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SPECTRAL REPRESENTATION OF LOCAL SEMIGROUPS
IN LOCALLY CONVEX SPACES

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1. INTRODUCTION

A classical problem in the theory of semigroups of continuous linear operators acting in a Hilbert space is to determine when the operators have a joint spectral integral representation. Devinatz [2] and Nussbaum [10], [11] extended the notion of semigroup to include certain one-parameter families of unbounded (symmetric or self-adjoint) operators acting in a Hilbert space, such as the Riesz potential operators in $L^2(\mathbb{R}^n)$ [11], which have the semigroup property and are weakly continuous on a suitable subspace. Their results yield various integral representations of such a one-parameter family; see also the recent paper [7].

Examples of one-parameter families of unbounded linear operators which have the semigroup property are also encountered in spaces other than Hilbert space. A classical example is the Riemann-Liouville fractional integral in $L^p((0, \infty))$, $1 < p < \infty$, [5]. Accordingly, criteria which yield integral representations of more general one-parameter families of operators are of interest. Such a criterion was recently established by Kantorovitz and Hughes [6] for one-parameter families acting in a reflexive Banach space.

The purpose of this note is to reformulate the criterion of Kantorovitz and Hughes so that it applies to one-parameter families of operators acting in more general spaces. This so extended criterion is based on a characterization of Fourier-Stieltjes transforms of vector measures analogous to the well known Bochner-Schoenberg test.

More precisely, let $X$ be a locally convex space. If $D$ is a dense subspace of $X$, denote by $\Pi(D)$ the algebra of all linear transformations with domain $D$ and range contained in $D$. Let $\Delta = [0, \alpha)$, where $0 < \alpha \leq \infty$. The system $\{T; D; \Delta\}$ is called a local semigroup on $\Delta$ if $T: \Delta \to \Pi(D)$ is a map such that $T(0)$ is the identity operator on $D$, $T(s + t) = T(s) T(t)$ whenever $s, t, s + t \in \Delta$, and $T(\cdot)(x)$ is a weakly continuous, $X$-valued function on $\Delta$, for each $x \in D$. This is essentially the definition given in [6].

A characterization will be presented of those local semigroups $\{T; D; \Delta\}$ for which there exists an equicontinuous spectral measure $P$, defined on the Borel $\sigma$-algebra $\mathcal{B}$...
of the real line $\mathbb{R}$, such that for each $x \in D$, the $X$-valued measure $E \mapsto P(E)(x)$, $E \in \mathcal{B}$, has compact support and

$$T(t)(x) = \int_{\mathbb{R}} e^{-ist} dP(s)(x), \quad t \in \Delta.$$ 

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## 2. PRELIMINARIES AND NOTATION

Throughout this note $X$ will denote a quasi-complete, locally convex Hausdorff space. If $A$ is a subset of $X$, then $bco(A)$ denotes the convex, balanced hull of $A$. Its closure is denoted by $\overline{bco}(A)$. The space of all continuous linear functionals on $X$ is denoted by $X^\prime$.

Let $C$ denote the complex number field. An entire function $f: \mathbb{C} \to X$ is said to be of exponential type if there exists $\beta > 0$ such that for every $\varepsilon > 0$ the set

$$\{e^{-(\beta + \varepsilon)|z|}; z \in \mathbb{C}\},$$

is bounded. If $A = [0, \alpha)$, where $0 < \alpha \leq \infty$, then a function $f: A \to X$ is said to be entire of exponential type if it can be extended to an $X$-valued, entire function of exponential type.

By a vector measure in $X$ is meant a $\sigma$-additive map $\mu: \mathcal{B} \to X$. For each $x' \in X'$, the complex-valued measure $E \mapsto \langle \mu(E), x' \rangle$, $E \in \mathcal{B}$, is denoted by $\langle \mu, x' \rangle$.

A complex-valued, $\mathcal{B}$-measurable function $f$ on $\mathbb{R}$ is said to be $\mu$-integrable if it is integrable with respect to every measure $\langle \mu, x' \rangle$, $x' \in X'$, and if, for every $E \in \mathcal{B}$, there exists an element $\int_E f \, d\mu$ of $X$ such that

$$\left\langle \int_E f \, d\mu, x' \right\rangle = \int_E f \, d\langle \mu, x' \rangle,$$

for each $x' \in X'$. Bounded measurable functions are always $\mu$-integrable [9; II Lemma 3.1]. Hence, the Fourier-Stieltjes transform, $\hat{\mu}$, of any vector measure $\mu: \mathcal{B} \to X$ can be defined by

$$\hat{\mu}(s) = \int_{\mathbb{R}} \exp(-ist) \, d\mu(t), \quad s \in \mathbb{R}.$$ 

Let $\mathcal{M}_d$ denote the linear space of all complex-valued measures on $\mathcal{B}$ with finite supports. The set of all measures $\omega \in \mathcal{M}_d$ such that $\|\omega\|_{\infty} \leq 1$ is denoted by $\Omega$ ($\|\cdot\|_{\infty}$ denotes the supremum norm).

The following result is a vector version of the Bochner-Schoenberg test. It is well known for Banach spaces [8]; its extension to more general spaces presents no difficulties.

**Bochner-Schoenberg Criterion.** Let $f: \mathbb{R} \to X$ be a bounded, weakly continuous function. Then there exists a (unique) vector measure $\mu: \mathcal{B} \to X$ such that $f = \hat{\mu}$.
if and only if,

\[ \left\{ \int_{\mathbb{R}} f(t) \, d\omega(t); \, \omega \in \Omega \right\} \]

is a relatively weakly compact subset of X.

Let \( L(X) \) denote the space of all continuous linear operators on \( X \), equipped with the topology of pointwise convergence. The space \( L(X) \) may not be quasi-complete. The identity operator is denoted by \( I \).

A map \( P: \mathcal{B} \rightarrow L(X) \) is called a spectral measure if it is \( \sigma \)-additive, multiplicative and \( P(E) = I \). Of course, the multiplicativity of \( P \) means that \( P(E \cap F) = P(E) P(F) \), for every \( E \in \mathcal{B} \) and \( F \in \mathcal{B} \). The spectral measure \( P \) is said to be equicontinuous if its range \( P(\mathcal{B}) = \{ P(E); \, E \in \mathcal{B} \} \) is an equicontinuous part of \( L(X) \). For such spectral measures, every bounded measurable function is \( P \)-integrable [13]. For each \( x \in X \), denote by \( P(\cdot)(x) \) the \( X \)-valued measure \( E \mapsto P(E)(x), \, E \in \mathcal{B} \).

Let \( \Lambda = \{ T; D; A \} \) be a local semigroup. Then \( \Lambda \) is said to be spectral if there exists an equicontinuous spectral measure \( P: \mathcal{B} \rightarrow L(X) \) such that for each \( x \in D \), each of the functions \( e^{-t(\cdot)}, \, t \in A \), is \( P(\cdot)(x) \)-integrable and the identity (1) is valid. If, in addition, each measure \( P(\cdot)(x), \, x \in D \), has compact support, then \( \Lambda \) is said to be of bounded type. This is equivalent to the existence of an increasing sequence of bounded Borel sets \( E_k, \, k = 1, 2, \ldots \), with \( E_k \uparrow \mathbb{R} \), such that \( D \subseteq \bigcup_{k=1}^{\infty} P(E_k)(X) \).

Let \( P: \mathcal{B} \rightarrow L(X) \) be an equicontinuous spectral measure and \( \Lambda = [0, \alpha) \), where \( 0 < \alpha \leq \infty \). Then \( D_0 = \bigcup_{k=1}^{\infty} P([-k, k]) (X) \) is a dense subspace of \( X \) such that for each \( x \in D_0 \), each of the functions \( e^{-t(\cdot)}, \, t \in A \), is \( P(\cdot)(x) \)-integrable. Accordingly, for each \( t \in A \), an element \( T(t) \) of \( \Pi(D_0) \) can be defined by the formula (1). In fact, for any dense subspace \( D \) of \( X \), contained in \( D_0 \), which is invariant for each of the operators \( T(t), \, t \in A \), the so constructed system \( \{ T; D; A \} \) is a spectral local semigroup of bounded type. It will be said to correspond to \( (P, D, A) \).

3. STATEMENT OF RESULTS

Let \( \Lambda = \{ T; D; A \} \) be a local semigroup. Let \( \mathcal{N} \) be a family of continuous seminorms determining the topology of \( X \). If \( c \) is a positive number belonging to \( \Lambda \), then define for each \( x \in D \) and \( q \in \mathcal{N} \) the quantities

\[ r_k(x, q, c) = \limsup_{n \to \infty} q([T(c/k) - I]^n(x))^{1/n}, \, k = 1, 2, \ldots . \]

An element \( x \in D \) is said to be a binomial vector for \( \Lambda \) with respect to \( c \), if there exists a positive integer \( k(x, c) \) such that

\[ r_k(x, q, c) < 1, \, q \in \mathcal{N}, \, k \geq k(x, c). \]
It is tacitly assumed that \( k(x, c) \) is the minimal positive integer specified by these inequalities.

If \( \omega \in \mathcal{M}_d \) is given by
\[
(2) \quad \omega = \sum_{k=1}^{N} c_k \delta_{t_k},
\]
where \( t_k \in \mathbb{R} \) and \( c_k \in \mathcal{C} \), for each \( k = 1, 2, \ldots, N \), and \( \delta_t \) denotes the Dirac point mass at \( t \in \mathbb{R} \), let
\[
\omega_n = \sum_{j=1}^{N} c_j \left( \frac{it_j}{n} \right),
\]
for each \( n = 0, 1, 2, \ldots \), where \( i = \sqrt{-1} \).

Let \( x \in D \) be a binomial vector for \( \Lambda \) with respect to \( c \). Since \( \limsup_{n \to \infty} \left| (2z)^{1/n} \right| \leq 1 \) for each complex number \( z \), the series
\[
b(x, c, \omega, k) = \sum_{n=0}^{\infty} \omega_n [T(c/k) - I]^n(x),
\]
is absolutely convergent for each \( \omega \in \mathcal{M}_d \) and each \( k \geq k(x, c) \). Accordingly, a subset \( B(x, c) \) of \( X \) can be defined by
\[
B(x, c) = \{ b(x, c, \omega, k); \omega \in \Omega, k \geq k(x, c) \}.
\]

The main result can now be stated. It will be proved, along with the other results of this section, in § 4.

**Theorem 1.** A local semigroup \( \Lambda = \{ T; D; \Delta \} \) is spectral and of bounded type if and only if for each \( x \in D \), the function \( T(\cdot)(x) \) is entire of exponential type and there exists a positive rational number \( c \in \Lambda \) such that the following conditions are satisfied.

(i) Every \( x \in D \) is a binomial vector for \( \Lambda \) with respect to \( c \).
(ii) For each \( x \in D \), the set \( B(x, c) \) is relatively weakly compact.
(iii) For each \( q \in \mathcal{N} \) there exists a positive number \( \alpha = \alpha(q) \) and seminorms \( q_1, \ldots, q_r \) in \( \mathcal{N} \) such that for each \( x \in D \),
\[
q(\xi) \leq \alpha \max \{ q_j(x); 1 \leq j \leq r \}, \quad \xi \in B(x, c).
\]

If the space \( X \) in Theorem 1 is a Banach space, then the hypothesis that \( T(\cdot)(x) \) is entire of exponential type for each \( x \in D \), can be omitted. This follows already from the conditions (i)–(iii) of the theorem (see the proof of Theorem 2). However, for non-normable spaces this is no longer the case. For example, let \( X \) denote the space of all complex sequences \( x = \{ x_n \}_{n=1}^{\infty} \), equipped with the topology of pointwise convergence. Let \( \Lambda = [0, \infty) \) and \( D = X \). For each \( t \in \Lambda \), define a continuous linear operator \( T(t) \) by \( T(t)(x) = y \), \( x \in X \), where \( y_n = e^{-tn}x_n \), for each \( n = 1, 2, \ldots \). Then \( \{ T; D; \Lambda \} \) is a local semigroup such that for any \( c > 0 \) the conditions (i)–(iii) of Theorem 1 are satisfied. However, there exist vectors \( x \in D \) for which \( T(\cdot)(x) \) has no entire extension of exponential type.
Theorem 2. Let $X$ be a Banach space and $\Lambda = \{T; D; \Delta\}$ a local semigroup. Then $\Lambda$ is spectral and of bounded type if and only if there exists a positive rational number $c \in \Lambda$, for which the conditions (i) and (ii) of Theorem 1 are satisfied, such that

$$b(T) = \sup \{\|\xi\|; \xi \in B(x, c), x \in D, \|x\| \leq 1\} < \infty.$$  

In a reflexive Banach space a set is relatively weakly compact if and only if it is bounded. Hence, for reflexive spaces, the relative weak compactness of the sets in condition (ii) of Theorem 1 can be replaced by their boundedness. But, the boundedness of each of the sets $B(x, c), x \in D$, follows from (3). Hence, Theorem 2 implies the following result due to Kantorovitz and Hughes [6; Theorem 3.3].

**Corollary.** Let $X$ be a reflexive Banach space and $\Lambda = \{T; D; \Delta\}$ a local semigroup. Then $\Lambda$ is spectral and of bounded type if and only if there exists a positive rational number $c \in \Lambda$ such that each $x \in D$ is a binomial vector for $\Lambda$ with respect to $c$ and (3) holds.

There is a class of spaces, including many non-normable ones, for which the conditions (i)–(iii) of Theorem 1 suffice to guarantee that a given local semigroup in such a space is spectral, but not necessarily of bounded type.

A locally convex space $X$ is said to be weakly $\Sigma$-complete if every sequence $\{x_n\}_{n=1}^{\infty}$ of its elements such that $\{\langle x_n, x' \rangle\}_{n=1}^{\infty}$ is absolutely summable for each $x' \in X'$, is itself summable to an element of $X$. In [9], such a space is said to have the B-P property. Weakly sequentially complete spaces, in particular reflexive spaces, are weakly $\Sigma$-complete. According to a theorem of Tumarkin [12], generalizing the well-known result of Bessaga and Pelczyński, a space is weakly $\Sigma$-complete if and only if it does not contain an isomorphic copy of the space $c_0$.

**Theorem 3.** Let $X$ be a weakly $\Sigma$-complete space and $\Lambda = \{T; D; \Delta\}$ be a local semigroup. If there exists a positive rational number $c \in \Lambda$ for which the conditions (i)–(iii) of Theorem 1 are satisfied, then $\Lambda$ is a spectral local semigroup.

4. PROOFS OF RESULTS

To prove the necessity of the conditions in Theorem 1, let $P: \mathcal{B} \to L(X)$ be an equicontinuous spectral measure and $\Lambda = \{T; D; \Delta\}$ be a spectral local semigroup of bounded type corresponding to $(P, D, \Delta)$.

Let $x \in D$. Then there exists a positive integer $m = m(x)$ such that $x \in P([-m, m])(X)$. Define an entire function with values in $X$ by

$$z \mapsto \int_{-m}^{m} e^{-zs} dP(s)(x) = \int_{\mathbb{R}} e^{-zs} dP(s)(x), \quad z \in \mathbb{C}.$$  

This function agrees with $T(\cdot)(x)$ on $\Lambda$ (cf. (1)). It is again denoted by $T(\cdot)(x)$. It
follows that for each \( x' \in X' \) and \( \varepsilon > 0 \) the inequalities

\[
|e^{-(m+\varepsilon)|z|} \langle T(z)(x), x' \rangle| \leq e^{-\varepsilon|z|} |\langle P(\cdot)(x), x' \rangle| ([-m, m]), \quad z \in \mathbb{C},
\]

are valid. This shows that \( T(\cdot)(x) \) is entire of exponential type.

Let \( c \) be any positive number in \( \Lambda \). For each \( t \in \mathbb{R} \), consider the series

\[
(4) \quad v \mapsto \sum_{n=0}^{\infty} \binom{it}{n} (e^{-cv/k} - 1)^n, \quad |v| \leq m.
\]

It follows, from the ratio test for example, that if \( k(x, c) \) is chosen to be the smallest integer \( k \) satisfying \( k > cm/\ln 2 \), then the series (4) is absolutely convergent for all \( t \in \mathbb{R} \) and all \( k \geq k(x, c) \), to the function

\[
(5) \quad v \mapsto [1 + (e^{-cv/k} - 1)]^{it} = \exp (-ictv/k), \quad |v| \leq m.
\]

Let \( q \in \mathcal{N} \). If \( U_q^0 \) denotes the polar of \( q^{-1} ([0, 1]) \), then

\[
(6) \quad q(v) = \sup \{ |\langle y, x' \rangle|; x' \in U_q^0 \}, \quad y \in X.
\]

Since the identity

\[
(7) \quad [T(c/k) - I]^n(x) = \int_{-m}^{m} (e^{-cv/k} - 1)^n dP(v)(x),
\]

is valid for each \( k = 1, 2, \ldots \), and \( n = 0, 1, 2, \ldots \), it follows from (6) and (7) that for each \( k = 1, 2, \ldots \), the inequality

\[
q([T(c/k) - I]^n(x)) \leq \gamma(x, q) (e^{cm/k} - 1)^n,
\]

is valid, where \( \gamma(x, q) = \sup \{ |\langle P(\cdot)(x), x' \rangle| (\mathbb{R}); x' \in U_q^0 \} \) is finite [9; II Lemma 1.1]. Accordingly,

\[
(8) \quad r_k(x, q, c) = \limsup_{n \to \infty} q([T(c/k) - I]^n(x))^{1/n} \leq (e^{cm/k} - 1) < 1,
\]

for all \( k \geq k(x, c) \). Since \( q \in \mathcal{N} \) was arbitrary, this shows that \( x \) is a binomial vector for \( \Lambda \) with respect to \( c \).

If \( t \in \mathbb{R} \), then it follows for each \( k \geq k(x, c) \), the partial sums of the series (4) are uniformly bounded. Accordingly, if \( \omega \in \Omega \) is given by (2), then the identities (4) and (5) and the Dominated Convergence Theorem for vector measures [9; II Theorem 4.2] imply that

\[
(8) \quad \int_{-m}^{m} \sum_{j=1}^{N} c_j \sum_{n=0}^{\infty} \binom{it_j}{n} (e^{-cv/k} - 1)^n dP(v)(x) = \int_{-m}^{m} \sum_{j=1}^{N} c_j \exp (-icv/k) dP(v)(x),
\]

for each \( k \geq k(x, c) \). However, for each \( k = 1, 2, \ldots \), we also have

\[
\sum_{j=1}^{N} c_j \sum_{n=0}^{\infty} \binom{it_j}{n} (e^{-cv/k} - 1)^n = \sum_{n=0}^{\infty} \omega_n (e^{-cv/k} - 1)^n, \quad |v| \leq m.
\]

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Again by the Dominated Convergence Theorem and (7) it follows that

\[ \int_{-m}^{m} \left( \sum_{j=1}^{N} c_j \sum_{n=0}^{\infty} i^n \right) \left( e^{-i\omega j} - 1 \right) \omega^n \, dP(v)(x) = \sum_{n=0}^{\infty} \omega_n \left[ T(c/k) - I \right]^n(x), \]

for all \( k \geq k(x, c) \). The identities (8) and (9) imply that

\[ b(x, c, \omega, k) = \int_{\mathbb{R}} \left( \sum_{j=1}^{N} c_j \exp(-icw_jk) \right) \, dP(v)(x), \]

for all \( k \geq k(x, c) \). Furthermore, the inequality \( \| \phi \|_\infty \leq 1 \) implies that the supremum norm of the integrand in (10) does not exceed 1. It follows [9; IV Lemma 6.1] that

\[ B(x, c) \subseteq \text{co}(P(\cdot)(x))(\mathbb{R}), \]

and hence, that \( B(x, c) \) is relatively weakly compact [9; IV Theorem 6.1].

To verify condition (iii) in Theorem 1, let

\[ \mathcal{A} = \left\{ \int_{\mathbb{R}} f \, dP; \, \|f\|_\infty \leq 1, \, f \text{ measurable} \right\}. \]

Then \( \mathcal{A} \) is an equicontinuous part of \( L(X) \), [13; Proposition 2.1]. Hence, if \( q \in \mathcal{N} \), then there exists \( \alpha = \alpha(q) > 0 \) and seminorms \( q_1, \ldots, q_r \) in \( \mathcal{N} \) such that

\[ q(S(x)) \leq \alpha \max \{q_j(x); 1 \leq j \leq r\}, \quad x \in X, \]

for all \( S \in \mathcal{A} \). Fix \( x \in D \). If \( \xi \in B(x, c) \), then it was noted (cf. (10)) that there exists a measurable function \( f \) with \( \|f\|_\infty \leq 1 \) such that

\[ \xi = \int_{\mathbb{R}} f(v) \, dP(v)(x) = \left( \int_{\mathbb{R}} f \, dP \right)(x). \]

It follows from (11), (12) and the definition of \( \mathcal{A} \) that \( q(\xi) \leq \alpha \max \{q_j(x); 1 \leq j \leq r\}. \) Hence, condition (iii) is verified. This completes the proof of necessity. \( \square \)

The proof of the sufficiency of the conditions in Theorem 1 is based on the following lemma. Its proof is a combination of the Bochner-Schoenberg Criterion and the proof of Lemma 3.6 of [6]. Even though some of the arguments and calculations are identical to those in the proof of Lemma 3.6 in [6], they are included for completeness and ease of reading. Firstly however, some notation.

If \( A = \{T; D; \mathcal{A}\} \) is a local semigroup and \( c \) is a positive rational number in \( A \), then for each binomial vector \( x \) of \( A \) with respect to \( c \) we can define an entire, \( X \)-valued function \( T_k(\cdot)(x) \), \( k \geq k(x, c) \), by

\[ T_k(z)(x) = \sum_{n=0}^{\infty} \binom{z}{n} \left[ T(c/k) - I \right]^n(x), \quad z \in \mathbb{C}. \]

Furthermore, if \( k \geq k(x, c) \) is fixed, then for each \( q \in \mathcal{N} \) there exists a positive
number $\beta = \beta(q)$ such that for each $\varepsilon > 0$ the set of numbers
\begin{equation}
\{e^{-(\beta + \varepsilon)|z|}q(T_k(z)(x)); \ z \in \mathbb{C}\},
\end{equation}
is bounded. However, there may not exist a single number $\beta > 0$ such that (14) is bounded for all $q \in \mathcal{N}$ (cf. example in §3). Accordingly, $T_k(\cdot)(x)$ may not be of exponential type, unless $X$ is a Banach space.

**Lemma 1.** Let $\Lambda = \{T; D; A\}$ be a local semigroup and $c$ a positive rational number in $\Lambda$. If $x \in D$ is a binomial vector for $\Lambda$ with respect to $c$ such that the set $B(x, c)$ is relatively weakly compact, then the function $T(\cdot)(x)$ has an entire extension and there exists a unique vector measure $\mu_x: \mathcal{B} \to X$, such that each measure $\langle \mu_x, x' \rangle$, $x' \in X'$, has compact support and
\begin{equation}
\langle T(z)(x), x' \rangle = \int_{\mathbb{R}} e^{-zs} d\langle \mu_x(s), x' \rangle, \quad z \in \mathbb{C},
\end{equation}
for each $x' \in X'$.

**Proof.** Let $c = d/e$. It will be shown that for each $k \geq k(x, c)$, the function $z \mapsto T_k(ekz)(x)$, $z \in \mathbb{C}$, is independent of $k$. Accordingly, if $T(\cdot)(x)$ is defined on $\mathbb{C}$ by
\begin{equation}
T(z)(x) = T_k(ekz)(x), \quad z \in \mathbb{C},
\end{equation}
for any $k \geq k(x, c)$, then $T(\cdot)(x)$ is entire and has the desired properties.

Let $\omega \in \Omega$ be given by (2). Then it follows from (13) that for each $k \geq k(x, c)$,
\[
\int_{\mathbb{R}} T_k(iv)(x) d\omega(v) = \sum_{j=1}^{N} c_j T_k(it_j)(x) = \sum_{n=0}^{\infty} \omega_n[T(c|k) - I]^n(x) \in B(x, c).
\]
Accordingly, for each $k \geq k(x, c)$ the set \{$\int_{\mathbb{R}} T_k(iv)(x) d\omega(v); \ \omega \in \Omega$\} is relatively weakly compact. Furthermore, since the function $v \mapsto T_k(vi)(x)$, $v \in \mathbb{R}$, is bounded and weakly continuous it follows from the Bochner-Schoenberg Criterion that there exists a unique measure $\nu_x(k): \mathcal{B} \to X$, with range contained in $\text{bco} \ B(x, c)$, such that
\[
T_k(is)(x) = \int_{\mathbb{R}} e^{-isv} d\nu_x(k)(v), \quad s \in \mathbb{R},
\]
for each $k \geq k(x, c)$. Furthermore, for each $x' \in X'$, $\langle \nu_x(k)(\cdot), x' \rangle$ is the unique Borel measure on $\mathbb{R}$ such that
\begin{equation}
\langle T_k(is)(x), x' \rangle = \int_{\mathbb{R}} e^{-isv} d\langle \nu_x(k)(v), x' \rangle, \quad s \in \mathbb{R},
\end{equation}
for each $k \geq k(x, c)$.

Since the function $\langle T_k(\cdot)(x), x' \rangle$ is entire of exponential type (cf. (14)) and is bounded on the real line, the Paley-Wiener-Schwartz theorem [3; Ch. 6, Theorem 5] implies that its Fourier transform (which is $2\pi \langle \mu_x(k), x' \rangle$ by (17)) has compact support. The bilateral Laplace transform
\[
\int_{\mathbb{R}} e^{-sz} d\langle \mu_x(k)(v), x' \rangle, \quad z \in \mathbb{C},
\]
is therefore well defined, entire and coincides with $\langle T_k(\cdot)(x), x' \rangle$ on the imaginary axis (by (17)). Hence,

$$\langle T_k(z)(x), x' \rangle = \int_{\mathbb{R}} e^{-zv} \, d\mu_x(k)(v), x',$$

for every $z \in \mathbb{C}$.

If $N$ is a positive integer, then for each $k \geq k(x, c)$,

$$T_k(N)(x) = \sum_{n=0}^{N} \binom{N}{n} [T(c/k) - 1]^n(x) = T(c/k)^N(x).$$

Since $c = d/e$ and $1/e$ belong to $A$, it follows that

$$T_{id}(ekN)(x) = T(1/ek)^{ekN}(x) = [T(1/ek)^{k}]^{ekN}(x) = [T(1/e)]^{ekN}(x),$$

for each $k \geq k(x, c)$ and each positive integer $N$.

Fix $x' \in X'$ and $k, l \geq k(x, c)$. The function

$$f(z) = \langle T_{id}(ekz)(x), x' \rangle - \langle T_{id}(elz)(x), x' \rangle, \quad z \in \mathbb{C},$$

is a Laplace-Stieltjes transform (by (18)) which vanishes for positive integral values of $z$ (cf. (20)). It follows from Lerch's theorem that $f(z) = 0$ for all $z \in \mathbb{C}$ [4; Theorem 6.2.2]. Accordingly, $T(\cdot)(x)$ is well defined by (16) and is entire.

If $r \in A$ is a positive rational number, we may write $r = f/k$, where $f$ and $k$ are positive integers and $k \geq k(x, c)$. Since $ef/ek = r \in A$, it follows from (19) that

$$T_{id}(ef)(x) = T(1/ek)^{ef}(x) = T(r)(x).$$

The weak continuity of $T(\cdot)(x)$ on $A$ then implies that it agrees with (16) on $A$. Hence, $T(\cdot)(x)$ has an entire extension.

Since $T_{id}(ekz)(x) = T_{id}(elz)(x)$, for all $k, l \geq k(x, c)$ and all $z \in \mathbb{C}$, it follows from the identity

$$T_{id}(ekz)(x) = T_{id}(elz)(x),$$

valid for all $s \in \mathbb{R}$ and $k \geq k(x, c)$, and the uniqueness of Fourier-Stieltjes transforms that we may define a vector measure $\mu_x: \mathcal{B} \rightarrow X$ by

$$\mu_x(E) = \mu_x(kd)(E/ke), \quad E \in \mathcal{B},$$

for any $k \geq k(x, c)$. It is clear from (16) and (21) that (15) is satisfied. This completes the proof of the lemma. $\square$

We now prove the sufficiency of the conditions in Theorem 1. So, let $A = \{T; D; A\}$ be a local semigroup and $c$ be a positive rational number in $A$ for which the conditions of Theorem 1 are satisfied.

For each $x \in D$, let $T(\cdot)(x)$ be the entire function and $\mu_x$ the $X$-valued measure
as constructed in Lemma 1. Then for each $E \in \mathcal{B}$, define a map $P(E): D \to X$ by

$$P(E)(x) = \mu_x(E), \quad x \in D.$$  

Since for each complex number $z$ the map $x \mapsto T(z)(x), \ x \in D,$ is linear (cf. (16)), it follows from (15) and the uniqueness of Laplace-Stieltjes transforms that the map $P(E)$ is linear.

Let $q \in \mathcal{N}$. Let $\alpha = \alpha(q) > 0$ and $q_1, \ldots, q_r \in \mathcal{N}$ be as given by condition (iii). For each $E \in \mathcal{B}$ it follows that

$$q(P(E)(x)) = q(\mu_x(E)) \leq \alpha \max \{q_j(x); 1 \leq j \leq r\}, \quad x \in D,$$

since $\mu_x(E) \in \text{bco} B(x, c)$. Hence, each operator $P(E), \ E \in \mathcal{B},$ is continuous on $D$ and so can be extended uniquely to a continuous operator on all of $X$, still denoted by $P(E)$, which satisfies

$$q(P(E)(x)) \leq \alpha \max \{q_j(x); 1 \leq j \leq r\}, \quad x \in X.$$  

Accordingly, $P(\mathcal{B}) = \{P(E); E \in \mathcal{B}\}$ is an equicontinuous part of $L(X)$. Since $P(\cdot)(x)$ is $\sigma$-additive for each $x$ in a dense subspace of $X$, it follows that $E \mapsto P(E), \ E \in \mathcal{B},$ is an $L(X)$-valued measure.

If $x \in D$, then it follows from (15) and (22) that

$$\langle x, x' \rangle = \langle T(0)(x), x' \rangle = \langle P(R)(x), x' \rangle, \quad x' \in X'.$$

Accordingly, $P(R) = I$. The next step is to show that $P$ is multiplicative. Since $\|e^{-ist}\|_{\infty} \leq 1$ for each $s \in \mathcal{R}$, it follows from (15) that for each $x \in D$ and $s \in \mathcal{R}$,

$$T(is)(x) = \int_{\mathcal{R}} e^{-ist} dP(v)(x) \in \text{bco} (P(\cdot)(x)) (\mathcal{B}) \subseteq \text{bco} B(x, c).$$

Hence, if $q \in \mathcal{N}$, then by condition (iii) there is $\alpha > 0$ and seminorms $q_1, \ldots, q_r$ in $\mathcal{N}$ such that

$$q(T(is)(x)) \leq \alpha \max \{q_j(x); 1 \leq j \leq r\}, \quad s \in \mathcal{R},$$

for each $x \in D$. Since $D$ is dense in $X$, each operator $T(is), \ s \in \mathcal{R}$, has a unique continuous extension to all of $X$, again denoted by $T(is)$, such that (24) is valid for all $x \in X$. Hence, $\{T(is); s \in \mathcal{R}\}$ is an equicontinuous part of $L(X)$ and it follows from (23) and the uniqueness of continuous extension that

$$T(is)(x) = \int_{\mathcal{R}} e^{-ist} dP(v)(x), \quad s \in \mathcal{R},$$

for each $x \in X$. Arguing as in the proof of Theorem 3.3 in [6] it follows that $T(\cdot): \mathcal{R} \to L(X)$ is an equicontinuous group. It then follows from the group property and (25) that $P$ is necessarily multiplicative.

It remains to show that for each $x \in D$, the measure $P(\cdot)(x) = \mu_x$ has compact support. Fix $x \in D$. By hypothesis, the entire extension of $T(\cdot)(x)$ (as constructed in Lemma 1), is of exponential type. Hence, there exists a positive number $\beta = \beta(x)$
such that for each $\epsilon > 0$ and $x' \in X'$ there is a number $M = M(x, \epsilon, x')$ such that

$$\langle T(z) (x), x' \rangle \leq Me^{(\beta + \epsilon)|z|}, \quad z \in \mathbb{C}.$$ 

Since for each $x' \in X'$, the function $\langle T(i \cdot) (x), x' \rangle$ is entire of exponential type and is bounded on $\mathbb{R}$, it follows from (26) and the Paley-Wiener-Schwartz theorem that its Fourier transform, which is $2\pi \langle \mu_x, x' \rangle$ by (15), has support contained in $[-\beta, \beta]$. Since $\beta$ is independent of $x'$, it follows that $\mu_x$ has compact support. Hence, $A$ is a spectral local semigroup of bounded type. This completes the proof of Theorem 1. □

**Proof of Theorem 2.** Suppose that $A$ is a spectral local semigroup of bounded type corresponding to $(P, D, A)$, where $P: \mathcal{B} \to L(X)$ is a spectral measure. Let $c$ be an arbitrary positive number in $A$. Theorem 1 implies that the conditions (i) -- (iii) are satisfied. Furthermore, by condition (iii) there is $\alpha > 0$ such that for each $x \in D$, $\|\xi\| \leq \alpha \|\xi\|$, $\xi \in B(x, c)$.

It follows easily that $b(T) \leq \alpha < \infty$. Hence, (3) is valid.

Conversely, suppose that there is a positive rational number $c \in A$ for which the stated requirements of Theorem 2 are satisfied. If $x \in D$, then it is easily verified that for each $\beta > 0$, $r_d(\beta x, \|\cdot\|, c) = r_d(x, \|\cdot\|, c)$, $k = 1, 2, \ldots$ and $k(\beta x, c) = k(x, c)$. It follows that if $\xi \in B(x, c)$, then $\beta \xi \in B(\beta x, c)$. Hence, fix $x \in D$. If $\xi \in B(x, c)$, then $\xi/\|\xi\|$ belongs to $B([x/\|\xi\|, c)$. It follows from (3) that $\|\xi/\|\xi\| \leq b(T)$. That is, for each $x \in D$, $\|\xi\| \leq b(T) \|\xi\|$, $\xi \in B(x, c)$.

Hence, conditions (i) -- (iii) of Theorem 1 are satisfied.

It follows (cf. proof of Theorem 1) that there exists a spectral measure $P: \mathcal{B} \to L(X)$ such that for each $x \in D$ and $x' \in X'$ the measure $\langle P(\cdot) (x), x' \rangle$ has compact support and satisfies

$$\langle T(i \cdot) (x), x' \rangle = \int_{\mathbb{R}} e^{-ts} d\langle P(s) (x), x' \rangle, \quad t \in A.$$ 

Fix $x \in D$. It follows from Rybakov’s theorem [9; VI Theorem 3.2] (or from a well known result of W. Bade [1; Theorem 3.1]) that there exists $x' \in X'$ such that $P(\cdot)(x)$ is absolutely continuous with respect to $\langle P(\cdot) (x), x' \rangle$. Hence, $P(\cdot)(x)$ has compact support. Then each of the functions $e^{-it\cdot(x)}$, $t \in A$, is $P(\cdot)(x)$-integrable and it follows from (27) that (1) is valid. Hence, $A$ is a spectral local semigroup of bounded type. □

**Proof of Theorem 3.** As noted above, it follows from the conditions (i) -- (iii) of Theorem 1 that there exists an equicontinuous spectral measure $P: \mathcal{B} \to L(X)$ such that for each $x \in D$ and $x' \in X'$ the measure $\langle P(\cdot) (x), x' \rangle$ has compact support and satisfies (27). That is, if $x \in D$, then for each $t \in A$ the function $e^{-it\cdot(x)}$ is integrable
with respect to each of the measures \( \langle P(\cdot)(x), x' \rangle \), \( x' \in X \). Hence, each function \( e^{-t(x)} \), \( t \in A \), is actually \( P(\cdot)(x) \)-integrable [9; II Theorem 5.1]. It then follows from (27) that the identity (1) is valid for each \( x \in D \) and \( t \in A \), that is, \( A \) is spectral.

References


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