable function provided the integration domain is an interval. Modifying his results a little, we can assert the following two properties of the GP-integral:

1. Let $L = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$, $L_1 = [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^{n-1}$, let $g: L \to \mathbb{R}^1$ be differentiable on $L$. Then $\frac{\partial g}{\partial x_1}$ is GP-integrable on $L$ and

$$\left(\text{GP}\right) \int_L \frac{\partial g}{\partial x_1} \, dx = \int_{L_1} \left[ g(b_1, x_2, \ldots, x_n) - g(a_1, x_2, \ldots, x_n) \right] \, dx_2 \ldots dx_n .$$

2. If, moreover, $\omega = g \, dx_2 \wedge \cdots \wedge dx_n$ then

$$\left(\text{GP}\right) \int_L \omega = \int_{\partial L} \omega ,$$

where $\partial L$ stands for the boundary of $L$.

On the other hand, the GP-integral has some not so nice properties. Firstly, it is not additive with respect to the integration domain, in the following sense:
If $L^1, L^2 \subset \mathbb{R}^n$, $n \geq 2$, are compact non-overlapping intervals, $L = L^1 \cup L^2$ an interval, $f: L \to \mathbb{R}^1$ GP-integrable on $L$, $i = 1, 2$, then $f$ need not be GP-integrable on $L$ (for a counterexample, see [2]).

Secondly, the GP-integral (like the Perron integral) strongly depends on the coordinate system, so that no comprehensive transformation theorem is available. For example, if $f$ is GP-integrable and $\varphi$ is a rotation of the coordinate system, then $f \circ \varphi = h$ need not be GP-integrable. (Cf. [3], 11.4 with $f = \zeta$.)

The first disadvantage was removed in [2] by modifying the class of admissible partitions (moreover, a dominated convergence theorem was established). The other drawback was dealt with in [4] for $n = 2$.

In the present paper we give a definition of an integral in $\mathbb{R}^n$ based on partition of unity. We shall establish a transformation theorem and a divergence theorem. Further, we shall prove that the integral is an extension of the Lebesgue integral. Since the divergence theorem implies that partial derivatives of differentiable functions are integrable, our integral exists for some non absolutely integrable functions, thus being a true extension of the Lebesgue integral. As concerns the relationship between our and the Perron integral, let us mention that a partial derivative of a differentiable function need not be Perron integrable, while the function $\zeta$ from [3], 11.4 is Perron integrable but our integral of $\zeta$ does not exist.

1. PU-PARTITIONS

Let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space, $\mathbb{R}^1 = \mathbb{R}$, $\mathbb{R}^+ = (0, +\infty)$. For $M \subset \mathbb{R}^n$, the symbols $\overline{M}$, $\text{Int} M$, $\partial M$ stand for the closure, interior and boundary of $M$, respectively. If $M$ is Lebesgue measurable, then $m_n(M)$ denotes its $n$-dimensional measure (the index is omitted if there is no danger of misunderstanding). If $y \in \mathbb{R}^n$, $r > 0$, we denote

$$
diam M = \sup \{ \| y - x \| ; \ x, y \in M \},$$

$$
dist (y, M) = \inf \{ \| y - x \| ; \ x \in M \},$$

$$
B(y, r) = \{ x \in \mathbb{R}^n ; \ \| y - x \| < r \},$$

$$
\Omega(M, r) = \{ x \in \mathbb{R}^n ; \ dist (x, M) < r \}.
$$

For a function $f: M \to \mathbb{R}$ we denote

$$
\text{supp} f = \overline{\{ x \in \mathbb{R}^n ; \ f(x) \neq 0 \}}
$$

(the support of $f$). Any function $\delta: M \to \mathbb{R}^+$ is called a gauge (on $M$).

**Definition 1.1.** Let $M \subset \mathbb{R}^n$ be bounded. A family

$$
\Delta = \{ (i^j, \delta_j) ; \ j = 1, \ldots, k \},
$$

where $k$ is a positive integer, $i^j \in M$, $\delta_j: \mathbb{R}^n \to [0, 1]$ are $C^1$ functions with compact
supports,

\[ 0 \leq \sum_{j=1}^{k} g_j(x) \leq 1 \quad \text{for} \quad x \in \mathbb{R}^n, \]

\[ \text{Int} \{ x \in \mathbb{R}^n; \sum_{j=1}^{k} g_j(x) = 1 \} \supset \text{Cl} M, \]

is called a PU-partition of M (the letters PU stand for “partition of unity”).

If \( \delta \) is a gauge on M, then a PU-partition (1.1) of M is said to be \( \delta \)-fine if

\[ \text{supp} \ g_j \subset B(t^j, \delta(t^j)), \quad j = 1, \ldots, k. \]

For any PU-partition (1.1) of M we define

\[ \Sigma(A) = \sum_{j=1}^{k} \int_{\mathbb{R}^n} \left\| x - t^j \right\| \sum_{i=1}^{n} \left| \frac{\partial g_j}{\partial x_i} \right| \, dx. \]

(Notice that in fact we integrate over supp \( g_j \). In similar situations we shall often omit the index indicating the integration domain.)

**Proposition 1.1.** Let \( M \subset \mathbb{R}^n \) be compact. Then there exists \( K > 0 \) such that for every gauge \( \delta \) on M there is a \( \delta \)-fine PU-partition \( A \) of M with \( \Sigma(A) \leq K. \)

**Proof.** Let us introduce an auxiliary function \( \omega: \mathbb{R} \rightarrow \mathbb{R} \) of class \( C^\infty \) such that \( \omega(s) = 0 \) for \( s \leq -1, \omega(s) = 1 \) for \( s \geq 1, \omega(s) + \omega(-s) + \omega(s) = 1 \) for \( s \in \mathbb{R}. \) Set

\[ h_{a,b,\sigma}(s) = \omega \left( \frac{s-a}{\sigma} \right) \omega \left( \frac{b-s}{\sigma} \right) \]

provided \( a < b. \) If \( J = [a_1, b_1] \times \ldots \times [a_n, b_n] \) is a compact \( n \)-dimensional interval, put

\[ \chi_{J,\sigma}(x) = \prod_{i=1}^{n} h_{a_i,b_i,\sigma}(x_i), \]

where \( x \in \mathbb{R}^n, \) \( x = (x_1, \ldots, x_n). \) Then we have, for \( t \in J: \)

\[ \lim_{\sigma \rightarrow 0+} \sum_{i=1}^{n} \int_{\mathbb{R}^n} \left\| x - t \right\| \left| \frac{\partial \chi_{J,\sigma}}{\partial x_i} \right| \, dx = \int_{\partial J} \left\| x - t \right\| dS_{n-1} \leq \text{diam} J \sum_{n-1}^{\delta J} \leq \text{diam} J \text{m}_{n-1}(\partial J). \]

Let us proceed to the proof proper of our proposition. Let \( I \subset \mathbb{R}^n \) be a compact interval, \( M \subset \text{Int} I. \) Given a gauge \( \delta \) on M, we can put \( \delta(x) = \frac{1}{2} \text{dist}(x, M) \) for \( x \in I \setminus M, \) thus extending \( \delta \) to the whole of I. It was shown in [2] that there exists \( K_0 > 0 \) (generally depending on I) such that for every gauge \( \delta_0 \) on I there is a \( \delta_0 \)-fine P-partition \( \Pi \) (i.e. a family \( \Pi = \{(t^j, J^j); j = 1, \ldots, k\} \) such that \( J^j \) are compact intervals, \( t^j \in J^j, J^j \subset B(t^j, \delta_0(t^j)) \) with \( \Sigma_0(\Pi) = \sum_{j=1}^{k} \text{diam} J^j \text{m}_{n-1}(\partial J^j) \leq K_0. \) (Such a P-partition is constructed by repeatedly halving the edges of I, thus obtaining — after a finite number of steps — subintervals with the required properties. For a more precise result see Lemma 6.1 below.)

118
For the given gauge $\delta$ (extended to $I$ in the way mentioned above) construct a $(\delta/2)$-fine P-partition $\Pi = \{(t^j, J^j); j = 1, \ldots, k\}$ with $\Sigma_0(\Pi) \leq K_0$. Then

$$\Delta = \{(t^j, \chi_{J^j, n}); j = 1, \ldots, k, t^j \in M\}$$

is a PU-partition of $M$ if $\sigma$ is sufficiently small; moreover, it is $\delta$-fine provided $\sigma \leq \frac{1}{2} \min \{\delta(t^j); j = 1, \ldots, k\}$. If we take $\sigma$ so small that

$$\sum_{i=1}^n \int_{\mathbb{R}^n} \|x - t^j\| \frac{\partial \chi_{J^j, n}}{\partial x_i} \, dx \leq \text{diam } J^j m^{-1}(\partial J^j) + \frac{1}{k},$$

we obtain

$$\Sigma(\Delta) \leq \sum_{j=1}^k \left[ \text{diam } J^j m^{-1}(\partial J^j) + \frac{1}{k} \right] \leq K_0 + 1.$$ 

Moreover, the PU-partition $\Delta$ fulfills $\sum_{j=1}^k \chi_{J^j, n}(x) \leq 1$ for $x \in \mathbb{R}^n$, which proves our proposition.

Remark 1.1. It is clear from the above proof that the constant $K$ from the proposition can be chosen the same for all sets $M$ which are subsets of a fixed compact interval.

2. DEFINITION AND ELEMENTARY PROPERTIES OF THE PU-INTEGRAL

Definition 2.1. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a function with compact support. For a PU-partition $\Delta$ of $\text{supp } f$ defined by (1.1) put

(2.1) $$S(f, \Delta) = \sum_{j=1}^k f(t^j) \int_{\mathbb{R}^n} \vartheta_j(x) \, dx.$$ 

Let $\gamma \in \mathbb{R}$ satisfy the following condition:

for every $\varepsilon > 0$, $K > 0$ there is a gauge $\delta$ on $\text{supp } f$ such that

(2.2) $$\left| \gamma - S(f, \Delta) \right| \leq \varepsilon$$

provided $\Delta$ is a $\delta$-fine PU-partition of $\text{supp } f$ with $\Sigma(\Delta) \leq K$.

Then $f$ is said to be PU-integrable and we write

(2.3) $$\gamma = (\text{PU}) \int f(x) \, dx;$$

$\gamma$ is called the PU-integral of $f$.

Remark 2.1. The above definition is a modification of the definitions of the Riemann and Perron integrals (cf. [3]). If we delete the requirement on $\Sigma(\Delta)$ and also that on the smoothness of the functions $\vartheta_j$, then, taking $\vartheta_j$ to be the characteristic functions of (semiclosed) intervals $J^j = (a^j, b^j] = (a^j_1, b^j_1] \times \ldots \times (a^j_n, b^j_n]$ and assuming $t^j \in \text{Cl } J^j$, we obtain the Perron integral. If, moreover, we admit only
constant gauges $\delta$, we arrive at the Riemann integral. (Evidently, $S(f, \Delta) = \sum_{j=1}^{k} f(t^j) \omega(J^j)$.)

Let us notice that the PU-integral has the elementary properties usually required of the concept of an integral, namely, monotonicity and linearity. Monotonicity is immediately seen from the definition if we observe that for a nonnegative function the integral sums, and hence also the integral itself, are nonnegative. The property of linearity can be formulated as follows: If $f_i: \mathbb{R}^n \to \mathbb{R}$ are PU-integrable, $c_i \in \mathbb{R}$, $i = 1, 2$, then $f: x \mapsto c_1 f_1(x) + c_2 f_2(x)$ is PU-integrable and

$$\int f \, dx = c_1 \int f_1 \, dx + c_2 \int f_2 \, dx.$$ 

Indeed, it is evident that if $f_i$ is PU-integrable and $c_i \in \mathbb{R}$, then $c_i f_i$ is PU-integrable and

$$\int c_i f_i \, dx = c_i \int f_i \, dx.$$ 

The additivity of the PU-integral immediately follows from an alternative equivalent definition of the PU-integral:

**Definition 2.2.** Let $I$ be a compact interval in $\mathbb{R}^n$, let $f: \mathbb{R}^n \to \mathbb{R}$, supp$f \subset I$. Let $\gamma \in \mathbb{R}$ satisfy the following condition:

for every $\varepsilon > 0$, $K > 0$ there is a gauge $\delta$ on $I$ such that (2.2) holds provided $\Delta$ is a $\delta$-fine PU-partition of $I$ with $\Sigma(\Delta) \leq K$.

Then $f$ is said to be PUI-integrable, $\gamma$ is its PUI-integral and we write

$$\gamma = \int f \, dx.$$ 

**Theorem 2.1.** Let $f: \mathbb{R}^n \to \mathbb{R}$ have a compact support, let $I \subset \mathbb{R}^n$ be a compact interval, $I \supset \text{supp}\, f$. Then $f$ is PU-integrable if and only if it is PUI-integrable; the identity

$$\int f \, dx = \int f \, dx$$ 

holds provided one of the integrals exists.

**Proof.** 1. Let $f$ be PU-integrable, $\varepsilon > 0$, $K > 0$. Find a gauge $\delta$ on supp$f$ such that

$$|S(f, \Delta) - \int f \, dx| \leq \varepsilon$$ 

whenever $\Delta$ is a $\delta$-fine PU-partition of supp$f$ with $\Sigma(\Delta) \leq K$. Let $\delta_0$ be a gauge on $I$, satisfying

(i) $\delta_0(x) = \delta(x)$ for $x \in \text{supp}\, f$;
(ii) $\delta_0(x) = \frac{1}{2} \text{dist}(x, \text{supp } f)$ for $x \in I \setminus \text{supp } f$. Let

$$A_0 = \{(\tau^j, \zeta_j); j = 1, \ldots, k\}$$

be a $\delta_0$-fine PU-partition of $I$ with $\Sigma(A_0) \leq K$ and denote

$$Q = \{j; \tau^j \in \text{supp } f\},$$

$$\Delta = \{(\tau^j, \zeta_j); j \in Q\}.$$ 

It follows from (ii) that $\Delta$ is a PU-partition of $\text{supp } f$ (in particular, $\sum_{j=1}^{k} \zeta_j(x) = 1$ for $x \in \Omega(\text{supp } f, \eta) \cap \{x; \sum_{j=1}^{k} \zeta_j(x) = 1\}$ where $\eta = \min \{\frac{1}{2} \text{dist}(\tau^j, \text{supp } f); j = 1, \ldots, k, j \notin Q_j\}$). Further, (i) implies that $\Delta$ is $\delta$-fine and, finally, $\Sigma(\Delta) \leq \Sigma(A_0) \leq K$. Evidently,

$$S(f, A_0) = S(f, \Delta),$$

which proves that $(\text{PUI}) \int f \, dx$ exists and (2.4) holds.

2. Let $f$ be PUI-integrable, $\varepsilon > 0$, $K > 0$. Find a gauge $\delta$ on $I$ such that

$$(2.5) \quad \left| S(f, \Delta) - (\text{PUI}) \int f \, dx \right| \leq \varepsilon$$

for every $\delta$-fine PU-partition $\Delta$ of $I$ with $\Sigma(\Delta) \leq K + K_0 + 1$, where $K_0$ is such a constant that for any gauge $\delta$ on $I$ there exists a $5$-fine PU-partition $\tilde{A}$ of $I$ with $\Sigma(\tilde{A}) \leq K_0$ (cf. Proposition 1.1). Let

$$\Delta = \{(\tau^j, \vartheta^j); j = 1, \ldots, k\}$$

be a $\delta$-fine (more precisely, $\delta|_{\text{supp } f}$-fine) PU-partition of $\text{supp } f$ with $\Sigma(\Delta) \leq K$, and set $\vartheta(x) = \sum_{j=1}^{k} \vartheta^j(x)$ for $x \in \mathbb{R}^n$. Then $\vartheta: \mathbb{R}^n \to [0, 1]$ is of class $C^1$, has a compact support, $\text{Int} \{x; \vartheta(x) = 1\} \supseteq \text{supp } f$ and $|\partial \vartheta/\partial x_i| \leq C$ for $i = 1, \ldots, n$, where $C > 0$ is a constant.

Let $\delta^*$ be a gauge on $I$ fulfilling

(i) $\delta^*(x) \leq \delta(x)$ for $x \in I$;

(ii) $B(x, 2\delta^*(x)) \subseteq \text{Int} \{x; \vartheta(x) = 1\}$ for $x \in \text{supp } f$;

(iii) $\delta^*(x) \leq \left[2 \omega(I) \text{Cn} \right]^{-1}$ for $x \in I$;

(iv) $\omega\left(\bigcup_{x \in I} B(x, \delta^*(x))\right) \leq 2 \omega(I)$.

Let $Z = \{(\tau^i, \zeta_i); i = 1, \ldots, s\}$ be a $\delta^*$-fine PU-partition of $I$ with $\Sigma(Z) \leq K_0$. Put

$$A^* = \Delta \cup \{(\tau^i, \vartheta_i(1 - \vartheta)); i = 1, \ldots, s\}.$$ 

Then, as a consequence of the identity $\sum_{i=1}^{s} \zeta_i(x) = 1$ that holds in a neighbourhood of $I$, $A^*$ is a $\delta$-fine PU-partition of $I$ since

$$\sum_{j=1}^{k} \vartheta^j(x) + \sum_{i=1}^{s} (1 - \vartheta(x)) \zeta_i(x) = \vartheta(x) + (1 - \vartheta(x)) \sum_{i=1}^{s} \zeta_i(x)$$

and

$$S(f, A) - (\text{PUI}) \int f \, dx = S(f, A^*) - (\text{PUI}) \int f \, dx \leq \varepsilon.$$
for $x \in \mathbb{R}^n$. (ii) implies that $\zeta_i(x)(1 - \vartheta(x)) = 0$ for $x \in \mathbb{R}^n$ if $\tau_i \in \text{supp } f$, so that

$$(2.6) \quad S(f, A^*) = S(f, A).$$

Moreover,

$$\Sigma(A^*) = \Sigma(A) + \sum_{i=1}^{s} \int \|x - \tau^i\| \sum_{i=1}^{n} \left| \frac{\partial}{\partial x_i} \left[ (1 - \vartheta(x)) \zeta_i(x) \right] \right| dx \leq$$

$$\leq \Sigma(A) + \Sigma(Z) + \sum_{i=1}^{s} \int \|x - \tau^i\| \zeta_i(x) \sum_{i=1}^{n} \left| \frac{\partial \vartheta(x)}{\partial x_i} \right| dx .$$

Each summand in the last term is integrated over $\text{supp } \zeta_i \subset B(\tau^i, \delta^*(\tau^i))$, hence $\|x - \tau^i\| \leq [2 \omega(I) Cn]^{-1}$ by (iii). Using, moreover, the estimate $|\partial \vartheta/\partial x_i| \leq C$ we obtain

$$\sum_{i=1}^{s} \int \|x - \tau^i\| \zeta_i(x) \sum_{i=1}^{n} \left| \frac{\partial \vartheta(x)}{\partial x_i} \right| dx \leq \frac{1}{2 \omega(I)} \sum_{i=1}^{s} \int \zeta_i(x) dx =$$

$$= \frac{1}{2 \omega(I)} \sum_{i=1}^{s} \zeta_i(x) dx .$$

Taking into account that $\sum_{i=1}^{s} \zeta_i(x) \leq 1$ for $x \in \mathbb{R}^n$, we conclude by virtue of (iv) that $\Sigma(A^*) \leq K + K_0 + 1$. Hence

$$\left| S(f, A^*) - \int f(x) dx \right| \leq \varepsilon$$

so that (2.6) implies (2.5). Our theorem is proved.

3. TRANSFORMATION THEOREM

**Theorem 3.1.** Let $f: \mathbb{R}^n \to \mathbb{R}$ with a compact support be PU-integrable, let $G \subset \mathbb{R}^n$ be an open bounded set. Let $\psi: G \to \psi(G)$ be a $C^1$-diffeomorphism, supp $f \subset \psi(G)$.

Then $(f \circ \psi)|\det D\psi|$ is PU-integrable and

$$\int f(x) dx = \int f(\psi(y))|\det D\psi(y)| dy .$$

**Proof** proceeds analogously to that of Theorem 1, [4]. Let $\varepsilon > 0$, $K > 0$. Let $c_1, c_2$ be positive constants to be fixed later. Find a gauge $\delta'$ on supp $f$ such that for every $\delta'$-fine PU-partition $A'$ of supp $f$ with $\Sigma(A') \leq K' = Kc_1$ we have

$$(3.2) \quad \left| \int f(x) dx - S(f, A') \right| < \frac{1}{4} \varepsilon .$$

Further, find a gauge $\delta$ on supp $f \circ \psi = \psi^{-1}(\text{supp } f)$ such that

$$\psi(B(y, \delta(y))) \subset B(\psi(y), \delta(\psi(y))) ,$$

122
\begin{align}
(3.4) \quad & \left| \det D \psi(\eta) - \det D \psi(y) \right| \leq \frac{\varepsilon}{2c_2[1 + |f(\psi(y))|]}, \\
(3.5) \quad & \| \psi(\eta) - \psi(y) \| \leq (1 + \| D \psi(y) \|) \| \eta - y \|
\end{align}

for \( y, \eta \in \text{supp} f \circ \psi \), \( \eta \in B(y, \delta(y)) \).

Now let \( A = \{(y^j, \zeta_j); j = 1, \ldots, k\} \) be a \( \delta \)-fine PU-partition of \( \text{supp} f \circ \psi \) with \( \Sigma(A) \leq K \). Without loss of generality we may suppose \( \zeta_j \subset G \). Put \( x^j = \psi(y^j) \), \( \vartheta_j = \zeta_j \circ \psi^{-1} \). Then

\[ A' = \{(x^j, \vartheta_j), \; j = 1, \ldots, k\} \]

is a \( \delta' \)-fine PU-partition of \( \text{supp} f \). Indeed, denote

\[ Z = \{y \in \mathbb{R}^n; \sum_{j=1}^k \zeta_j(y) = 1\}, \quad \Theta = \{x \in \mathbb{R}^n; \sum_{j=1}^k \vartheta_j(x) = 1\} . \]

Then \( \Theta = \psi(Z) \) since \( \vartheta_j = \zeta_j \circ \psi^{-1} \). Moreover, since \( \text{Int} Z \supset \text{supp} f \circ \psi \) and \( \psi \) is a diffeomorphism, we have

\[ \text{Int} \Theta \supset \psi(\text{supp} f \circ \psi) = \text{supp} f . \]

Secondly, we have to prove \( \text{supp} \vartheta_j \subset B(x^j, \delta'(x^j)) \). But if \( x \in \text{supp} \vartheta_j \), then \( \psi^{-1}(x) \in \text{supp} \zeta_j \), that is, \( \psi^{-1}(x) \in B(y^j, \delta(y^j)) \), and hence, in virtue of (3.3), \( x \in \psi(B(y^j, \delta(y^j)) \subset B(\psi(y^j), \delta'(\psi(y^j))) = B(x^j, \delta'(x^j)) \). Finally, for \( \Sigma(A') \) we have

\begin{align*}
\Sigma(A') &= \sum_{j=1}^k \int_{\mathbb{R}^n} \| x - x' \| \sum_{i=1}^n \left| \frac{\partial \vartheta_j}{\partial x_i} \right| \, dx \\
&\leq \frac{k}{c} \sum_{j=1}^k \int_{\mathbb{R}^n} \| \psi(y) - \psi(y') \| \sum_{s=1}^n \sum_{i=1}^n \left| \frac{\partial \zeta_j}{\partial y_s} \frac{\partial \psi^{-1}}{\partial x_i} \right| \left| \det D \psi(y) \right| \, dy \\
&\leq c \sum_{j=1}^k \int_{\mathbb{R}^n} \| \psi(y) - \psi(y') \| \left| \det D \psi(y) \right| \sum_{s=1}^n \left| \frac{\partial \zeta_j}{\partial y_s} \right| \, dy ,
\end{align*}

where \( c \) is a positive constant. By (3.5) we obtain

\[ \Sigma(A') \leq K \max \left\{ 1 + \| D \psi(y') \| ; \; y \in \text{supp} \zeta_j \right\} \sup \left\{ \left| \det D \psi(y) \right| ; \; y \in \text{supp} \zeta_j \right\} , \]

\begin{align*}
&= K c \max \left\{ 1 + \| D \psi(y') \| ; \; y \in \text{supp} \zeta_j \right\} \sup \left\{ \left| \det D \psi(y) \right| ; \; y \in \text{supp} \zeta_j \right\} , \\
&= K c \max \left\{ 1 + \| D \psi(y') \| ; \; y \in \text{supp} \zeta_j \right\} \sup \left\{ \left| \det D \psi(y) \right| ; \; y \in \text{supp} \zeta_j \right\} ,
\end{align*}

where \( G_0 = \bigcup_{j=1}^k \text{supp} \zeta_j \subset G \) is a compact set.

Thus, choosing

\[ c_1 = c \max \left\{ 1 + \| D \psi(y') \| ; \; y \in \text{supp} \zeta_j \right\} \sup \left\{ \left| \det D \psi(y) \right| ; \; y \in \text{supp} \zeta_j \right\} , \]

we have \( \Sigma(A') \leq K c_1 = K' \).
Now we have to estimate the difference

\[ \left| \left( \text{PU} \right) \int f(x) \, dx - S(f \circ \psi | \det D\psi, \Delta) \right| \leq \]

\[ \leq \left| \left( \text{PU} \right) \int f(x) \, dx - S(f, \Delta') \right| + \left| S(f, \Delta') - S(f \circ \psi | \det D\psi, \Delta) \right| \leq \]

\[ \leq \frac{1}{2} \varepsilon + \left| S(f, \Delta') - S(f \circ \psi | \det D\psi, \Delta) \right| \]

(cf. (3.2)). The last term is estimated as follows:

\[ \left| S(f, \Delta') - S(f \circ \psi | \det D\psi, \Delta) \right| = \]

\[ = \sum_{j=1}^{k} \left| \int_{\mathbb{R}^n} \partial f(x) \, dx - f(y') \right| \left| \det D\psi(y') \right| \int_{\mathbb{R}^n} \zeta_{j}(y) \, dy \right| \leq \]

\[ \leq \sum_{j=1}^{k} \left| \int_{\mathbb{R}^n} \partial f(x) \, dx - \left| \det D\psi(y') \right| \int_{\mathbb{R}^n} \zeta_{j}(y) \, dy \right| \leq \]

\[ \leq \sum_{j=1}^{k} \left| \int_{\mathbb{R}^n} \partial \psi(y') \right| \left| \det D\psi(y') - \zeta_{j}(y) \right| \left| \det D\psi(y') \right| \, dy \leq \]

\[ \leq \sum_{j=1}^{k} \left| \int_{\mathbb{R}^n} \zeta_{j}(y) \right| \left| \det D\psi(y) - \det D\psi(y') \right| \, dy \leq \]

\[ \leq \sum_{j=1}^{k} \left| \int_{\mathbb{R}^n} \zeta_{j}(y) \right| \left| \det D\psi(y') \right| \, dy \leq \frac{1}{2} \varepsilon e_{2} \sum_{j=1}^{k} \int_{\mathbb{R}^n} \zeta_{j}(y) \, dy \]

(cf. (3.4)). By choosing the constant \( c_{2} \) suitably (notice that \( 0 \leq \sum_{j=1}^{k} \zeta_{j}(y) \leq 1 \) and we have assumed \( \text{supp} \zeta_{j} \subset G \) for \( j = 1, \ldots, k \)) we conclude that the estimated difference is less than \( \frac{1}{2} \varepsilon \). Consequently,

\[ \left| \left( \text{PU} \right) \int f(x) \, dx - S(f \circ \psi | \det D\psi, \Delta) \right| \leq \varepsilon , \]

which proves (3.1).

4. STOKES' THEOREM

**Theorem 4.1.** Let \( g: \mathbb{R}^{n} \to \mathbb{R} \) with a compact support have the differential \( Dg \) at every \( x \in \mathbb{R}^{n} \). Put \( f_{p} = \frac{\partial g}{\partial x_{p}} \), \( p = 1, \ldots, n \). Then \( f_{p} \) is \( \text{PU} \)-integrable and

\[ (\text{PU}) \int f_{p} \, dx = 0 . \]

**Proof.** Let \( \varepsilon > 0, K > 0 \). For every \( t \in \mathbb{R}^{n} \) find \( \delta(t) > 0 \) such that

\[ \| g(x) - g(t) - Dg(t)(x - t) \| \leq \frac{\varepsilon}{K} \| x - t \| \]

124
for $x \in B(t, \delta(t))$. Then $\delta : t \mapsto \delta(t)$ is a gauge on $\mathbb{R}^n$. Let (1.1) be a $\delta$-fine PU-partition of supp $g$ with $\Sigma(A) \leq K$. Denote

$$q_i(x) = g(t) + Dg(t)(x - t),$$

Then

$$\left| \int \sum_{j=1}^{k} (g(x) - q_i(x)) \frac{\partial \theta_j}{\partial x_p} \, dx \right| \leq \frac{\varepsilon}{K} \int \sum_{j=1}^{k} \left\| x - t \right\| \left\| \frac{\partial \theta_j}{\partial x_p} \right\| \, dx \leq \varepsilon$$

since $\Sigma(A) \leq K$. Further, since $\sum_{j=1}^{k} \theta_j(x) = 1$ for $x \in \text{supp } g$, we have

$$\sum_{j=1}^{k} \int g \frac{\partial \theta_j}{\partial x_p} \, dx = 0,$$

hence

$$\left(4.4\right) \quad \left| \sum_{j=1}^{k} \int q_i(x) \frac{\partial \theta_j}{\partial x_p} (x) \, dx \right| \leq \varepsilon .$$

On the other hand, integration by parts (with respect to $x_p$) yields

$$\left(4.5\right) \quad \int q_i(x) \frac{\partial \theta_j}{\partial x_p} (x) \, dx_p = - \frac{\partial g}{\partial x_p} (t) \int \theta_j(x) \, dx_p ,$$

hence

$$\left(4.6\right) \quad \int q_i(x) \frac{\partial \theta_j}{\partial x_p} (x) \, dx = - \frac{\partial g}{\partial x_p} (t) \int \theta_j(x) \, dx .$$

Combining (4.4) and (4.6) we conclude

$$\left| \sum_{j=1}^{k} f_p(t) \int \theta_j(x) \, dx \right| \leq \varepsilon ,$$

which proves (4.1).

**Corollary.** If $g, f_p$ satisfy the assumptions of Theorem 4.1 and $h : \mathbb{R}^n \to \mathbb{R}$ is of class $C^1$, then (PU) $\int h f_p \, dx$ exists, $p = 1, \ldots, n$.

**Proof.** By Theorem 4.1 we have

$$0 = (\text{PU}) \int \frac{\partial}{\partial x_p} (hg) \, dx = \int \frac{\partial h}{\partial x_p} g \, dx + (\text{PU}) \int hf_p \, dx$$

and the first integral on the right hand side exists since the integrand is continuous.

**Theorem 4.2.** Let $g : \mathbb{R}^n \to \mathbb{R}$ with a compact support have the differential $Dg$ at every $x \in \mathbb{R}^n$. Put again $f_p = \partial g/\partial x_p$, $p = 1, \ldots, n$, and

$$\chi(x) = \begin{cases} 1 & \text{for } x_1 \geq 0, \\ 0 & \text{for } x_1 < 0 \end{cases}$$
(of course we write \( x = (x_1, \ldots, x_n) \) for \( x \in \mathbb{R}^n \). Then

\[
\text{(PU)} \int \chi f_p \, dx = 0 \quad \text{for} \quad p = 2, \ldots, n,
\]

\[
\text{(PU)} \int \chi f_1 \, dx = - \int_{\mathbb{R}^{n-1}} g(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n.
\]

Proof. We proceed similarly as in the proof of Theorem 4.1. Let \( 0 < \varepsilon < 1, K > 0, \varrho > 1, \text{supp} \, g \subset B(0, \varrho - 1) \). Let \( \delta: \mathbb{R}^n \to \mathbb{R}^+ \) be a gauge found in the same way as above so that (4.2) holds. Moreover, let us suppose that \( \delta(t) \leq \|t\|/2 \) provided \( t = (t_1, \ldots, t_n), t_1 \neq 0 \), and that the following condition holds:

(4.7) Let \( t \in \text{supp} \, g, \, t = (0, t_2, \ldots, t_n) \). If \( \max \{|f_p(t)|; \, p = 1, 2, \ldots, n\} \leq 1 \), then

\[
\delta(t) \leq 2^{-2} \varepsilon (2\varrho)^{-n+1}; \quad \text{if} \quad 2^{m-1} < \max \{|f_p(t)|; \, p = 1, 2, \ldots, n\} \leq 2^m, \, m = 1, 2, \ldots,
\]

then \( \delta(t) \leq 2^{-2m-2} \varepsilon (2\varrho)^{-n+1} \).

Let (1.1) be a \( \delta \)-fine PU-partition of \( \text{supp} \, g \) with \( \Sigma(A) \leq K \). Let us denote \( Q_+ = \{j; \, t_1' > 0\}, \, Q_0 = \{j; \, t_1' = 0\}, \, Q = Q_+ \cup Q_0 \). Then

\[
S(\chi f_p, A) = \sum_{j \in Q} f_p(t') \int_{\mathbb{R}^n} \vartheta_j \, dx.
\]

In the same way as in the proof of Theorem 4.1 we establish an estimate analogous to (4.4), namely,

(4.8) \[
\sum_{j \in Q} \left| \int_{\mathbb{R}^n} \chi q_x \frac{\partial \vartheta_j}{\partial x_p} \, dx \right| \leq \varepsilon.
\]

If \( p \neq 1 \), we integrate by parts with respect to \( x_p \), thus obtaining (4.6), which yields

\[
\sum_{j \in Q} f_p(t') \int_{\mathbb{R}^n} \chi \vartheta_j \, dx \leq \varepsilon.
\]

Since \( \chi \vartheta_j = \vartheta_j \) for \( j \in Q_+ \), we have

\[
S(\chi f_p, A) = \sum_{j \in Q} f_p(t') \int_{\mathbb{R}^n} \chi \vartheta_j \, dx + \sum_{j \in Q_0} f_p(t') \int_{\mathbb{R}^n} (1 - \chi) \vartheta_j \, dx,
\]

so that

(4.9) \[
S(\chi f_p, A) \leq \varepsilon + \sum_{j \in Q_0} |f_p(t')| \int_{\mathbb{R}^n} \vartheta_j \, dx.
\]

In order to estimate the sum on the right hand side, put

\[Q_{00} = \{j \in Q_0; \max \{|f_p(t')|; \, p = 1, 2, \ldots, n\} \leq 1\},\]

\[Q_{0m} = \{j \in Q_0; \, 2^{m-1} < \max \{|f_p(t')|; \, p = 1, 2, \ldots, n\} \leq 2^m\}, \, m = 1, 2, \ldots\]
Since
\[ 0 \leq \sum_{j \in \mathbb{Q}_0} \vartheta_j(x) \leq 1 , \]
supp \[ \sum_{j \in \mathbb{Q}_0} \vartheta_j \subset [-2^{-2m-2} \varepsilon(2\varepsilon)^{1-n}, 2^{-2m-2} \varepsilon(2\varepsilon)^{1-n+1}] \times \]
we have
\[ \sum_{j \in \mathbb{Q}_0} |f_p(t^j)| \int_{\mathbb{R}^n} \vartheta_j \, dx \leq \sum_{m=0}^{\infty} 2^m \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Q}_0} \vartheta_j \, dx \leq \sum_{m=0}^{\infty} 2^m \cdot 2^{-2m-1} \varepsilon(2\varepsilon)^{1-n} (2\varepsilon)^{n-1} = \varepsilon , \]
which together with (4.9) yields
\[ |S(xf_p, A)| \leq 2\varepsilon . \]

If \( p = 1 \), integration by parts yields again
\[ \left\lfloor \chi(x) q_i(x) \frac{\partial \vartheta_j}{\partial x_1} (x) \right\rfloor R \int_{\mathbb{R}^n} = - f_i(t_i) \int_{\mathbb{R}^n} \vartheta_j(x) \, dx_1 \]
if \( j \in \mathbb{Q}_+ \) (recall that \( \delta(t^j) \leq |t_i^j|/2 \) in this case), and
\[ \int_{\mathbb{R}^n} \chi(x) q_i(x) \frac{\partial \vartheta_j}{\partial x_1} (x) \, dx_1 = \]
\[ = - q_i(0, x_2, \ldots, x_n) \vartheta_j(0, x_2, \ldots, x_n) - f_i(t_i) \int_{\mathbb{R}^n} \chi(x) \vartheta_j(x) \, dx_1 \]
if \( j \in \mathbb{Q}_0 \). This yields
\[ S(xf_1, A) = \sum_{j \in \mathbb{Q}} f_i(t_i) \int_{\mathbb{R}^n} \vartheta_j \, dx = \]
\[ = - \sum_{j \in \mathbb{Q}} \int_{\mathbb{R}^n} \chi(x) q_i(x) \frac{\partial \vartheta_j}{\partial x_1} (x) \, dx + \sum_{j \in \mathbb{Q}_0} f_i(t_i) \int_{\mathbb{R}^n} (1 - \chi) \vartheta_j \, dx \]
\[ - \sum_{j \in \mathbb{Q}_0} \int_{\mathbb{R}^n-1} q_i(0, x_2, \ldots, x_n) \vartheta_j(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n . \]

By (4.8), (4.2) and (4.3) we obtain
\[ (4.10) \quad |S(xf_1, A) + \int_{\mathbb{R}^n} g(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n| \leq \]
\[ \leq \varepsilon + \sum_{j \in \mathbb{Q}_0} |f_i(t_i)| \int_{\mathbb{R}^n} \vartheta_j \, dx + \]
\[ + \int_{\mathbb{R}^n-1} g(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n - \sum_{j \in \mathbb{Q}_0} \int_{\mathbb{R}^n-1} q_i(0, x_2, \ldots, x_n) \vartheta_j(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n | . \]

In the same way as the sum on the right hand side of (4.9) was estimated we obtain
that
\[ \sum_{j \in Q_0} |f_j(t)| \int_{R^n} \partial_j dx \leq \varepsilon. \]

We still have to estimate the last term on the right hand side of (4.10).

Since \( \sum_{j \in Q_0} \partial_j(0, x_2, \ldots, x_n) = 1 \) for \((0, x_2, \ldots, x_n) \in \text{supp} \ g \), we have
\[ \int_{R^n-1} g(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n = \sum_{j \in Q_0} \int_{R^n-1} g(0, x_2, \ldots, x_n) \partial_j(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n. \]

It follows from (4.7) that \( \max \{ \hat{\delta}(t^j) ; j \in Q_0 \} \leq 2^{-2} e(2q)^{-n+1} \) and that \( |x_j| < q \) for \( j = 2, 3, \ldots, n \) provided \( \quad \sum_{j \in Q_0} \partial_j(0, x_2, \ldots, x_n) > 0 \). Therefore
\[ \left| \int_{R^n-1} g(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n - \sum_{j \in Q_0} \int_{R^n-1} q_{ij}(0, x_2, \ldots, x_n) \partial_j(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n \right| \leq \]
\[ \leq \sum_{j \in Q_0} \int_{R^n-1} |g(0, x_2, \ldots, x_n) - q_{ij}(0, x_2, \ldots, x_n)| \partial_j(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n \leq \]
\[ \leq \sum_{j \in Q_0} \int_{R^n-1} \frac{e}{K} \| (0, x_2, \ldots, x_n) - t^j \| \partial_j(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n \leq \]
\[ \leq \frac{e}{K} \max \{ \hat{\delta}(t^j) ; j \in Q_0 \} \int_{R^n-1} \sum_{j \in Q_0} \partial_j(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n \leq \]
\[ \leq \frac{e}{K} 2^{-2} e(2q)^{-n+1} (2q)^{n-1} = e^2(4K)^{-1}. \]

This completes the proof of Theorem 4.2.

Let \( N \) be an \( n \)-manifold of class \( C^1 \) (without boundary or with a boundary \( \partial N \)).

Let \( \{(U_{\lambda}, h_{\lambda}); \lambda \in \Lambda \} \) be a system of charts. Put \( V_\lambda = h_\lambda(U_\lambda) \subset R^n \). It may be assumed that for \( \lambda \in \Lambda \) one of the following conditions is fulfilled:

(4.11) \( V_\lambda \) is open in \( R^n \);

(4.12) \( V_\lambda \) is relatively open in the half-space \( \{ x \in R^n; x_1 \geq 0 \} \), the points \( x = (0, x_2, \ldots, x_n) \in V_\lambda \) being images of points of the boundary \( \partial N \).

Let \( \xi \) be a \( p \)-form on \( N \). For \( \lambda \in \Lambda \) let us represent \( (h_\lambda)^* \xi \) in the form
\[ (h_\lambda)^* \xi = \sum_{i_1 < i_2 < \ldots < i_p} a^{(\lambda)}_{i_1, \ldots, i_p}(x) \, dx_{i_1} \wedge \ldots \wedge dx_{i_p}. \]

\( \xi \) is called differentiable if \( a^{(\lambda)}_{i_1, \ldots, i_p} \) is differentiable on \( V_\lambda \) for all \( \lambda, i_1, \ldots, i_p \). Assume
that $\xi$ is an $n$-form with a compact support. Let the following conditions be fulfilled:

(4.13) if $\lambda \in \Lambda$ and if (4.11) holds, then $(PU) \int \psi a_1, \ldots, a_n \, dx$ exists for every $C^1$ function $\psi: \mathbb{R}^n \to \mathbb{R}$ such that $\text{supp} \psi \subset V_\lambda$;

(4.14) if $\lambda \in \Lambda$ and if (4.12) holds, then $(PU) \int \varrho \, dx$ exists provided $\varrho$ is defined by

$$
\varrho(x) = \begin{cases} 
\psi(x) a_1, \ldots, a_n(x) & \text{for } x \in V_\lambda, \\
0 & \text{otherwise}, 
\end{cases}
$$

$\psi: \mathbb{R}^n \to \mathbb{R}$ being such a $C^1$ function that

$$
\text{supp} \psi \cap \{x \in \mathbb{R}^n; x_1 \geq 0\} \subset V_\lambda.
$$

Then $\xi$ is called $PU$-integrable.

Let $\xi$ be $PU$-integrable. Let $\varphi_i: N \to [0, 1], \ i = 1, \ldots, m$ be such $C^1$ functions that

$$
\text{supp} \xi \subset \text{Int} \{y \in N; \sum_{i=1}^m \varphi_i(y) = 1\}
$$

and that for every $i$ there exists such a $\lambda_i \in \Lambda$ that $\text{supp} \varphi_i \subset U_{\lambda_i}$. Put

$$
I(\xi, \varphi_1, \ldots, \varphi_m, \lambda_1, \ldots, \lambda_m) = \sum_{i=1}^m (PU) \int (h_{\lambda_i})^*(\varphi_i \xi) \, dx
$$

(we put $(h_{\lambda_i})^*(\varphi_i \xi)(x) = 0$ for $x \in \mathbb{R}^n \setminus V_{\lambda_i}$). It can be proved that $I$ is independent of $\varphi_1, \ldots, \varphi_m, \lambda_1, \ldots, \lambda_m$, so that it is called the $PU$-integral of $\xi$ and denoted by $(PU) \int_N \xi$.

**Theorem 4.3.** (Stokes). Let $\eta$ be a differentiable $(n - 1)$-form on $N$ with a compact support. Then $d\eta$ is a $PU$-integrable $n$-form and

(4.15)

$$(PU) \int_N d\eta = \int_{\partial N} \eta.$$

Theorem 4.3 can be proved in a standard manner from Theorems 4.1 and 4.2 and Corollary of Theorem 4.1. The identity (4.15) also holds if $N$ is replaced by a singular chain (cf. [5], [6]).

5. RELATION TO THE LEBESGUE INTEGRAL

**Theorem 5.1.** Let $f: \mathbb{R}^n \to \mathbb{R}$ with a compact support have a convergent Lebesgue integral. Then it is $PU$-integrable and

$$(PU) \int f \, dx = (L) \int f \, dx.$$

**Proof.** Let $I \subset \mathbb{R}^n$ be a compact interval, $\text{Int} I \supset \text{supp} f$. Suppose first that $f$ is bounded. Then for every $\varepsilon > 0$ there exist functions $u, v: I \to \mathbb{R}$ satisfying the following conditions:
(i) \( u \) is upper semicontinuous, \( v \) is lower semicontinuous;
(ii) \( u(x) \leq f(x) \leq v(x) \) for \( x \in \mathbb{R}^n \);
(iii) \( \int u \, dx + \alpha \geq \int f \, dx \geq \int v \, dx - \alpha \).

Moreover, choosing \( \eta > 0 \) such that \( B(x, 2\eta) \subset I \) for \( x \in \text{supp} f \), we may assume \( u(x) = 0 = v(x) \) for \( x \in \mathbb{R}^n \setminus \Omega(\text{supp} f, \eta) \supset \mathbb{R}^n \setminus I \).

Find a gauge \( \delta \) on \( I \) such that
\[
\delta(x) \leq \eta \quad \text{for} \quad x \in I,
\]
\[
u(x) \leq u(t) + \alpha, \quad v(x) \geq v(t) - \alpha \quad \text{for} \quad x \in B(t, \delta(t)).
\]

Let \( A = \{(t^l, \delta_j); \ j = 1, \ldots, k\} \) be a \( \delta \)-fine PU-partition of \( I \). Denote \( g(x) = \sum_{j=1}^{k} f(t^l) \delta_j(x) \). Then, applying the identities \( \sum_{j=1}^{k} \delta_j(x) = 1 \) for \( x \in I \), \( u(x) = v(x) = g(x) = 0 \) for \( x \in \mathbb{R}^n \setminus I \), we obtain
\[
u(x) = \sum_{j=1}^{k} (u(x) - f(t^l)) \delta_j(x) \leq \alpha \sum_{j=1}^{k} \delta_j(x) \leq \alpha,
\]
\[
u(x) = \sum_{j=1}^{k} (f(t^l) - v(x)) \delta_j(x) \leq \alpha \sum_{j=1}^{k} \delta_j(x) \leq \alpha
\]
for \( x \in \mathbb{R}^n \). Consequently,
\[
\int \nu \, dx - S(f, A) \leq \alpha m(I), \quad S(f, A) - \int \nu \, dx \leq \alpha m(I),
\]
and (iii) implies
\[
\int f \, dx - S(f, A) \leq \int \nu \, dx - S(f, A) + \alpha \leq \alpha m(I) + 1),
\]
\[
S(f, A) - \int f \, dx \leq S(f, A) - \int \nu \, dx + \alpha \leq \alpha m(I) + 1.
\]

Thus, given \( \varepsilon > 0 \), we find a gauge \( \delta \) and a constant \( \alpha \) such that
\[
\left| \int f \, dx - S(f, A) \right| \leq \varepsilon
\]
holds for any \( \delta \)-fine PU-partition \( A \) of \( I \), which proves our theorem provided \( f \) is bounded.

Let us now suppose that \( f \) is not bounded. For simplicity, assume that \( f: \mathbb{R}^n \rightarrow \{0\} \cup [1, \infty) \). (The general case then immediately follows by the linearity property of the integrals.) Let again \( \alpha > 0 \) and let \( \eta_l, l = 1, 2, \ldots \) be a sequence of positive
reals. Denote
\[ E_i = \{ x \in \mathbb{R}^n; (1 + \alpha)^{l-1} \leq f(x) < (1 + \alpha)^l \}, \quad l = 1, 2, \ldots, \]
\[ E_0 = \text{supp } f \setminus \bigcup_{i=1}^{\infty} E_i. \]
(Obviously \( f(x) = 0 \) for \( x \in E_0 \).) Since \( E_i \) are measurable, there exist open sets \( V_i, V_i \supset E_i \) with \( \mu(V_i \setminus E_i) \leq \eta_i, \ l = 1, 2, \ldots \). Let \( \delta \) be a gauge on \( I \) such that
\[ B(x, \delta(x)) \subset V_i \quad \text{for} \quad x \in E_i. \]
Let again \( \Delta = \{(t^n, \theta_j); j = 1, \ldots, k\} \) be a \( \delta \)-fine PU-partition of \( I \). Denote
\[ g_i(x) = \left\{ \begin{array}{ll}
\sum_{t^n \in E_i} f(t^n) \theta_j(x) & \text{if} \quad \{t^n \in E_i\} \neq \emptyset, \\
0 & \text{otherwise},
\end{array} \right. \]
so that
\[ g(x) = \sum_{j=1}^{k} f(t^n) \theta_j(x) = \sum_{i=1}^{\infty} g_i(x). \]
Evidently,
\[ \int_{E_i} g_i \, dx \leq (1 + \alpha)^l \mu(V_i), \]
\[ \int_{E_i} f \, dx \geq (1 + \alpha)^{l-1} \mu(E_i), \]
hence
\[ \int_{E_i} g_i \, dx \leq (1 + \alpha) \int_{E_i} f \, dx + (1 + \alpha)^l \mu(V_i \setminus E_i), \]
\[ S(f, \Delta) = \int g \, dx \leq (1 + \alpha) \int f \, dx + \sum_{i=1}^{\infty} (1 + \alpha)^i \eta_i. \]
A suitable choice of \( \alpha, \eta_i \) evidently yields
\[ S(f, \Delta) \leq \int f \, dx + \epsilon. \]
Since \( f \) is bounded from below, the other estimate is obtained as in the first part of the proof by approximating the function \( f \) from below by an upper semicontinuous function. Hence (5.1) again holds and the proof of Theorem 5.1 is complete.

Observe that the function \( \Sigma \) was not needed in the proof of Theorem 5.1. Therefore an analogous result (Theorem 5.2 below) can be proved in the same way for the following modification of the PU-integral:

Let \( M \subset \mathbb{R}^n \) be bounded. A family (1.1) is called a PUL-partition of \( M \) if \( t^n \in M, \ \theta_j: \mathbb{R}^n \to [0, 1] \) are measurable functions with compact supports and if (1.2), (1.3) hold. If \( \delta \) is a gauge on \( M \), then a PUL-partition of \( M \) is said to be \( \delta \)-fine if (1.4) holds.

**Definition 5.1.** Let \( f: \mathbb{R}^n \to \mathbb{R} \) have a compact support. Let \( \gamma \in \mathbb{R} \) satisfy the following condition:
for every $\varepsilon > 0$ there is a gauge $\delta$ on $\text{supp} \ f$ such that (2.2) holds provided $\Delta$ is a $\delta$-fine PUL-partition of $\text{supp} \ f$. Then $f$ is said to be \textit{PUL-integrable} and we write

$$\gamma = (\text{PUL}) \int f \, dx.$$ 

\textbf{Theorem 5.2.} Let $f: \mathbb{R}^n \to \mathbb{R}$ with a compact support be Lebesgue integrable with a convergent integral. Then it is PUL-integrable and

$$(\text{PUL}) \int f \, dx = (L) \int f \, dx.$$ 

Let us introduce SL-partitions, SL-integrable functions and the SL-integral in the analogous way as the PUL-partitions etc. with the only change that $\mathcal{G}_t$ are characteristic functions of semiclosed intervals (cf. Remark 2.1). Let $f: \mathbb{R}^n \to \mathbb{R}$ have a compact support, let $I \subseteq \mathbb{R}^n$ be a compact interval, $\text{Int} \ I \supset \text{supp} \ f$. The integral $(S_\varnothing) \int f \, d\Lambda$ was introduced in [3], 7.4 (put $\Lambda(J) = (b_1 - a_1) \cdot (b_2 - a_2) \cdots (b_n - a_n)$ for $J = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$). It was proved in [3], Theorem 7.6 that $(S_\varnothing) \int f \, d\Lambda$ exists if and only if $(L) \int f \, dx$ exists and is finite, and that both the integrals coincide in this case. It is easy to prove that $(SL) \int f \, dx$ exists if and only if $(S_\varnothing) \int f \, d\Lambda$ exists and that both the integrals coincide in this case. Obviously, $(SL) \int f \, dx$ exists and is equal to $(PUL) \int f \, dx$ if the latter exists. We conclude:

\textbf{Theorem 5.3.} Let $f: \mathbb{R}^n \to \mathbb{R}$ have a compact support. Then the following conditions are equivalent:

(i) $f$ is PUL-integrable;
(ii) $f$ is SL-integrable;
(iii) $f$ is Lebesgue integrable, with a finite integral.

If one of the above conditions is fulfilled, then

$$(PUL) \int f \, dx = (SL) \int f \, dx = (L) \int f \, dx.$$ 

\section{Some Results on the M_2-Integral}

The M_2-integral was introduced in [2]. In an equivalent manner it can be described as follows: Let $I \subseteq \mathbb{R}^n$ be a compact interval. A system

$$\Gamma = \{(t^i, J^i) \mid i = 1, 2, \ldots, k\}$$

is called an $L$-\textit{partition} of $I$, if $J^i$ are nonoverlapping intervals, $t^i \in I$, $i = 1, 2, \ldots, k$, $\bigcup_{i=1}^{k} J^i = I$. An $L$-partition $\Gamma$ is called a $P$-\textit{partition}, if $t^i \in J^i$ for $i = 1, 2, \ldots, k$. 

132
Let $\delta$ be a gauge on $I$. $\Gamma$ is called $\delta$-fine, if $J^i \subset B(t^i, \delta(t^i))$ for $i = 1, 2, \ldots, k$. Put
\[ \Sigma(\Gamma) = \sum_{i=1}^{k} \int_{J^i} \| t^i - x \| \, d\mu_{n-1}, \]
$\mu_{n-1}$ being the $(n-1)$-dimensional Lebesgue measure. (If $n = 1$, $J^i = [a^i, b^i]$.
then $\Sigma(\Gamma) = \sum_{i=1}^{k} (|t^i - a^i| + |t^i - b^i|).$) For $f : I \to R$ put
\[ S(f, \Gamma) = \sum_{i=1}^{k} f(t^i) \mu_n(J^i). \]

**Definition 6.1.** Let $\gamma \in R$, $f : I \to R$, $I \subset R^n$ being a compact interval. Let the following condition be fulfilled:

for every $\varepsilon > 0$ and $K > 0$ there exists such a gauge $\delta$ on $I$ that
\[ |\gamma - S(f, \Gamma)| \leq \varepsilon \]
provided $\Gamma$ is a $\delta$-fine $L$-partition of $I$ with $\Sigma(\Gamma) \leq K$.

Then $f$ is said to be $M_2$-integrable and we write
\[ \gamma = (M_2) \int_I f \, dx; \]
$\gamma$ is called the $M_2$-integral of $f$.

The equivalence of Definition 6.1 and Definition 4 of [2] follows immediately from Corollary of Theorem 8 of [2].

**Theorem 6.1.** Let $I \subset R^n$ be a compact interval, $f : R^n \to R$, $\text{supp} f \subset I$. Let $f$ be $PU$-integrable. Then $f$ is $M_2$-integrable and
\[ (M_2) \int_I f \, dx = (PU) \int_I f \, dx. \]

**Proof.** Let $I = [a, b] = [a_1, b_1] \times \ldots \times [a_n, b_n]$. By Theorem 2.1 $f$ is PU-integrable. Let $\varepsilon > 0$, $K > 0$. By Definition 2.2 there is such a gauge $\delta$ on $I$ and $\gamma \in R$
that
\[ |\gamma - S(f, A)| \leq \varepsilon \]
for every $(2\delta)$-fine $PU$-partition $A$ of $I$ with $\Sigma(A) \leq K + 1$. Let $\Gamma$ be a $\delta$-fine $L$-partition of $I$ with $\Sigma(\Gamma) \leq K$,
\[ \Gamma = \{(t^i, J^i) ; \ i = 1, 2, \ldots, k\}. \]
For a fixed $i$ write $J^i$ in the form
\[ J^i = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n] \]
and for $\sigma > 0$, $x = (x_1, x_2, \ldots, x_n) \in R^n$ put
\[ \delta^i_\sigma(x) = \prod_{i=1}^{n} h_{c_i,d_i,\sigma}(x_i) \]
where
\[ c_i = a_i \quad \text{if} \quad a_i > a_i, \]
\[ c_i = a_i - 2\sigma \quad \text{if} \quad a_i = a_i, \]
\[ d_i = b_i \quad \text{if} \quad b_i < b_i, \]
\[ d_i = b_i + 2\sigma \quad \text{if} \quad b_i = b_i, \quad i = 1, 2, \ldots, n. \]

For the definition of \( h_{a,b,a} \), see (1.6) and above. Obviously \( \Delta = \{(t^j, \theta^j)\}; j = 1, 2, \ldots, k \) is a PU-partition of \( I \) for \( \sigma \) sufficiently small and
\[ (6.5) \quad \lim_{\sigma \to 0} S(f, \Delta) = S(f, \Gamma). \]

Moreover,
\[ \lim_{\sigma \to 0} \sum_{i=1}^n \int \|x - t^i\| \, \frac{\partial \theta^i}{\partial x_1} \, dx = \int \|x - t^i\| \, d\omega_{n-1}, \]
so that
\[ (6.6) \quad \lim_{\sigma \to 0} \Sigma(\Delta) = \Sigma(\Gamma). \]

If \( \sigma \) is sufficiently small, \( \Delta \) is \( 2\delta \)-fine, \( \Sigma(\Delta) < K + 1 \) so that
\[ |y - S(f, \Delta)| \leq \varepsilon \]
any by (6.5) we obtain
\[ |y - S(f, \Gamma)| \leq \varepsilon, \]
which makes the proof complete.

Let \( f: I \to R \) be \( M_2 \)-integrable. The following assertions are consequences of [2], Theorem 1, Corollary and Remark 5.
\[ (6.7) \quad \text{If } H \subset I \text{ is an interval, then} \]
\[ (M_2) \int_H f \, dx \text{ exists.} \]
\[ (6.8) \quad \text{Let } H, H^1, \ldots, H^s \text{ be intervals, } \bigcup_{i=1}^s H^i = H \subset I \text{ and let the intervals } H^1, H^2, \ldots, \]
\[ \ldots, H^s \text{ be nonoverlapping. Then} \]
\[ (M_2) \int_H f \, dx = \sum_{i=1}^s (M_2) \int_{H^i} f \, dx. \]

The following theorem is an analogue of the Saks-Henstock Lemma, which was proved in [2], p. 372.

**Theorem 6.2.** Let \( I \subset R^n \) be a compact interval and let \( f: I \to R \) be \( M_2 \)-integrable. Let \( \gamma = (M_2) \int_I f \, dx, \varepsilon > 0, K > 0 \) and let \( \delta \) be such a gauge on \( I \) that
\[ |y - S(f, \Gamma)| \leq \varepsilon. \]
for every \( \delta \)-fine \( L \)-partition \( \Gamma \) of \( I \) with

\[ \Sigma(\Gamma) \leq K + 4n^2 m_n(I). \]

Let \( L^s, s = 1, 2, \ldots, r \) be nonoverlapping intervals, \( t^s \in I, L^s \subseteq B(t^s, \delta(t^s)) \) for \( s = 1, 2, \ldots, r \),

\[ \sum_{s=1}^r \int_{L^s} \|x - t^s\| \, d\mu_{n-1} \leq K \]

(in case that \( n = 1 \), \( L^s = [a^s, b^s] \) condition (6.9) reads)

\[ \sum_{s=1}^r (|t^s - a^s| + |t^s - b^s|) \leq K. \]

Then

\[ \left| \sum_{s=1}^r \left( f(t^s) \mu_n(L^s) - (M_2) \int_{L^s} f \, dx \right) \right| \leq \varepsilon. \]

The following lemma will be needed in the proof:

**Lemma 6.1.** Let \( I \subseteq \mathbb{R}^n \) be a compact interval and let \( \delta \) be a gauge on \( I \). Then there exists such a \( \delta \)-fine \( P \)-partition \( \Gamma = \{(t^i, J^i); i = 1, 2, \ldots, k\} \) of \( I \) that

\[ \Sigma_0(\Gamma) = \sum_{i=1}^k \text{diam}(J^i) \mu_{n-1}(\partial J^i) \leq 4n^2 m_n(I). \]

**Proof.** Without loss of generality we may assume that

\[ I = [0, b_1] \times [0, b_2] \times \cdots \times [0, b_n], \]

\[ 0 < b_1 \leq b_j \quad \text{for} \quad j = 2, 3, \ldots, n. \]

There are such nonnegative integers \( l_j \) that \( 2^{l_j}b_1 \leq b_j < 2^{l_j+1}b_1 \) for \( j = 2, 3, \ldots, n \). Let \( J \) be the set of such \( j \in \{2, 3, \ldots, n\} \) that \( l_j > 0 \). If \( J \neq \emptyset \), cut \( I \) by the hyperplanes \( x_j = pb_j 2^{-l_j}, p = 1, 2, \ldots, 2^{l_j} - 1, j \in J \). Thus \( I \) is cut into intervals \( I^1, I^2, \ldots, I^q \) with \( q = \prod_{j=1}^n 2^{l_j} \) and the lengths of edges of each interval \( I^m \) are \( b_1, b_2 2^{-l_2}, b_3 2^{-l_3}, \ldots, b_n 2^{-l_n}, m = 1, 2, \ldots, q \). If we find \( \delta \)-fine \( P \)-partitions \( \Gamma_m \) of intervals \( I^m \) fulfilling \( \Sigma_0(\Gamma_m) \leq 4n^2 m_n(I^m), m = 1, 2, \ldots, q \), we may put \( \Gamma = \bigcup_{m=1}^q \Gamma_m \). \( \Gamma \) is a \( P \)-partition of \( I \) fulfilling (6.11).

Thus we may assume without loss of generality that

\[ b_j < 2b_1 \quad \text{for} \quad j = 2, 3, \ldots, n. \]

If there is such a \( t \in I \) that \( I \subseteq B(t, \delta(t)) \), we put \( \Gamma = \{(t, I)\} \). Otherwise we cut \( I \) by hyperplanes \( x_j = \frac{1}{2}b_j, j = 1, 2, \ldots, n \); thus \( I \) is cut into intervals \( I^p, p = 1, 2, \ldots, 2^n \). Let \( P \) be the set of such \( p \) that there is \( t^p \in I^p \) fulfilling \( I^p \subseteq B(t^p, \delta(t^p)) \). Couples \((t^p, I^p), p \in P\) are elements of \( \Gamma \); intervals \( I^p, p \notin P \) are cut in an analogous
way etc. After a finite number of steps we obtain a $\delta$-fine $P$-partition

$$
\Gamma = \{(t_i, J_i) \mid i = 1, 2, \ldots, k\}
$$

of $I$. Moreover, there exist such nonnegative integers $r_i$ that the lengths of edges of $J^i$ are $2^{-r_i}b_j$, $j = 1, 2, \ldots, n$. We have

$$
\mu_{n-1}(\partial J^i) \text{diam } J^i = 2^{1-nr_i} \sum_{j=1}^{n} b_1 \ldots b_{j-1} b_{j+1} \ldots b_n \sqrt{b_1^2 + \ldots + b_n^2} \leq 2 \cdot 2^{-nr_1} b_1 b_2 \ldots b_n \sum_{j=1}^{n} \frac{1}{b_1} \sqrt{b_1^2 + \ldots + b_n^2} \leq 2 \mu_n(J^i) \sqrt{n} \leq 4n^2 \mu_n(J^i).
$$

Hence $\Sigma_{0}(\Gamma) \leq 4n^2 \mu_n(I)$ which proves the lemma.

**Proof of Theorem 6.2.** There are such intervals $G^1, G^2, \ldots, G^p \subset I$ that the intervals $G^1, G^2, \ldots, G^p$, $L^1, L^2, \ldots, L'$ are nonoverlapping and

$$
I = \bigcup_{s=1}^{r} (t_s, L_s') \cup \bigcup_{i=1}^{p} G^i.
$$

Let $\eta > 0$. There exist gauges $\delta_i$ on $G^i$ such that $|S(f, \Gamma_i) - (M_2) \int_{G^i} f \, dx| \leq \eta/p$ for every $\delta_i$-fine $L$-partition $\Gamma_i$ of $G^i$ with $\Sigma(\Lambda_i) \leq 4n^2 \mu_n(G^i)$, $i = 1, 2, \ldots, p$. Moreover, we can assume that $\delta_i(x) \leq \delta(x)$ for $x \in G^i$. Let $\tilde{\Gamma}_i$ be $\delta_i$-fine $P$-partitions of $G^i$ with $\Sigma_{0}(\tilde{\Gamma}_i) \leq 4n^2 \mu_n(G^i)$ (cf. Lemma 6.1). Put

$$
\Gamma = \bigcup_{s=1}^{r} \{(t_s, L_s')\} \cup \bigcup_{i=1}^{p} \tilde{\Gamma}_i.
$$

As every $P$-partition is simultaneously an $L$-partition, $\Gamma$ is a $\delta$-fine $L$-partition of $I$. Moreover,

$$
\Sigma(\Gamma) = \sum_{s=1}^{r} \int_{s \in L_s} \|x - t_s\| \, dm_{n-1} + \sum_{i=1}^{p} \Sigma(\tilde{\Gamma}_i).
$$

Since $\tilde{\Gamma}_i$ are $P$-partitions of $G^i$, we have $\Sigma(\tilde{\Gamma}_i) \leq \Sigma_{0}(\tilde{\Gamma}_i)$ for $i = 1, 2, \ldots, p$ (cf. (6.11)), so that

$$
\Sigma(\Gamma) \leq K + 4n^2 \sum_{i=1}^{p} \mu_n(G^i) \leq K + 4n^2 \mu_n(I).
$$

It follows that

$$
|\left( M_2 \right) \int_{I} f \, dx - S(f, \Gamma)| \leq \varepsilon.
$$

Simultaneously we have (cf. (6.8))

$$
\left( M_2 \right) \int_{I} f \, dx = \sum_{s=1}^{r} (M_2) \int_{s \in L_s} f \, dx + \sum_{i=1}^{p} (M_2) \int_{G^i} f \, dx,
$$

$$
S(f, \Gamma) = \sum_{s=1}^{r} S(f, t_s, L_s') + \sum_{i=1}^{p} S(f, \tilde{\Gamma}_i),
$$

$$
\left| M_2 \right| \int_{G^i} f \, dx - S(f, \tilde{\Gamma}_i) \right| \leq \eta/p.
$$
It follows that
\[
\left| \sum_{s=1}^{r} (M_s) \int_{L_s} f \, dx - \sum_{s=1}^{r} f(t^s) \omega_n(L^s) \right| \leq \varepsilon + \eta
\]
and (6.10) holds, since \( \eta > 0 \) was arbitrary.

7. EXAMPLES

A. Let \( n = 1, \alpha > 0, \beta > 0 \). Let \( \omega \) be defined as before (see (1.6) and above). Put
\[
\xi(x) = \begin{cases} 
  x^\alpha \cos \pi x^{-\beta} & \text{for } x > 0, \\
  0 & \text{for } x \leq 0,
\end{cases}
\]
\[
\eta(x) = \xi(x) \omega(3 - x) \text{ for } x \in \mathbb{R}.
\]
If \( \alpha > 1 \), then \( \eta \) is differentiable and \( (PU) \int \eta' \, dx \) exists by Theorem 4.1, \( \eta' \) being the derivative of \( \eta \), \( \eta'(x) = \pi x^{\alpha-1} \cos \pi x^{-\beta} + \pi \beta x^{\alpha-\beta-1} \sin \pi x^{-\beta} \) for \( 0 < x \leq 2 \).

Let \( \lambda > 0 \). Put
\[
v(x) = \begin{cases} 
  x^{-\lambda} \sin \pi x^{-\beta} & \text{for } 0 < x \leq 1, \\
  0 & \text{for } x \leq 0 \text{ and for } x > 1.
\end{cases}
\]

It is not difficult to conclude that
\[
(PU) \int v \, dx \text{ exists if } 0 < \lambda < 1 \text{ or if } 0 < \lambda < \beta.
\]
Indeed, for \( 0 < \lambda < 1 \), \( v \) is even Lebesgue integrable; the other part of (7.1) follows from the fact that the difference \( v - (\pi \beta)^{-1} \eta' \) is absolutely integrable for \( 0 < \lambda < \beta \) if we put \( \alpha = \beta - \lambda + 1 \).

Our aim is to prove that
\[
(M_2) \int_{[0,1]} v \, dx \text{ does not exist if } \lambda \geq 1, \ 0 < \beta \leq \lambda.
\]

Observe that (7.2) and Theorem 6.1 imply
\[
(PU) \int v \, dx \text{ does not exist if } \lambda \geq 1, \ 0 < \beta \leq \lambda.
\]

The zeros \( \sigma_l \) of \( v \) in \( (0, 1] \) are given by \( \sigma_l = (l)^{-1/\beta} \), \( l = 1, 2, \ldots \). Let \( p \) be a positive odd integer and for \( s = 1, 2, \ldots \) put
\[
L_s = [\sigma_{p+2s+1}, \sigma_{p+2s}].
\]

We have \( v(x) \geq 0 \) for \( x \in L_s \), \( s = 1, 2, \ldots \) and there is such a \( c > 0 \) that
\[
(M_2) \int_{L_s} v \, dx \geq \frac{c}{\beta} (p + 2s)^{(\lambda-\beta-1)/\beta}, \ s = 1, 2, \ldots.
\]

In order to prove (7.2) assume that \( \lambda \geq 1, \ 0 < \beta \leq \lambda \) and that \( (M_2) \int_{[0,1]} v \, dx \).
exists. In Theorem 6.2 put $K = 1$, $e = c/4\beta$, $y = (M_2) \int_{[0,1]} v \, dx$ and let $\delta$ be such a gauge on $[0,1]$ that $|y - S(v, I)| \leq c/4\beta$ for every $\delta$-fine $L$-partition of $[0,1]$ with $\Sigma(I) \leq 5$.

Let $p$ be so large that $\sigma_p < \delta(0), \sigma_p < \frac{1}{4}$. Put $t_s = 0$ and define $L_s$ by (7.4), $s = 1, 2, \ldots, r$, where $r$ will be chosen later. Observe that (under our choice) the left hand side in (6.9) is $\sum_{s=1}^{r} (\sigma_{p+2s} + \sigma_{p+2s+1})$ and the left hand side in (6.10) is

$$\sum_{s=1}^{r} (M_2) \int_{L_s} v \, dx$$

since $f(t_s) = 0$ for $s = 1, 2, \ldots, r$.

Let two cases be treated separately:

1. $1 \leq \beta \leq \lambda$. Then by (7.5)

$$\sum_{s=1}^{r} (M_2) \int_{L_s} v \, dx \geq \frac{c}{\beta} \sum_{s=1}^{r} (p + 2s)^{-1/\beta} \geq \frac{c}{2\beta} \sum_{s=1}^{r} (\sigma_{p+2s} + \sigma_{p+2s+1})$$

Since $\sum_{s=1}^{\infty} (\sigma_{p+2s} + \sigma_{p+2s+1}) = \infty$, $\sigma_p < \frac{1}{4}$, the number $r$ may be chosen in such a way that $\frac{1}{2} < \sum_{s=1}^{r} (\sigma_{p+2s} + \sigma_{p+2s+1}) = 1$ so that $\sum_{s=1}^{r} (M_2) \int_{L_s} v \, dx > c/4\beta$. Thus (6.9) is fulfilled and (6.10) is not fulfilled, which is a contradiction.

2. $0 < \beta < 1 \leq \lambda$. As $\sum_{s=1}^{\infty} (\sigma_{p+2s} + \sigma_{p+2s+1}) < \infty$, $p$ can be chosen so large that

$$\sum_{s=1}^{r} (\sigma_{p+2s} + \sigma_{p+2s+1}) < 1 \quad \text{for} \quad r = 1, 2, \ldots$$

On the other hand $(M_2) \int_{L_s} v \, dx \geq \frac{c}{\beta} (p + 2s)^{-1} / \beta$ by (7.5), so that $\sum_{s=1}^{\infty} (M_2) \int_{L_s} v \, dx = \infty$ and thus (6.10) does not hold for $r$ sufficiently large.

Remark 7.1. Let $\lambda \geq 1$. Since $v$ is continuous on $[0,1]$ and $\lim_{T \to 0^+} \int_{[0,T]} v(x) \, dx$ exists iff $\beta > \lambda - 1$, we may conclude that the Perron integral $(P) \int_{[0,1]} v \, dx$ exists iff $\beta > \lambda - 1$.

B. Let $n = 2$. Put

$$\zeta(x) = \zeta(x_1, x_2) = \zeta(x_1) \omega(x_2 + 1) \omega(2 - x_2),$$

$$\varrho(x) = \varrho(x_1, x_2) = \varrho(x_1) \omega(x_2 + 1) \omega(2 - x_2) \quad \text{for} \quad x = (x_1, x_2) \in \mathbb{R}^2.$$ If $\alpha > 1$, then $\zeta$ is differentiable and $(PU) \int (\partial \zeta / \partial x_1) \, dx$ exists by Theorem 6.1. Again (cf. Part A) it is not difficult to conclude that

$$(P) \int \varrho \, dx \quad \text{exists if} \quad 0 < \lambda < 1 \quad \text{or if} \quad 0 < \lambda < \beta.$$ Analogously to Part A we have

$$(M_2) \int_I \varrho \, dx \quad \text{does not exist, if} \quad \lambda \geq 1,$$

$$0 < \beta \leq \lambda, \quad I = [0,1] \times [0,1].$$
and (7.7) and Theorem 6.1 imply

\[(7.8) \quad (PU) \int q \, dx \text{ does not exist, if } \lambda \geq 1, \quad 0 < \beta \leq \lambda. \]

In order to indicate the proof of (7.7) let us start with the following lemma:

**Lemma 7.1.** Let \( x > 0, E \subset [0, 1], m_\varepsilon(E) > \frac{1}{\varepsilon}, m_\varepsilon(E) \) being the outer Lebesgue measure of \( E \). Then there are \( \tau_1, \tau_2, \ldots, \tau_q \in E \) such that \( q \geq 1/8x \) and that the intervals \([\tau_i - x, \tau_i + x], i = 1, 2, \ldots, q\) are disjoint.

**Proof.** \( \tau_i \in E \) can be chosen arbitrarily. If \( \tau_i \in E \) are known, \( i = 1, 2, \ldots, k \) such that \( k < 1/8x \) and that the intervals \([\tau_i - x, \tau_i + x]\) are disjoint for \( i = 1, 2, \ldots, k \), then

\[E \setminus \left( \bigcup_{i=1}^{k} [\tau_i - 2x, \tau_i + 2x] \right) \neq \emptyset, \quad \tau_{k+1} \in E \setminus \left( \bigcup_{i=1}^{k} [\tau_i - 2x, \tau_i + 2x] \right)\]

may be chosen arbitrarily and the intervals \([\tau_i - x, \tau_i + x], i = 1, 2, \ldots, k + 1\) are disjoint.

(7.7) can be proved in an analogous way as (7.2). If \( \delta \) is a gauge on \( I \), put \( Q(\delta) = \{x_2 \in [0, 1]; \delta(0, x_2) \geq 2x\} \). Since \( Q(\kappa_1) \supseteq Q(x_2) \) for \( 0 < \kappa_1 < \kappa_2 \) and \( \bigcup_{\kappa > 0} Q(\kappa) = [0, 1] \), there is \( \kappa_0 > 0 \) such that \( m_\varepsilon(Q(\kappa_0)) > \frac{1}{\varepsilon} \). Let \( p \) be such an even positive integer that \( \kappa_0 < \kappa_p \). For every \( k = 1, 2, \ldots, r \) put \( E = Q(x), \kappa = \kappa_p + 2k \), and define \( \tau_1^{(s)}, \tau_2^{(s)}, \ldots, \tau_q^{(s)} \) by Lemma 7.1. Theorem 6.2 can be applied in an analogous way, the role of couples \((t_s, L_s), s = 1, 2, \ldots, r\) being played by the couples

\[
((0, \tau_1^{(s)}), [\sigma_{p+2s+1}, \sigma_{p+2s}] \times [\tau_i^{(s)} - \sigma_{p+2s}, \tau_i^{(s)} + \sigma_{p+2s}]),
\]
i = 1, 2, \ldots, \( q \), \( s = 1, 2, \ldots, r \) with a suitable choice of \( r \) (and possibly of \( p \)).

**Remark 7.2.** Again it can be proved that

\[ (P) \int_I q \, dx \text{ exists if } \lambda \geq 1, \beta > \lambda - 1. \]

**References**


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