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Czechoslovak Mathematical Journal, Vol. 35 (1985), No. 1, 167–172

Persistent URL: http://dml.cz/dmlcz/102006

Czechoslovak Mathematical Journal, 35 (110) 1985, Praha

ON A POWER OF RELATIONAL STRUCTURES

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(Received May 28, 1984)

The aim of this paper is to define direct operations and an operation of a power for relational structures and to prove their properties. In particular, a power satisfies the expected rules with the exception of $(\boldsymbol{G}^{\boldsymbol{H}})^{\boldsymbol{K}} \simeq \boldsymbol{G}^{\boldsymbol{H}.\boldsymbol{K}}$. We derive sufficient conditions for the validity of that law.

1. Let $I \neq \emptyset$ be a set, let n_i be a positive integer for any $i \in I$. A family $(n_i;\ i \in I)$ will be called a *type*. The types $(n_i;\ i \in I)$, $(m_j;\ j \in J)$ are *similar* iff there exists a bijection $\varphi: I \to J$ such that $m_{\varphi(i)} = n_i$ for all $i \in I$.

2. Definition. Let $G \neq \emptyset$ be a set, let $(n_i;\ i \in I)$ be a type. Let C_i be an n_i-ary relation on the set G for any $i \in I$, i.e. $C_i \subseteq G^{n_i}$. Then $\boldsymbol{G} = (G, (C_i;\ i \in I))$ is called a *relational structure* of type $(n_i;\ i \in I)$.

If $\boldsymbol{G} = (G, (C_i;\ i \in I))$ is a relational structure, then the set G is called a *carrier* of $\boldsymbol{G}$ and C_i are called *relations* of $\boldsymbol{G}$. Sometimes we denote by $\mathscr{R}_i(\boldsymbol{G})$ the i^{th} relation of $\boldsymbol{G} = (G, (C_i;\ i \in I))$, i.e. $\mathscr{R}_i(\boldsymbol{G}) = C_i$.

Two relational structures $\boldsymbol{G} = (G, (C_i;\ i \in I))$ and $\boldsymbol{H} = (H, (D_j;\ j \in J))$ of types $(n_i;\ i \in I)$ and $(m_j;\ j \in J)$, respectively, are called *similar* iff their types $(n_i;\ i \in I)$ and $(m_j;\ j \in J)$ are similar.

If $\boldsymbol{G} = (G, (C_i;\ i \in I)$, $\boldsymbol{H} = (H, (D_j;\ j \in J))$ are similar relational structures, then we can assume without loss of generality that $I = J$ and that the mapping φ in Sec. 1 is an identity on I, i.e. that $m_i = n_i$ for all $i \in I$.

3. Definition. Let $\boldsymbol{G} = (G, (C_i;\ i \in I))$, $\boldsymbol{H} = (H, (D_i;\ i \in I))$ be similar relational structures of type $(n_i;\ i \in I)$. Let $f: G \to H$ be a mapping which has the following property: for any $i \in I$ and any $x_1, \ldots, x_{n_i} \in G$ the implication $(x_1, \ldots, x_{n_i}) \in C_i \Rightarrow$ $\Rightarrow (f(x_1), \ldots, f(x_{n_i})) \in D_i$ holds. Then f is called a *homomorphism* of the relational structure $\boldsymbol{G}$ into the relational structure $\boldsymbol{H}$.

We denote by Hom $(\boldsymbol{G}, \boldsymbol{H})$ the set of all homomorphisms of $\boldsymbol{G}$ into $\boldsymbol{H}$.

A bijective homomorphism f of $\boldsymbol{G}$ onto $\boldsymbol{H}$ such that f^{-1} is a homomorphism of $\boldsymbol{H}$ onto $\boldsymbol{G}$ is called an *isomorphism* of $\boldsymbol{G}$ onto $\boldsymbol{H}$. Two similar relational structures $\boldsymbol{G}, \boldsymbol{H}$ are called *isomorphic* iff there exists an isomorphism of $\boldsymbol{G}$ onto $\boldsymbol{H}$; in that case we write $\boldsymbol{G} \simeq \boldsymbol{H}$.

4. Definition. Let $K \neq \emptyset$ be a set, let $(G_k;\ k \in K)$ be a family of similar relational structures of type $(n_i;\ i \in I)$. Let $\boldsymbol{G}_k = (G_k, (C_{ik};\ i \in I))$ for any $k \in K$ and let $G_{k_1} \cap G_{k_2} = \emptyset$ for $k_1, k_2 \in K$, $k_1 \neq k_2$. The *direct sum* $\sum_{k \in K} \boldsymbol{G}_k$ of the family $(\boldsymbol{G}_k;\ k \in K)$ is the relational structure $\boldsymbol{G} = (G, (C_i;\ i \in I))$ of type $(n_i;\ i \in I)$ for which $G = \bigcup_{k \in K} G_k$ and $C_i = \bigcup_{k \in K} C_{ik}$ for any $i \in I$.

If $K = \{1, \ldots, n\}$ then we write $\sum_{k \in K} \boldsymbol{G}_k = \boldsymbol{G}_1 + \ldots + \boldsymbol{G}_n$.

5. Remark. Let $(\boldsymbol{G}_k;\ k \in K) = ((G_k, (C_{ik};\ i \in I));\ k \in K)$ be a family of similar relational structures and let $\boldsymbol{G} = (G, (C_i;\ i \in I)) = \sum_{k \in K} \boldsymbol{G}_k$. Then the canonical insertion $j_k\colon G_k \to G$ defined by $j_k(x) = x$ for $x \in G_k$ is an isomorphic embedding of $\boldsymbol{G}_k$ into $\boldsymbol{G}$.

6. Let $\boldsymbol{G} = (G, (C_i;\ i \in I))$, $\boldsymbol{H} = (G, (D_i;\ i \in I))$ be similar relational structures of type $(n_i;\ i \in I)$ with the same carrier G. Put $\boldsymbol{G} \prec \boldsymbol{H}$ iff $C_i \subseteq D_i$ for all $i \in I$. Clearly $\prec$ is a (partial) order on the class of all relational structures of type $(n_i;\ i \in I)$ with the same carrier G.

7. Lemma. *Let $(\boldsymbol{G}_k;\ k \in K) = ((G_k, (C_{ik};\ i \in I));\ k \in K)$ be a family of similar relational structures of type $(n_i; i \in I)$ with $G_{k_1} \cap G_{k_2} = \emptyset$ for $k_1 \neq k_2$ and let $\boldsymbol{G} = (G, (C_i;\ i \in I)) = \sum_{k \in K} \boldsymbol{G}_k$. Then $\boldsymbol{G}$ is the least element (with respect to $\prec$) in the class of such relational structures $\boldsymbol{H}$ of type $(n_i;\ i \in I)$ and with carrier G, for which all canonical insertions j_k $(k \in K)$ are homomorphisms of $\boldsymbol{G}_k$ into $\boldsymbol{H}$.*

Proof. By Sec. 5 all canonical insertions $j_k\colon G_k \to G$ are homomorphisms of $\boldsymbol{G}_k$ into $\boldsymbol{G}$. Let $\boldsymbol{H} = (G, (D_i;\ i \in I))$ be a relational structure of type $(n_i;\ i \in I)$ with carrier G and such that all canonical insertions j_k are homomorhisms of $\boldsymbol{G}_k$ into $\boldsymbol{H}$. Let $i \in I$ and let $x_1, \ldots, x_{n_i} \in G$ be such elements that $(x_1, \ldots, x_{n_i}) \in C_i$. Then there exists $k \in K$ such that $x_1, \ldots, x_{n_i} \in G_k$ and $(x_1, \ldots, x_{n_i}) \in C_{ik}$. By assumption then $(x_1, \ldots, x_{n_i}) = (j_k(x_1), \ldots, j_k(x_{n_i})) \in D_i$. Thus $C_i \subseteq D_i$ for all $i \in I$ and $\boldsymbol{G} \prec \boldsymbol{H}$.

8. Definition. Let $K \neq \emptyset$ be a set, let $(\boldsymbol{G}_k; k \in K) = ((G_k, (C_{ik};\ i \in I));\ k \in K)$ be a family of similar relational structures of type $(n_i;\ i \in I)$. The *direct product* $\prod_{k \in K} \boldsymbol{G}_k$ of the family $(\boldsymbol{G}_k;\ k \in K)$ is the relational structure $\boldsymbol{G} = (G, (C_i;\ i \in I))$ of type $(n_i;\ i \in I)$ for which $G = \mathsf{X}_{k \in K} G_k$ and $C_i = \mathsf{X}_{k \in K} C_{ik}$ for any $i \in I$.

Note that $\mathsf{X}_{k \in K} G_k$ means here the cartesian product of sets and $\mathsf{X}_{k \in K} C_{ik}$ means the direct product of relations, i.e. if $x_1, \ldots, x_{n_i} \in \mathsf{X}_{k \in K} G_k$, then $(x_1, \ldots, x_{n_i}) \in \mathsf{X}_{k \in K} C_{ik}$ is equivalent to $(\mathrm{pr}_k\, x_1, \ldots, \mathrm{pr}_k\, x_{n_i}) \in C_{ik}$ for all $k \in K$. If $K = \{1, \ldots, n\}$, then we write $\prod_{k \in K} \boldsymbol{G}_k = \boldsymbol{G}_1 \ldots \boldsymbol{G}_n$.

9. Lemma. *Let $(\boldsymbol{G}_k;\ k \in K) = ((G_k, (C_{ik}; i \in I));\ k \in K)$ be a family of similar* *relational structures of type $(n_i;\ i \in I)$ and let $\boldsymbol{G} = (G, (C_i;\ i \in I)) = \prod_{k \in K} \boldsymbol{G}_k$. Then $\boldsymbol{G}$ is the greatest element (with respect to $\prec$) in the class of such relational structures $\boldsymbol{H}$ of type $(n_i;\ i \in I)$ and with carrier G, for which all projections $\mathrm{pr}_k (k \in K)$ are homomorphisms of $\boldsymbol{H}$ onto $\boldsymbol{G}_k$.*

Proof. From the definition of the direct product it follows directly that any projection pr_k is a homomorphism of $\boldsymbol{G}$ onto $\boldsymbol{G}_k$. Let $\boldsymbol{H} = (G, (D_i;\ i \in I))$ be a relational structure of type $(n_i;\ i \in I)$ and with carrier G such that all projections $\mathrm{pr}_k\ (k \in K)$ are homomorphisms of $\boldsymbol{H}$ onto $\boldsymbol{G}_k$. Let $i \in I$ and let $x_1, \ldots, x_{n_i} \in G$ be such elements that $(x_1, \ldots, x_{n_i}) \in D_i$. Then by the assumption $(\mathrm{pr}_k\, x_1, \ldots, \mathrm{pr}_k\, x_{n_i}) \in C_{ik}$ for all $k \in K$ and this implies by Sec. 8 $(x_1, \ldots, x_{n_i}) \in C_i$. Thus $D_i \subseteq C_i$ for all $i \in I$ and $\boldsymbol{H} \prec \boldsymbol{G}$.

10. Definition. Let $\boldsymbol{G} = (G, (C_i;\ i \in I))$, $\boldsymbol{H} = (H, (D_i;\ i \in I))$ be similar relational structures of type $(n_i;\ i \in I)$. The *power* $\boldsymbol{G}^{\boldsymbol{H}}$ is the relational structure $\boldsymbol{K} = (K, (E_i;\ i \in I))$ of type $(n_i;\ i \in I)$ for which $K = \mathrm{Hom}(\boldsymbol{H}, \boldsymbol{G})$ and for any $i \in I$, $f_1, \ldots, f_{n_i} \in K$ we have $(f_1, \ldots, f_{n_i}) \in E_i$ iff $(f_1(x), \ldots, f_{n_i}(x)) \in C_i$ for all $x \in H$.

11. Theorem. *Let $K \neq \emptyset$ be a set, let $(\boldsymbol{G}_k;\ k \in K) = ((G_k, (C_{ik};\ i \in I));\ k \in K)$ be a family of similar relational structures of type $(n_i;\ i \in I)$ and let $\boldsymbol{H} = (H, (D_i;\ i \in I))$ be a relational structure of type $(n_i;\ i \in I)$. Then*

$$\Big(\prod_{k \in K} \boldsymbol{G}_k\Big)^{\boldsymbol{H}} \simeq \prod_{k \in K} \boldsymbol{G}_k^{\boldsymbol{H}}.$$

Proof. For any $f \in \mathrm{Hom}(\boldsymbol{H}, \prod_{k \in K} \boldsymbol{G}_k)$ and any $k \in K$ denote $f_k = \mathrm{pr}_k f$. We easily see that $f_k \in \mathrm{Hom}(\boldsymbol{H}, \boldsymbol{G}_k)$. On the other hand, if $f_k \in \mathrm{Hom}(\boldsymbol{H}, \boldsymbol{G}_k)$ for all $k \in K$, then $f = \mathsf{X}_{k \in K} f_k \in \mathrm{Hom}(\boldsymbol{H}, \prod_{k \in K} \boldsymbol{G}_k)$. This shows that the correspondence $f \to (f_k;\ k \in K)$ is a bijective mapping of $\mathrm{Hom}(\boldsymbol{H}, \prod_{k \in K} \boldsymbol{G}_k)$ onto $\mathsf{X}_{k \in K} \mathrm{Hom}(\boldsymbol{H}, \boldsymbol{G}_k)$. We prove that this mapping is an isomorphism of $(\prod_{k \in K} \boldsymbol{G}_k)^{\boldsymbol{H}}$ onto $\prod_{k \in K} \boldsymbol{G}_k^{\boldsymbol{H}}$. Let $i \in I$, $f_1, \ldots, f_{n_i} \in \mathrm{Hom}(\boldsymbol{H}, \prod_{k \in K} \boldsymbol{G}_k)$ and $(f_1, \ldots, f_{n_i}) \in \mathscr{R}_i(\prod_{k \in K} \boldsymbol{G}_k)^{\boldsymbol{H}}$. Then $(f_1(x), \ldots, f_{n_i}(x)) \in \mathscr{R}_i(\prod_{k \in K} \boldsymbol{G}_k)$ for all $x \in H$ so that $(\mathrm{pr}_k f_1(x), \ldots, \mathrm{pr}_k f_{n_i}(x)) \in C_{ik}$ for all $k \in K$ and all $x \in H$, i.e. $((f_1)_k(x), \ldots, (f_{n_i})_k(x)) \in C_{ik}$ for all $k \in K$ and all $x \in H$ and this implies $(((f_1)_k, \ldots, (f_{n_i})_k;\ k \in K) \in \mathscr{R}_i(\prod_{k \in K} \boldsymbol{G}_k^{\boldsymbol{H}})$. We have shown that a mapping $f \to (f_k;\ k \in K) = (\mathrm{pr}_k f;\ k \in K)$ is a homomorphism of $(\prod_{k \in K} \boldsymbol{G}_k)^{\boldsymbol{H}}$ onto $\prod_{k \in K} \boldsymbol{G}_k^{\boldsymbol{H}}$. However, the last consideration can be reversed and thus this mapping is an isomorphism.

12. Theorem. *Let $\boldsymbol{G} = (G, (C_i;\ i \in I)$ be a relational structure of type $(n_i;\ i \in I)$, let $(\boldsymbol{H}_k;\ k \in K) = ((H_k, (D_{ik};\ i \in I);\ k \in K)$ be a family of relational structures of type $(n_i;\ i \in I)$ and let $H_{k_1} \cap H_{k_2} = \emptyset$ for $k_1, k_2 \in K$, $k_1 \neq k_2$. Then*

$$\boldsymbol{G}^{\sum_{k \in K} \boldsymbol{H}_k} \simeq \prod_{k \in K} \boldsymbol{G}^{\boldsymbol{H}_k}.$$

Proof. Let $f \in \operatorname{Hom}(\sum_{k\in K} \boldsymbol{H}_k, \boldsymbol{G})$ be any element and let $k \in K$. We denote by f_k the restriction of f onto H_k, i.e. $f_k = f \cap (H_k \times G)$. Then clearly $f_k \in \operatorname{Hom}(\boldsymbol{H}_k, \boldsymbol{G})$. Conversely, if $f_k \in \operatorname{Hom}(\boldsymbol{H}_k, \boldsymbol{G})$ for all $k \in K$, then $f = \bigcup_{k\in K} f_k \in \operatorname{Hom}(\sum_{k\in K} \boldsymbol{H}_k, \boldsymbol{G})$. Thus, the correspondence $f \to (f_k;\ k \in K)$ is a bijective mapping of the set $\operatorname{Hom}(\sum_{k\in K} \boldsymbol{H}_k, \boldsymbol{G})$ onto the set $\mathsf{X}_{k\in K} \operatorname{Hom}(\boldsymbol{H}_k, \boldsymbol{G})$. We show that this mapping is an isomorphism of $\boldsymbol{G}^{\sum_{k\in K} \boldsymbol{H}_k}$ onto $\prod_{k\in K} \boldsymbol{G}^{\boldsymbol{H}_k}$. Let $i \in I$, $f_1, \ldots, f_{n_i} \in \operatorname{Hom}(\sum_{k\in K} \boldsymbol{H}_k, \boldsymbol{G})$ and $(f_1, \ldots, f_{n_i}) \in \mathscr{R}_i(\boldsymbol{G}^{\sum_{k\in K} \boldsymbol{H}_k})$. Then $(f_1(x), \ldots, f_{n_i}(x)) \in C_i$ for all $x \in \bigcup_{k\in K} H_k$, so that $((f_1)_k(x), \ldots, (f_{n_i})_k(x)) \in C_i$ for all $k \in K$ and all $x \in H_k$, which implies $((f_1)_k, \ldots$ $\ldots, (f_{n_i})_k) \in \mathscr{R}_i(\boldsymbol{G}^{\boldsymbol{H}_k})$ for all $k \in K$ and $(((f_1)_k, \ldots, (f_{n_i})_k;\ k \in K) \in \mathscr{R}_i(\prod_{k\in K} \boldsymbol{G}^{\boldsymbol{H}_k})$. We have proved that the mapping $f \to (f_k;\ k \in K)$ is a homomorphism of $\boldsymbol{G}^{\sum_{k\in K} \boldsymbol{H}_k}$ onto $\prod_{k\in K} \boldsymbol{G}^{\boldsymbol{H}_k}$. By a reverse argument we show that the inverse mapping is a homomorphism of $\prod_{k\in K} \boldsymbol{G}^{\boldsymbol{H}_k}$ onto $\boldsymbol{G}^{\sum_{k\in K} \boldsymbol{H}_k}$ and the theorem is proved.

Let G be a set and C an n-ary relation on G. We call this relation *weakly reflexive* iff $(x, x, \ldots, x) \in C$ for any $x \in G$. Note that if C is unary, then C is weakly reflexive iff $C = G$ and if C is binary, then weak reflexivity of C denotes the reflexivity in the usual sense.

13. Theorem. *Let* $\boldsymbol{G} = (G, (C_i;\ i \in I))$, $\boldsymbol{H} = (H, (D_i;\ i \in I))$, $\boldsymbol{K} = (K, (E_i;\ i \in I))$ *be similar relational structures of type* $(n_i;\ i \in I)$. *Let all relations* D_i *and all relations* E_i $(i \in I)$ *be weakly reflexive. Then there exists an isomorphic embedding of the relational structure* $\boldsymbol{G}^{\boldsymbol{H}.\boldsymbol{K}}$ *into the relational structure* $(\boldsymbol{G}^{\boldsymbol{H}})^{\boldsymbol{K}}$.

Proof. Let $f \in \operatorname{Hom}(\boldsymbol{H} . \boldsymbol{K}, \boldsymbol{G})$ be any element, thus $f: H \times K \to G$. For any $y \in K$ denote by f_y the mapping $f_y: H \to G$ defined by $f_y = f(\cdot, y)$, i.e. $f_y(x) =$ $= f(x, y)$ for $x \in H$. We show that $f_y \in \operatorname{Hom}(\boldsymbol{H}, \boldsymbol{G})$. Let $i \in I$, $x_1, \ldots, x_{n_i} \in H$ and $(x_1, \ldots, x_{n_i}) \in D_i$. Then $((x_1, y), \ldots, (x_{n_i}, y)) \in \mathscr{R}_i(\boldsymbol{H} . \boldsymbol{K})$ so that $(f(x_1, y), \ldots$ $\ldots, f(x_{n_i}, y)) \in C_i$, i.e. $(f_y(x_1), \ldots, f_y(x_{n_i})) \in C_i$. Thus, $f_y \in \operatorname{Hom}(\boldsymbol{H}, \boldsymbol{G})$. Further, let $x \in H$ be any element, $i \in I$ and $y_1, \ldots, y_{n_i} \in K$, $(y_1, \ldots, y_{n_i}) \in E_i$. Then $((x, y_1), \ldots$ $\ldots, (x, y_{n_i})) \in \mathscr{R}_i(\boldsymbol{H} . \boldsymbol{K})$ so that $(f(x, y_1), \ldots, f(x, y_{n_i}) \in C_i$, i.e. $(f_{y_1}(x), \ldots, f_{yn_i}(x)) \in$ $\in C_i$. This shows that $(f_{y_1}, \ldots, f_{yn_i}) \in \mathscr{R}_i(\boldsymbol{G}^{\boldsymbol{H}})$ so that the mapping $y \to f_y$ is a homomorphism of $\boldsymbol{K}$ into $\boldsymbol{G}^{\boldsymbol{H}}$, i.e. an element of the set $\operatorname{Hom}(\boldsymbol{K}, \boldsymbol{G}^{\boldsymbol{H}})$. Thus, if we write $\varphi(f)(y) = f_y$ for any $f \in \operatorname{Hom}(\boldsymbol{H} . \boldsymbol{K}, \boldsymbol{G})$ and any $y \in K$, then $\varphi: \operatorname{Hom}(\boldsymbol{H} . \boldsymbol{K}, \boldsymbol{G}) \to$ $\to \operatorname{Hom}(\boldsymbol{K}, \boldsymbol{G}^{\boldsymbol{H}})$. We show that φ is an isomorphic embedding of $\boldsymbol{G}^{\boldsymbol{H}.\boldsymbol{K}}$ into $(G^{\boldsymbol{H}})^{\boldsymbol{K}}$. Let $f, g \in \operatorname{Hom}(\boldsymbol{H} . \boldsymbol{K}, \boldsymbol{G})$ and $f \neq g$. Then there exists $(x, y) \in H \times K$ with $f(x, y) \neq g(x, y)$. Thus $f_y(x) \neq g_y(x)$ for some $y \in K$ and some $x \in H$, so that $f_y \neq g_y$ for some $y \in K$ and $\varphi(f) \neq \varphi(g)$. Hence φ is injective. Let $i \in I$, $f_1, \ldots, f_{n_i} \in$ $\in \operatorname{Hom}(\boldsymbol{H} . \boldsymbol{K}, \boldsymbol{G})$, $(f_1, \ldots, f_{n_i}) \in \mathscr{R}_i(\boldsymbol{G}^{\boldsymbol{H}.\boldsymbol{K}})$. Then $(f_1(x, y), \ldots, f_{n_i}(x, y)) \in C_i$ for all $(x, y) \in H \times K$, thus $((f_1)_y(x), \ldots, (f_{n_i})_y(x)) \in C_i$ for all $x \in H$ and all $y \in K$, which implies $((f_1)_y, \ldots, (f_{n_i})_y) \in \mathscr{R}_i(\boldsymbol{G}^{\boldsymbol{H}})$ for all $y \in K$ and hence $(\varphi(f_1), \ldots, \varphi(f_{n_i})) \in \mathscr{R}_i(\boldsymbol{G}^{\boldsymbol{H}})^{\boldsymbol{K}}$.

Conversely, if $(\varphi(f_1), \ldots, \varphi(f_{n_i})) \in \mathscr{R}_i((\boldsymbol{G}^{\boldsymbol{H}})^{\boldsymbol{K}})$, then by the reverse argument we find that $(f_1, \ldots, f_{n_i}) \in \mathscr{R}_i(\boldsymbol{G}^{\boldsymbol{H}.\boldsymbol{K}})$. Thus φ is an isomorphic embedding of $\boldsymbol{G}^{\boldsymbol{H}.\boldsymbol{K}}$ into $(\boldsymbol{G}^{\boldsymbol{H}})^{\boldsymbol{K}}$.

14. Let $G \neq \emptyset$ be a set, C an n-ary relation on G. We say that C has the *diagonal property* iff the following holds: For any family $(x_{ik};\ i, k = 1, \ldots, n)$ of elements of G such that $(x_{i1}, x_{i2}, \ldots, x_{in}) \in C$ for all $i = 1, \ldots, n$ and $(x_{1k}, x_{2k}, \ldots, x_{nk}) \in C$ for all $k = 1, \ldots, n$ we have $(x_{11}, x_{22}, \ldots, x_{nn}) \in C$.

In other words, if in the matrix

$$\begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{pmatrix}$$

all rows and all columns are in the relation C, then its diagonal is also in the relation C.

15. Examples. (1) Any unary relation on a set G has the diagonal property.

(2) Let C be a binary relation on a set G. Then C has the diagonal property iff C is transitive.

Proof. If C is transitive and $x_{11}, x_{12}, x_{21}, x_{22} \in G$ are elements satisfying the condition in Sec. 14, then in particular $(x_{11}, x_{12}) \in C$, $(x_{12}, x_{22}) \in C$ and transitivity of C yields $(x_{11}, x_{22}) \in C$. Conversely, if C has the diagonal property and $x, y, z \in G$, $(x, y) \in C$, $(y, z) \in C$, then the matrix

$$\begin{pmatrix} x & y \\ y & z \end{pmatrix}$$

satisfies the condition of Sec. 14 and thus $(x, z) \in C$.

(3) As an example of a ternary relation with the diagonal property, let $(G, <)$ be an ordered set and let C be the ternary relation on G given by $(x, y, z) \in C \Leftrightarrow x < < y < z$. More generally, if C is a transitive binary relation on a set G and $n \geqq 3$, then the n-ary relation D on G given by $(x_1, \ldots, x_n) \in D$ iff $(x_i, x_{i+1}) \in C$ for $i = = 1, \ldots, n - 1$ has the diagonal property.

16. Theorem. *Let* $\boldsymbol{G} = (G, (C_i;\ i \in I))$, $\boldsymbol{H} = (H, (D_i;\ i \in I))$, $\boldsymbol{K} = (K, (E_i;\ i \in I))$ *be similar relational structures of type* $(n_i;\ i \in I)$. *Let all relations* D_i *and all relations* E_i $(i \in I)$ *be weakly reflexive and let all relations* C_i $(i \in I)$ *have the diagonal property. Then* $(\boldsymbol{G}^{\boldsymbol{H}})^{\boldsymbol{K}} \simeq \boldsymbol{G}^{\boldsymbol{H}.\boldsymbol{K}}$.

Proof. By the proof of Sec. 13, the mapping $\varphi : \mathrm{Hom}\,(\boldsymbol{H}\,.\,\boldsymbol{K}, \boldsymbol{G}) \to \mathrm{Hom}\,(\boldsymbol{K}, \boldsymbol{G}^{\boldsymbol{H}})$, where $\varphi(f)(y) = f_y$, is an isomorphic embedding of the relational structure $\boldsymbol{G}^{\boldsymbol{H}.\boldsymbol{K}}$ into the relational structure $(\boldsymbol{G}^{\boldsymbol{H}})^{\boldsymbol{K}}$. Thus, it suffices to show hat φ is a surjective mapping. Let $g \in \mathrm{Hom}\,(\boldsymbol{K}, \boldsymbol{G}^{\boldsymbol{H}})$ be any element. Put $f(x, y) = g(y)(x)$ for any $x \in H$, $y \in K$. We show that $f \in \mathrm{Hom}\,(\boldsymbol{H}\,.\,\boldsymbol{K}, \boldsymbol{G})$. Let $i \in I$, $x_1, \ldots, x_{n_i} \in H$, $y_1, \ldots, y_{n_i} \in \in K$, $((x_1, y_1), \ldots, (x_{n_i}, y_{n_i})) \in \mathscr{R}_i(\boldsymbol{H}\,.\,\boldsymbol{K})$. Then $(x_1, \ldots, x_{n_i}) \in D_i$, $(y_1, \ldots, y_{n_i}) \in E_i$ and hence

$$((x_j, y_1), \ldots, (x_j, y_{n_i})) \in \mathscr{R}_i(\boldsymbol{H} \,.\, \boldsymbol{K}) \quad \text{for all} \quad j = 1, \ldots, n_i\,,$$

$$((x_1, y_k), \ldots, (x_{n_i}, y_k)) \in \mathscr{R}_i(\boldsymbol{H} \,.\, \boldsymbol{K}) \quad \text{for all} \quad k = 1, \ldots, n_i\,.$$

As $g \in \operatorname{Hom}(\boldsymbol{K}, \boldsymbol{G}^{\boldsymbol{H}})$, we have $(g(y_1), \ldots, g(y_{n_i})) \in \mathscr{R}_i(\boldsymbol{G}^{\boldsymbol{H}})$, so that $(g(y_1)(x), \ldots \ldots, g(y_{n_i})(x)) \in C_i$ for all $x \in H$, in particular $(g(y_1)(x_j), \ldots, g(y_{n_i})(x_j)) \in C_i$ for all $j = 1, \ldots, n_i$. $(*)$ Further, $g(y) \in \operatorname{Hom}(\boldsymbol{H}, \boldsymbol{G})$ for any $y \in K$, in particular $g(y_k) \in \operatorname{Hom}(\boldsymbol{H}, \boldsymbol{G})$ for all $k = 1, \ldots, n_i$, Consequently, we have $(g(y_k)(x_1), \ldots, g(y_k)(x_{n_i})) \in C_i$ for all $k = 1, \ldots, n_i$. $(**)$ As C_i has the diagonal property, $(*)$ and $(**)$ yield $(g(y_1)(x_1), g(y_2)(x_2), \ldots, g(y_{n_i})(x_{n_i})) \in C_i$, i.e. $(f(x_1, y_1), \ldots, f(x_{n_i}, y_{n_i})) \in C_i$. Thus $f \in \operatorname{Hom}(\boldsymbol{H} \,.\, \boldsymbol{K}, \boldsymbol{G})$ and the definition of the mapping φ implies $\varphi(f) = g$.

Let us call a set with one binary relation a *binary structure*. Such a structure can be called reflexive or transitive iff its relation is reflexive or transitive, respectively. From Secs. 13 and 16 we immediately obtain

17. Corollary. 1. *Let $\boldsymbol{G}, \boldsymbol{H}, \boldsymbol{K}$ be binary structures and let $\boldsymbol{H}, \boldsymbol{K}$ be reflexive. Then there exists an isomorphic embedding of the binary structure $\boldsymbol{G}^{\boldsymbol{H}.\boldsymbol{K}}$ into the binary structure $(\boldsymbol{G}^{\boldsymbol{H}})^{\boldsymbol{K}}$.*

2. *Let $\boldsymbol{G}, \boldsymbol{H}, \boldsymbol{K}$ be binary structures. Let $\boldsymbol{H}, \boldsymbol{K}$ be reflexive and let $\boldsymbol{G}$ be transitive. Then $(\boldsymbol{G}^{\boldsymbol{H}})^{\boldsymbol{K}} \simeq \boldsymbol{G}^{\boldsymbol{H}.\boldsymbol{K}}$.*

3. *Let $\boldsymbol{G}, \boldsymbol{H}, \boldsymbol{K}$ be quasiordered sets. Then $(\boldsymbol{G}^{\boldsymbol{H}})^{\boldsymbol{K}} \simeq \boldsymbol{G}^{\boldsymbol{H}.\boldsymbol{K}}$.*

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