CHARACTERIZATIONS OF CONFORMALLY FLAT HYPERSURFACES

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1. INTRODUCTION

In this paper we study hypersurfaces of Euclidean space satisfying one of the conditions $C \cdot C = 0$, $C \cdot R = 0$ or $Q \cdot C = 0$, where $R$ denotes the Riemann-Christoffel curvature tensor, $Q$ the Ricci tensor and $C$ the Weyl conformal curvature tensor of the hypersurface and where the first tensor acts on the second as a derivation.

Semi-symmetric spaces, i.e. Riemannian manifolds for which $R \cdot R = Q$, have been studied by various authors. For references one can consult the recent work of Z. I. Szabo on this subject [12].

In [8] K. Nomizu studied semi-symmetric hypersurfaces of Euclidean space and P. J. Ryan investigated semi-symmetric hypersurfaces of space-forms [9]. Y. Matsuyama [6], I. Mogi and H. Nakagawa [7], P. J. Ryan [10], S. Tanno [13], S. Tanno and T. Takahashi [14] studied hypersurfaces of space forms satisfying one of the conditions $R \cdot Q = 0$ or $VQ = 0$. In [16] two of the authors characterized hypercylinders in Euclidean spaces by the condition $Q \cdot R = 0$. For hypersurfaces in Euclidean space with $R \cdot C = 0$ or $C \cdot R = 0$, see [11]. Complex hypersurfaces in complex space forms satisfying similar conditions have been investigated by P. J. Ryan [11], T. Takahashi [15] and the authors [4].

We prove the following theorem.

Theorem. Let $M^n$ be a hypersurface in an $(n + 1)$-dimensional Euclidean space $(n > 3)$ and denote by $R$ the Riemann-Christoffel curvature tensor, by $Q$ the Ricci tensor and by $C$ the Weyl conformal curvature tensor of $M^n$. Then the following assertions are equivalent:

(i) $C \cdot C = 0$,
(ii) $C \cdot R = 0$,
(iii) $Q \cdot C = 0$,
(iv) $M^n$ is conformally flat.

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2. BASIC FORMULAS

Let $M$ be a hypersurface of an $(n + 1)$-dimensional Euclidean space $E^{n+1}$. Let $\xi$ be a local normal section on $M$. In the following $X, Y, Z$ denote vector fields which are tangent to $M$. Then the formulas of Gauss and Weingarten are given by

$$\nabla^E_X Y = \nabla_X Y + h(X, Y) \xi$$

and

$$\nabla^E_X \xi = -AX,$$

where $\nabla$ is the Euclidean connection on $E^{n+1}$ and $\nabla$ is the Levi Civita connection on $M$. The second fundamental tensor $A$ is related to the second fundamental form $h$ by $h(X, Y) = g(AX, Y)$, where $g$ is the Riemannian metric on $M$. Let $X \wedge Y$ denote the endomorphism $Z \mapsto g(Z, Y)X - g(Z, X)Y$. Then the curvature tensor $R$ of $M$ is given by the equation of Gauss:

$$R(X, Y) = AX \wedge AY.$$

Since $A$ is symmetric there exists an orthonormal frame $e_1, e_2, \ldots, e_n$ consisting of eigenvectors, i.e. such that

$$(2.1) \quad Ae_i = \lambda_i e_i,$$

$(i \in \{1, 2, \ldots, n\}$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the principal curvatures of $M$).

The hypersurface $M$ is said to be quasiumbilical when $M$ has a principal curvature with multiplicity $\geq n - 1$. Let $C$ denote the Weyl conformal curvature tensor of $M$. For $n \geq 4 M$ is conformally flat iff $C$ vanishes identically. If $n \geq 4$, E. Cartan proved that a hypersurface $M$ of $E^{n+1}$ is conformally flat if and only if it is quasiumbilical [2]. We recall that every surface is conformally flat and that for every 3-dimensional Riemannian manifold the Weyl conformal curvature tensor $C$ vanishes identically. If $n = 3$ there exist nonquasiumbilical hypersurfaces $M$ of $E^{n+1}$ which are conformally flat [5]. By Theorem 1 in [3] $M$ is conformally flat if and only if $(\lambda_i - \lambda_j) \cdot (\lambda_k - \lambda_l) = 0$ for all mutually distinct $i, j, k, l$ in $\{1, 2, \ldots, n\}$.

By (2.1) the equation of Gauss implies that

$$R(e_i, e_j) = c_{ij} e_i \wedge e_j,$$

where

$$c_{ij} = \lambda_i \lambda_j,$$

and consequently

$$C(e_i, e_j) = a_{ij} e_i \wedge e_j,$$

where

$$a_{ij} = c_{ij} - \frac{1}{n - 2} \left( \sum_{t \neq i} c_{ti} + \sum_{t \neq j} c_{tj} \right) + \frac{2}{(n - 1)(n - 2)} \sum_{t < s} c_{ts},$$

(see also [3], $i, j \in \{1, \ldots, n\}$ and $i \neq j$).

Further,

$$Qe_i = \mu_i e_i.$$
where
\[ \mu_i = \lambda_i (\text{tr} A - \lambda_i) . \]

By \( C . C = 0 \) we mean that \( C(X, Y) . C = 0 \) for all vector fields \( X \) and \( Y \) tangent to \( M \), where \( C(X, Y) \) acts as a derivation on the algebra of tensor fields on \( M \), i.e.

\[ (C(X, Y) . C)(Z, W) V = \]
\[ = [C(X, Y), C(Z, W)] V - C(C(X, Y) Z, W) V - C(Z, C(X, Y) W) V \]

for \( X, Y, Z, V, W \) tangent to \( M \).

Because this derivation commutes with contractions the implication (ii) \( \Rightarrow \) (i) holds good. Furthermore (iv) trivially implies (i), (ii) and (iii).

3. PROOF OF (i) \( \Rightarrow \) (iv)

First we state that assertion (i) implies the following: (*) for all mutually distinct \( i, j, k \): \( a_{ij}(a_{ik} - a_{jk}) = 0 \). In fact, for \( i, j, k \) mutually distinct indices, we have

\[ (C(e_i, e_j) . C)(e_i, e_k) e_k = a_{ij}(a_{jk} - a_{ik}) e_j . \]

Let \( \lambda_1, \ldots, \lambda_p \) be the (mutually) distinct eigenvalues of \( A \) with multiplicities \( s_1, \ldots, s_p \) respectively. Denote by \( V_\alpha \) the space of eigenvectors with eigenvalue \( \lambda_\alpha \). If \( e_i, e_k \in V_\alpha \) and \( e_j, e_l \in V_\beta \) for \( i \neq j \) and \( k \neq l \), then \( a_{ij} = a_{kl} \). We define numbers \( b_{\alpha \beta} = a_{ij} \), where \( i \neq j, e_i \in V_\alpha \) and \( e_j \in V_\beta (\alpha, \beta \in \{1, \ldots, p\}) \). To prove (iv) it is sufficient to show that \( b_{\alpha \beta} = 0 \) for all \( \alpha, \beta \) such that \( b_{\alpha \beta} \) is defined. In the following we will prove that the assumption \( b_{\alpha \beta} \neq 0 \) for some \( \alpha \) and \( \beta \) in \( \{1, \ldots, p\} \) always leads to a contradiction.

First we consider the case \( p \geq 4 \). If there are distinct indices \( \alpha \) and \( \beta \) such that \( b_{\alpha \beta} \neq 0 \), (*) implies that there exist indices \( \gamma \) and \( \delta \) such that \( \alpha, \beta, \gamma, \delta \) are mutually distinct and

\[ b_{\alpha \gamma} = b_{\beta \gamma}, \]

and

\[ b_{\alpha \delta} = b_{\beta \delta} . \]

This gives

\[ (\lambda_\alpha - \lambda_\beta) \left( \lambda_\gamma - \frac{1}{n - 2} \left( \text{tr} A - \lambda_\alpha - \lambda_\beta \right) \right) = 0 \]

and

\[ (\lambda_\alpha - \lambda_\beta) \left( \lambda_\delta - \frac{1}{n - 2} \left( \text{tr} A - \lambda_\alpha - \lambda_\beta \right) \right) = 0 . \]

Substraction yields

\[ (\lambda_\alpha - \lambda_\beta) (\lambda_\gamma - \lambda_\delta) = 0 , \]

which is a contradiction.
If $b_{\alpha \beta} = 0$ for all distinct indices $\alpha$ and $\beta$ in $\{1, \ldots, p\}$, we obtain (2.2) and (2.3) in a trivial way. As before, this leads to a contradiction.

Next, we treat the case $p = 3$. Suppose first that there are distinct indices $\alpha, \beta$ in $\{1, 2, 3\}$, such that $b_{\alpha \beta} \neq 0$, say $b_{12} \neq 0$. If $s_1 \geq 2$ then (*) implies
\[
b_{11} = b_{12}
\]
and
\[
b_{31} = b_{32}.
\]
As in the first paragraph this yields
\[
(\lambda_1 - \lambda_2) (\lambda_1 - \lambda_3) = 0,
\]
which gives a contradiction. The case $s_2 \geq 2$ can be handled analogously. If $s_1 = 1$ and $s_2 = 1$, (*) gives
\[
(b_{12} - b_{13}) b_{23} = 0,
\]
from which we find that
\begin{align*}
(2.4) & \quad b_{23} = 0 \\
(2.5) & \quad b_{13} = b_{12}.
\end{align*}
(2.4) yields
\[
b_{23} = \frac{-(n - 3) (\lambda_3 - \lambda_1) (\lambda_3 - \lambda_2)}{(n - 1) (n - 2)},
\]
which gives a contradiction.

From (2.5) we obtain
\[
(\lambda_2 - \lambda_3) \left( \lambda_1 - \frac{1}{n - 2} (\text{tr} A - \lambda_2 - \lambda_3) \right) = 0,
\]
and thus
\[
(n - 3) (\lambda_1 - \lambda_3) = 0,
\]
which again contradicts $\lambda_1 \neq \lambda_3$.

If $b_{12} = b_{23} = b_{13} = 0$, we obtain from $b_{12} - b_{13} = 0$ and $b_{23} - b_{13} = 0$ that
\[
(n - 2) \lambda_1 - \text{tr} A + \lambda_2 + \lambda_3 = 0
\]
and
\[
(n - 2) \lambda_3 - \text{tr} A + \lambda_1 + \lambda_2 = 0.
\]
Subtraction yields $(n - 3) (\lambda_1 - \lambda_3) = 0$, which gives a contradiction.

Suppose $p = 2$. If $s_1 = 1$ or $s_2 = 1$, [3] gives that all $b_{\alpha \beta} = 0$, which contradicts the initial assumption. Thus we may suppose $s_1 \geq 2$ and $s_2 \geq 2$. If $b_{12} \neq 0$, (*) gives
\[
b_{11} = b_{12}
\]
and
\[
b_{22} = b_{12}.
\]
This leads to a contradiction $b_{12} = 0$ gives
\[
b_{12} = -\frac{(s_1 - 1)(s_2 - 1)(\lambda_1 - \lambda_2)^2}{(n - 1)(n - 2)} = 0,
\]
which is in contradiction with $\lambda_1 \neq \lambda_2$.

If $p = 1$, then all $b_{a\beta} = 0$, which gives again a contradiction.

4. PROOF OF (iii) ⇒ (iv)

The condition $Q \cdot C = 0$ implies that: (***) for all distinct $i, j \in \{1, \ldots, n\}$:
\[
\lambda_i(\text{tr } A - \lambda_i) a_{ij} = 0. \text{ Indeed we have}
\]
\[
(Q \cdot C) (e_i, e_j) e_i = 2\mu_i a_{ij} e_j,
\]
where $i, j \in \{1, \ldots, n\}$ and $i \neq j$.

We use the same conventions concerning the eigenvalues of $A$ as in Sec. 3 and we define numbers $b_{a\beta}$ in the same way.

First we consider the case that $\lambda_x \neq 0$ and $\lambda_x \neq \text{tr } A$ for all $x$ in $\{1, \ldots, p\}$.

(***), implies that all $b_{a\beta} = 0$.

Suppose that there are distinct $x$ and $\beta$ in $\{1, \ldots, p\}$ such that $\lambda_x = 0$ and $\lambda_\beta = = \text{tr } A$, say $\lambda_1 = 0$ and $\lambda_2 = \text{tr } A$. If $p \geq 3$, then (**) yields $a_{13} = a_{23} = 0$. From $a_{13} - a_{23} = 0$, we obtain
\[
(\lambda_2 - \lambda_1) ((n - 2) \lambda_3 - \text{tr } A + \lambda_2 + \lambda_1) = 0,
\]
which gives
\[
(n - 2) \lambda_3 = 0.
\]
This is in contradiction with $\lambda_1 \neq \lambda_3$. If $p = 2$, we have $\lambda_2 = \text{tr } A = s_2 \lambda_2$. Since $\lambda_2 \neq 0$, this yields $s_2 = 1$. This implies that all $b_{a\beta} = 0$.

If there is an $\alpha$ in $\{1, \ldots, p\}$ such that $\lambda_\alpha = 0$ and $\lambda_\beta \neq \text{tr } A$ for all $\beta \neq \alpha$ in $\{1, \ldots, p\}$ or if there is an $\alpha$ in $\{1, \ldots, p\}$ such that $\lambda_\alpha = \text{tr } A$ and $\lambda_\beta \neq 0$ for all $\beta \neq \alpha$ in $\{1, \ldots, p\}$, then $b_{a\beta} = 0$ for all $x$ in $\{1, \ldots, p\}$ and all $\beta$ in $\{2, \ldots, p\}$ for which $b_{a\beta}$ exists. If $p \geq 3$, then we obtain from $b_{12} - b_{13} = 0$ and $b_{12} - b_{23} = 0$ that
\[
(n - 2) \lambda_1 - \text{tr } A + \lambda_2 + \lambda_3 = 0
\]
and
\[
(n - 2) \lambda_2 - \text{tr } A + \lambda_1 + \lambda_3 = 0.
\]
Substraction yields
\[
(n - 3) (\lambda_1 - \lambda_2) = 0,
\]
which gives a contradiction.

We consider the case $p = 2$. If $s_2 = 1$ [3] implies that all $b_{a\beta} = 0$. If $s_2 \geq 2$, then (**) yields $b_{12} = b_{22}$. This gives
\[
(\lambda_2 - \lambda_1) (\lambda_2 (n - 2) - \text{tr } A + \lambda_1 + \lambda_2) = 0,
\]
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and thus

\[(s_1 - 1)(\lambda_2 - \lambda_1) = 0.\]

This gives a contradiction.

The case \(p = 1\) is trivial. This completes the proof of the theorem.

\section*{Bibliography}

[1] D. E. Blair, P. Verheyen and L. Verstraelen: Hypersurfaces satisfaisant à \(R \cdot C = 0\) ou \(C \cdot R = 0\), to appear.


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