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CONGRUENCE PAIRS FOR ALGEBRAS ABSTRACTING KLEENE AND STONE ALGEBRAS

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1. Introduction. Various notions of congruence pairs have been intensively studied and proved to be a useful tool in the study of lattices with pseudocomplementation (alias, p-algebras) and double p-algebras (see [12] and [2] and the references therein). In applications, much of the success of congruence pairs derives from the fact that every congruence relation on a distributive (double) p-algebra can be represented by a pair of congruences, one from each of the congruence lattices of a pair of simpler substructures. Recently, T. S. Blyth and J. C. Varlet introduced MS-algebras which are algebras of type $\langle 2, 2, 1, 0, 0\rangle$ abstracting de Morgan algebras and Stone algebras. In [5] they exhibit the Hasse diagrams of the subdirectly irreducible members of the variety $\boldsymbol{MS}$ of all MS-algebras while in [6] the lattice of subvarieties of $\boldsymbol{MS}$ is drawn and each of its members is characterized by identities. In a forthcoming paper [7] they consider a certain subvariety $\boldsymbol{K}_2$ of $\boldsymbol{MS}$ whose members may be thought of as algebras abstracting Kleene algebras and Stone algebras. Each member of $\boldsymbol{K}_2$ contains two simpler substructures, one being a Kleene algebra and the other being a distributive lattice with unit, and they develop a 'Chen-Grätzer' style construction theorem for the members of $\boldsymbol{K}_2$ utilizing methods similar to those employed by T. Katriňák [11] for Stone algebras. The purpose of this note is twofold. First, we supplement the various characterizations of $\boldsymbol{K}_2$ and its subvarieties obtained in [6] by ones expressed in terms of prime ideals and which lead to duality theories for the associated algebraic categories. Second, we introduce a suitable notion of congruence pair for the class $\boldsymbol{K}_2$ which generalizes that for Stone algebras and facilitates the representation of congruences on algebras in $\boldsymbol{K}_2$ in terms of pairs of congruences, one from each of the underlying simpler structures.

2. Preliminaries. An *MS-algebra* is an algebra $\langle L, \vee, \wedge, {}^\circ, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ whose reduct $\langle L, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and such that, for all $x, y \in L$,

$$x \leqq x^{\circ\circ}\,, \quad (x \wedge y)^\circ = x^\circ \vee y^\circ\,, \quad 1^\circ = 0\,.$$

Obviously, the class $\boldsymbol{MS}$ of all MS-algebras is a variety. The members of the subvariety $\boldsymbol{M}$ of $\boldsymbol{MS}$ defined by the identity $x = x^{\circ\circ}$ are called *de Morgan algebras* and the members of the subvariety $\boldsymbol{K}$ of $\boldsymbol{M}$ defined by the 'identity' $x \wedge x^\circ \leqq y \vee y^\circ$ are called *Kleene algebras.*

A *Stone algebra* is an algebra $\langle L, \vee, \wedge, {}^\circ, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ whose reduct $\langle L, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and whose unary operation $^\circ$ is usually denoted by * and characterized by

$$a \wedge x = 0 \Leftrightarrow x \leqq a^* \, .$$

The class $\boldsymbol{S}$ of Stone algebras is, in fact, a subvariety of $\boldsymbol{MS}$ and is characterized by the identity $x \wedge x^\circ = 0$. The subvariety $\boldsymbol{B}$ of $\boldsymbol{MS}$ characterized by the identity $x \vee x^\circ = 1$ is the class of Boolean algebras.

Some elementary properties which were proved in [5] and hold for all x, y in any MS-algebra L are:

$$0^\circ = 1$$

$$x \leqq y \Rightarrow x^\circ \geqq y^\circ \quad \text{and} \quad x^{\circ\circ} \leqq y^{\circ\circ}$$

$$x^\circ = x^{\circ\circ\circ}$$

$$(x \vee y)^\circ = x^\circ \wedge y^\circ$$

$$(x \vee y)^{\circ\circ} = x^{\circ\circ} \vee y^{\circ\circ}, \quad (x \wedge y)^{\circ\circ} = x^{\circ\circ} \wedge y^{\circ\circ} \, .$$

Consequently, $L^{\circ\circ} := \{x \in L;\ x = x^{\circ\circ}\} = \{x^\circ;\ x \in L\}$ is a de Morgan subalgebra of L and $L^\vee := \{x \vee x^\circ;\ x \in L\} = \{x \in L;\ x \geqq x^\circ\}$ is an increasing subset (i.e. order filter) of L.

In keeping with the notation of [6], we will denote by $\boldsymbol{K}_2$ the subvariety of $\boldsymbol{MS}$ generated by the four-element algebra K_2 whose Hasse diagram is depicted in figure 1. In passing, we record that, as a consequence of results from [5] and [6], the subdirectly irreducible members of $\boldsymbol{K}_2$ are precisely all subalgebras of K_2 and the Hasse diagram of the lattice of subvarieties of $\boldsymbol{K}_2$ is as depicted in figure 2.

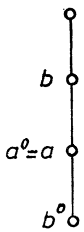

Fig. 1.

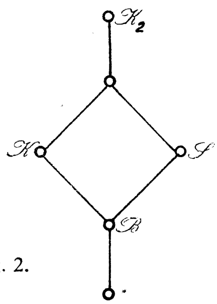

Fig. 2.

Two other recent contributions to the theory of MS-algebras are [3], in which alternative approaches to a generalization of the main result of [5] are expounded, and [4], in which the injectives in each of the subvarieties of $\boldsymbol{MS}$ are characterized. For all other unexplained notation and terminology we refer to [1] or [9].

3. Characterizations of K_2. The identity $x = x^{\circ\circ} \wedge (x \vee x^{\circ})$ is a familiar one which holds in the variety $\boldsymbol{S}$ of Stone algebras and which holds trivially in the variety $\boldsymbol{K}$ of Kleene algebras. Any MS-algebra in which it holds is called *firm* in [6] and it is not difficult to show that an MS-algebra L is firm if and only if $x^{\circ\circ} \wedge x^{\circ} = x \wedge x^{\circ}$, for all $x \in L$.

We begin by recording the following characterizations of $\boldsymbol{K}_2$ from [6]:

Theorem 1. *For an algebra $L \in \boldsymbol{MS}$, the following are equivalent:*

(i) $L \in \boldsymbol{K}_2$

(ii) *L is firm and $x \wedge x^{\circ} \leqq y \vee y^{\circ}$, for all $x, y \in L$*

(iii) *L is firm and $L^{\vee}$ is a filter*

(iv) *L is firm and $L^{\circ\circ} \in \boldsymbol{K}$.*

In this section, our aim is to give prime ideal characterizations of the class $\boldsymbol{K}_2$ and its subvarieties. If $\mathscr{P}(L)$ denotes the poset of prime ideals of an algebra $L \in \boldsymbol{MS}$ then it is easily verified that the mapping $g: \mathscr{P}(L) \to \mathscr{P}(L)$ defined by $g(P) = \{x \in L; x^{\circ} \notin P\}$ is well-defined and will play an important role in our characterization. In order to prepare the ground, we first prove

Lemma 2. *Let $L \in \boldsymbol{MS}$. Then*

(i) *$g^2(P) \subseteq P$, for all $P \in \mathscr{P}(L)$*

and (ii) *L satisfies $x \wedge x^{\circ} \leqq y \vee y^{\circ}$ if and only if P and $g(P)$ are comparable, for all $P \in \mathscr{P}(L)$.*

Proof. (i) Let $P \in \mathscr{P}(L)$. If $x \in g^2(P)$ then $x^{\circ} \notin g(P)$ which implies that $x^{\circ\circ} \in P$ and therefore $x \in P$, since $x \leqq x^{\circ\circ}$. Thus, $g^2(P) \subseteq P$.

(ii) Suppose that $x \wedge x^{\circ} \leqq y \vee y^{\circ}$, for all $x, y \in L$. Let $P \in \mathscr{P}(L)$. If $P \nsubseteq g(P)$ then there exists $p \in P$ such that $p^{\circ} \in P$. Now let $q \in g(P)$. Then $q^{\circ} \notin P$ and $q \wedge q^{\circ} \leqq \leqq p \vee p^{\circ} \in P$ which together imply that $q \in P$. Hence, $g(P) \subseteq P$ and it follows that P and $g(P)$ are comparable. If, conversely, P and $g(P)$ are comparable, for all $P \in \mathscr{P}(L)$, but $x \wedge x^{\circ} \nleqq y \vee y^{\circ}$, for some $x, y \in L$, then we can find a prime ideal P of L such that

$$y \vee y^{\circ} \in P \quad \text{and} \quad x \wedge x^{\circ} \notin P\,.$$

But $y \vee y^{\circ} \in P$ implies that $y \in P \setminus g(P)$ whereas $x \wedge x^{\circ} \notin P$ implies that $x \in g(P) \setminus P$. Thus, P and $g(P)$ are incomparable and we have a contradiction.

Theorem 3. *Let $L \in \boldsymbol{MS}$. Then $L \in \boldsymbol{K}_2$ if and only if, for all $P \in \mathscr{P}(L)$, we have*

(i) *P and $g(P)$ are comparable*

and (ii) *$P \subseteq g(P) \Rightarrow P = g^2(P)$.*

Proof. If $L \in \boldsymbol{K}_2$ then L satisfies $x \wedge x^{\circ} \leqq y \vee y^{\circ}$ and so, by Lemma 2(ii), P and $g(P)$ are comparable, for all $P \in \mathscr{P}(L)$. Suppose, now, that $P \subseteq g(P)$ but $P \neq g^2(P)$, for some $P \in \mathscr{P}(L)$. By Lemma 2(i), $P \nsubseteq g^2(P)$ and so there exists an element $a \in$

$\in P \setminus g^2(P)$. Now, since L is firm, we have $a^{\circ\circ} \wedge a^\circ = a \wedge a^\circ \leqq a$ so that $a^{\circ\circ} \wedge \wedge a^\circ \in P$ and, therefore, either $a^{\circ\circ} \in P$ or $a^\circ \in P$. But $a^{\circ\circ} \in P$ if and only if $a \in g^2(P)$ and so it follows that $a^\circ \in P$. Thus, $a \notin g(P)$ which is absurd because $a \in P \subseteq g^{\prime}(P)$.

Conversely, suppose that conditions (i) and (ii) hold. By Lemma 2(ii), L satisfies $x \wedge x^\circ \leqq y \vee y^\circ$ and so, by Theorem 1, it remains only to show that L is firm. Clearly, it is enough to show that $x^{\circ\circ} \wedge x^\circ \leqq x$, for all $x \in L$. Suppose, to the contrary, that $x^{\circ\circ} \wedge x^\circ \nleqq x$, for some $x \in L$. Then we can find a prime ideal P such that $x \in P$ but $x^{\circ\circ} \wedge x^\circ \notin P$. However, $x^{\circ\circ} \wedge x^\circ \notin P$ implies that $x^\circ \notin P$ and $x^{\circ\circ} \notin P$. But the latter condition is equivalent to $x^0 \in g(P)$, which, in turn, is equivalent to $x \notin g^2(P)$. It follows, now, that $g(P) \nsubseteq P$, since $x^\circ \in g(P) \setminus P$. Therefore, by hypothesis, $P = g^2(P)$ which is absurd because $x \in P \setminus g^2(P)$. Thus, $x^{\circ\circ} \wedge x^\circ \leqq x$, for all $x \in L$, and we conclude that L is firm.

Although nothing would have been gained, we could have used the duality of Ockham algebras, developed by A. Urquhart [14] (see also [8]), to prove the last theorem. However, it is worthwhile to point out that the algebraic category $\mathscr{K}_2$ associated with $\boldsymbol{K}_2$ is isomorphic to the dual of a certain category of ordered topological spaces. More precisely, let us call a pair $\langle X; g \rangle$, where X is a compact, totally order-disconnected space and g is a continuous, order reversing map from X into itself, a *K_2-space* if it satisfies, for all $x \in X$,

(i) $g^2(x) \leqq x$

(*ii*) x and $g(x)$ are comparable

and (iii) $x = g^2(x)$ whenever $x \leqq g(x)$.

It is not difficult to show, using the results from [14] and the last theorem, that $\mathscr{K}_2$ and the category whose objects are K_2-spaces and whose morphisms are continuous, order preserving maps which commute with g are dual categories.

Next, we give prime ideal characterizations of the proper, non-trivial subvarieties of $\boldsymbol{K}_2$ which, naturally enough, lead to dualities for each of the associated algebraic categories. First, we observe that by [6], $\boldsymbol{K} \vee \boldsymbol{S}$ can be characterized (relative to $\boldsymbol{K}_2$) by the identity $x \vee y^\circ \vee y^{\circ\circ} = x^{\circ\circ} \vee y^\circ \vee y^{\circ\circ}$, $\boldsymbol{S}$ by $x \wedge x^\circ = 0$, $\boldsymbol{K}$ by $x = x^{\circ\circ}$ and $\boldsymbol{B}$ by $x \vee x^\circ = 1$.

Theorem 4. *Let $L \in \boldsymbol{K}_2$. Then*

(i) *$L \in \boldsymbol{K} \vee \boldsymbol{S}$ if and only if, for all $P \in \mathscr{P}(L)$, $g^2(P) = P$ or $g^2(P) = g(P)$.*

(ii) *$L \in \boldsymbol{S}$ if and only if, for all $P \in \mathscr{P}(L)$, $g(P) \subseteq P$.*

(iii) *$L \in \boldsymbol{K}$ if and only if, for all $P \in \mathscr{P}(L)$, $g^2(P) = P$.*

(iv) *$L \in \boldsymbol{B}$ if and only if, for all $P \in \mathscr{P}(L)$, $g(P) = P$.*

Proof. (i) Suppose that the condition on $\mathscr{P}(L)$ holds but $L \notin \boldsymbol{K} \vee \boldsymbol{S}$. Then there are elements $x, y \in L$ such that $x^{\circ\circ} \nleqq x \vee y^\circ \vee y^{\circ\circ}$. Choose $P \in \mathscr{P}(L)$ such that $x \vee \vee y^\circ \vee y^{\circ\circ} \in P$ and $x^{\circ\circ} \notin P$. Then $x \in P \setminus g^2(P)$ and $y \in g^2(P) \setminus g(P)$ so $g^2(P) \neq P$ and $g^2(P) \neq g(P)$. Thus, $L \in \boldsymbol{K} \vee \boldsymbol{S}$. Now suppose that $L \in \boldsymbol{K} \vee \boldsymbol{S}$ but, for some $P \in \mathscr{P}(L)$, we have $g^2(P) \neq P$ and $g^2(P) \neq g(P)$. Since $L \in \boldsymbol{K}_2$, the former condition implies that $g(P) \subseteq P$ which, in conjunction with the latter condition and the fact that g is order reversing, shows that $g(P) \subset g^2(P)$. Consequently, we can find $y \in L$ such that $y^{\circ\circ} \in P$ and $y^\circ \in P$. Moreover, by lemma 2(i), $g^2(P) \subset P$ and so we can find $x \in L$ such that $x \in P$ and $x^{\circ\circ} \notin P$. But then $x \vee y^\circ \vee y^{\circ\circ} \in P$ which, since $x^{\circ\circ} \leqq x \vee y^\circ \vee y^{\circ\circ}$, implies that $x^{\circ\circ} \in P$ and we have a contradiction.

(ii) Let $L \in \boldsymbol{S}$. If $P \in \mathscr{P}(L)$ and $x \in g(P) \setminus P$ then $x^\circ \notin P$ which is absurd because $x \wedge x^\circ \in P$. Thus, $g(P) \subseteq P$. Conversely, if the condition on $\mathscr{P}(L)$ holds but $x \wedge x^\circ \neq 0$, for some $x \in L$, then there is $P \in \mathscr{P}(L)$ such that $x \wedge x^\circ \notin P$. It follows that $x \in g(P) \setminus P$ which is contrary to $g(P) \subseteq P$. Thus, $L \in S$.

The proofs of (iii) and (iv) are straightforeward and left to the reader.

4. Congruence pairs. Every algebra $L \in \boldsymbol{K}_2$ has two auxiliary substructures; namely, the Kleene subalgebra $L^{\circ\circ}$ and the sublattice $L^\vee$. We can associate with any $\theta \in \operatorname{Con}(L)$, the congruence lattice of L, the pair

$$\langle \theta_1, \theta_2 \rangle \in \operatorname{Con}(L^{\circ\circ}) \times \operatorname{Con}(L^\vee),$$

where θ_1 is the restriction $\theta \mid L^{\circ\circ}$ of θ to $L^{\circ\circ}$ and θ_2 is the restriction $\theta \mid L^\vee$ of θ to $L^\vee$. Clearly, the pair $\langle \theta_1, \theta_2 \rangle$ satisfies the following two conditions:

(CP_1) $c \equiv d(\theta_2) \Rightarrow c^\circ \equiv d^\circ(\theta_1)$

(CP_2) $a \equiv b(\theta_1) \,\&\, c \in L^\vee \Rightarrow a \vee c \equiv b \vee c(\theta)$.

Henceforth, any pair $\langle \theta_1, \theta_2 \rangle \in \operatorname{Con}(L^{\circ\circ}) \times \operatorname{Con}(L^\vee)$ that satisfies (CP_1) and (CP_2) will be called a *K_2-congruence pair.*

In order to prepare the way for the main theorem, we prove

Lemma 5. *Let* $\langle \theta_1, \theta_2 \rangle \in \operatorname{Con}(L^{\circ\circ}) \times \operatorname{Con}(L^\vee)$ *satisfy* (CP_2) *then*
(i) $a \equiv b(\theta_1) \,\&\, c \equiv d(\theta_2) \Rightarrow a \vee c \equiv b \vee d(\theta_2)$
and (ii) $a \equiv b(\theta_1) \Rightarrow a \vee a^0 \equiv b \vee b^0(\theta)$.

Proof (i). Suppose that $a \equiv b(\theta_1)$, $c \equiv d(\theta_2)$ and, without loss of generality, that $c \leqq d$. Then $a \vee c \equiv b \vee c(\theta_2)$ by (CP_2). This and $c \equiv d(\theta_2)$ imply $a \vee c \equiv b \vee d(\theta_2)$, since $c \leqq d$.

(ii) Let $a \equiv b(\theta_1)$. Then $a \vee a^\circ \equiv b \vee b^\circ(\theta_1)$, since $a^\circ \equiv b^\circ(\theta_1)$. Therefore, $a \vee a^\circ \equiv a \vee a^\circ \vee b \vee b^\circ(\theta_2)$ and $b \vee b^\circ \equiv a \vee a^\circ \vee b \vee b^\circ(\theta_2)$ by (i). Thus, $a \vee a^\circ \equiv b \vee b^\circ(\theta_2)$.

Theorem 6. *Every congruence relation θ on an algebra $L \in \boldsymbol{K}_2$ determines a K_2-congruence pair. Conversely, every K_2-congruence pair $\langle \theta_1, \theta_2 \rangle$ uniquely determines a congruence relation θ on L satisfying $\theta \mid L^{\circ\circ} = \theta_1$ and $\theta \mid L^\vee = \theta_2$ by the rule*

$$x \equiv y(\theta) \Leftrightarrow x^\circ \equiv y^\circ(\theta_1) \,\&\, x \vee x^\circ \equiv y \vee y^\circ(\theta_2)$$

or, equivalently, by the rule

$$x \equiv y(\theta) \Leftrightarrow x^\circ \equiv y^\circ(\theta_1) \,\&\, x \vee u \equiv y \vee u(\theta_2),$$

for all $u \in L^\vee$.

Proof. Let θ be the relation defined by the first rule. Clearly, θ is an equivalence relation. To show that it is, indeed, a congruence on L, let $a \equiv b(\theta)$ and $c \equiv d(\theta)$ so that

$$a^\circ \equiv b^\circ(\theta_1), \quad c^\circ \equiv d^\circ(\theta_1)$$

$$\text{and} \quad a \vee a^\circ \equiv b \vee b^\circ(\theta_2), \quad c \vee c^\circ \equiv d \vee d^\circ(\theta_2).$$

Then $(a \wedge c)^\circ = a^\circ \vee c^\circ \equiv b^\circ \vee d^\circ(\theta_1)$ and so $(a \wedge c)^\circ \equiv (b \wedge d)^\circ (\theta_1)$. Also, by distributivity, we have

$$(a \wedge c) \vee (a \wedge c)^\circ = (a \wedge c) \vee (a^\circ \vee c^\circ) = (a \vee a^\circ \vee c^\circ) \wedge (c \vee c^\circ \vee a^\circ).$$

Using Lemma 5(i), we see that

$$a \vee a^\circ \vee c^\circ \equiv b \vee b^\circ \vee d^\circ(\theta_2) \quad \text{and} \quad c \vee c^\circ \vee a^\circ \equiv d \vee d^\circ \vee b^\circ(\theta_2).$$

Therefore,

$$(a \wedge c) \vee (a \wedge c)^\circ \equiv (b \vee b^\circ \vee d^\circ) \wedge (d \vee d^\circ \vee b^\circ)(\theta_2) = (b \wedge d) \vee (b \wedge d)^\circ.$$

Consequently, θ preserves the meet operation.

Next, we show that θ preserves joins. First, observe that

$$(a \vee c)^\circ = a^\circ \wedge c^\circ \equiv b^\circ \wedge d^\circ(\theta_1)$$

and so $(a \vee c)^\circ \equiv (b \vee d)^\circ (\theta_1)$. In addition, we have

$$(a \vee c) \vee (a \vee c)^\circ = (a \vee c) \vee (a^\circ \wedge c^\circ) = (a \vee a^\circ \vee c) \wedge (c \vee c^\circ \vee a),$$

by distributivity.

Clearly, it is enough to show that $a \vee a^\circ \vee c \equiv b \vee b^\circ \vee d(\theta_2)$ and $c \vee c^\circ \vee$ $\vee\, a \equiv d \vee d^\circ \vee b(\theta_2)$. Using the fact that L is distributive and firm, we see that

$$a \vee a^\circ \vee c = (a \vee a^\circ) \vee [c^{\circ\circ} \wedge (c \vee c^\circ)] =$$

$$= (a \vee a^\circ \vee c^{\circ\circ}) \wedge [(a \vee a^\circ) \vee (c \vee c^\circ)].$$

But $a \vee a^\circ \equiv b \vee b^\circ(\theta_2)$, $c^{\circ\circ} \equiv d^{\circ\circ}(\theta_1)$ and $c \vee c^0 \equiv d \vee d^\circ(\theta_2)$ so that

$$a \vee a^\circ \vee c^{\circ\circ} \equiv b \vee b^\circ \vee d^{\circ\circ}(\theta_2),$$

by Lemma 5(i), and

$$(a \vee a^\circ) \vee (c \vee c^\circ) \equiv (b \vee b^\circ) \vee (d \vee d^\circ)(\theta_2).$$

Therefore,

$$a \vee a^\circ \vee c \equiv (b \vee b^\circ \vee d^{\circ\circ}) \wedge [(b \vee b^\circ) \vee (d \vee d^\circ)](\theta_2)$$

from which it follows that $a \vee a^\circ \vee c \equiv b \vee b^\circ \vee d(\theta_2)$. Similarly, $c \vee c^\circ \vee a \equiv$ $\equiv d \vee d^\circ \vee b(\theta_2)$. Therefore,

$$(a \vee c) \vee (a \vee c)^\circ \equiv (b \vee d) \vee (b \vee d)^\circ(\theta_2)$$

and we can conclude that θ preserves the join operation.

That θ preserves the unary operation $^\circ$ is easily seen. Indeed, if $a \equiv b(\theta)$ then

$a^{\circ\circ} \equiv b^{\circ\circ}(\theta_1)$ and so $a^{\circ\circ} \vee (a^{\circ\circ})^\circ \equiv b^{\circ\circ} \vee (b^{\circ\circ})^\circ (\theta_2)$, by Lemma 5(ii). Thus, $a^\circ \vee a^{\circ\circ} \equiv b^\circ \vee b^{\circ\circ}(\theta_2)$ and we conclude that $a^\circ \equiv b^\circ(\theta)$.

Next, we show that $\theta \mid L^{\circ\circ} = \theta_1$ and $\theta \mid L^\vee = \theta_2$. Let $a, b \in L^{\circ\circ}$. If $a \equiv b(\theta_1)$ then $a^\circ \equiv b^\circ(\theta_1)$ and, by Lemma 5(ii), $a \vee a^\circ \equiv b \vee b^\circ(\theta_2)$ so that $a \equiv b(\theta \mid L^{\circ\circ})$. Conversely, if $a \equiv b(\theta \mid L^{\circ\circ})$ then $a^\circ \equiv b^\circ(\theta_1)$ so that $a = a^{\circ\circ} \equiv b^{\circ\circ}(\theta_1) = b$. Therefore, $\theta \mid L^{\circ\circ} = \theta_1$. Now let $c, d \in L^\vee$. If $c \equiv d(\theta_2)$ then $c^\circ \equiv d^\circ(\theta_1)$, by (CP_1), and so $c \vee c^\circ \equiv d \vee d^\circ(\theta_2)$, by Lemma 5(i). Thus, $c \equiv d(\theta \mid L^\vee)$. Conversely, if $c \equiv d(\theta \mid L^\vee)$ then $c = c \vee c^\circ \equiv d \vee d^\circ(\theta_2) = d$, since $c, d \in L^\vee$, and so $\theta \mid L^\vee \leqq \theta_2$.

For the uniqueness part of the theorem, suppose that $\theta, \psi \in \mathrm{Con}(L)$, $\theta \mid L^{\circ\circ} = \psi \mid L^{\circ\circ}$ and $\theta \mid L^\vee = \psi \mid L^\vee$. Let $x \equiv y(\theta)$. Then $x^{\circ\circ} \equiv y^{\circ\circ}(\theta \mid L^{\circ\circ})$, so that $x^{\circ\circ} \equiv y^{\circ\circ}(\psi \mid L^{\circ\circ})$, and $x \vee x^\circ \equiv y \vee y^\circ(\theta \mid L^\vee)$, so that $x \vee x^\circ \equiv y\, y^\circ(\psi \mid L^\vee)$. Therefore,

$$x = x^{\circ\circ} \wedge (x \vee x^\circ) \equiv y^{\circ\circ} \wedge (y \vee y^\circ)(\psi),$$

since L is firm, and we have $x \equiv y(\psi)$. Similarly, we can show that $\psi \leqq \theta$. Hence, $\theta = \psi$.

Finally, we show that, for a given K_2-congruence pair $\langle \theta_1, \theta_2 \rangle$, the two rules for θ are equivalent. First, suppose that $x^\circ \equiv y^\circ(\theta_1)$, $x \vee x^\circ \equiv y \vee y^\circ(\theta_2)$ and $u \in L^\vee$. Since L is distributive and firm, we have

$$x \vee u = (x^{\circ\circ} \vee u) \wedge (x \vee x^\circ \vee u).$$

But $x^{\circ\circ} \equiv y^{\circ\circ}(\theta_1)$ and so $x^{\circ\circ} \vee u \equiv y^{\circ\circ} \vee u(\theta_2)$, by (CP_2). Obviously, $x \vee x^\circ \vee u \equiv y \vee y^\circ \vee u(\theta_2)$. Therefore,

$$x \vee u \equiv (y^{\circ\circ} \vee u) \wedge (y \vee y^\circ \vee u)(\theta_2)$$

from which it follows that $x \vee u \equiv y \vee u(\theta_2)$. Thus, the first rule implies the second. Next, suppose that $x^\circ \equiv y^\circ(\theta_1)$ and $x \vee u \equiv y \vee u(\theta_2)$, for all $u \in L^\vee$. Then, by Lemma 5(i),

$$x^\circ \vee (x \vee u) \equiv y^\circ \vee (y \vee u)(\theta_2), \quad \text{for all} \quad u \in L^\vee.$$

On taking $u = x \vee x^\circ$ and $u = y \vee y^\circ$ in turn, we obtain $x \vee x^\circ \equiv x \vee y \vee x^\circ \vee y^\circ(\theta_2)$ and $y \vee y^\circ \equiv x \vee y \vee x^\circ \vee y^\circ(\theta_2)$ from which it follows that $x \vee x^\circ \equiv y \vee y^\circ(\theta_2)$. Thus, the second rule implies the first.

Corollary 7. *If $L \in K_2$ then the set $\mathrm{Con}_2(L)$ of K_2-congruence pairs of L is a sublattice of $\mathrm{Con}(L^{\circ\circ}) \times \mathrm{Con}(L^\vee)$ and $\theta \mapsto \langle \theta \mid L^{\circ\circ}, \theta \mid L^\vee \rangle$ is an isomorphism from $\mathrm{Con}(L)$ to $\mathrm{Con}_2(L)$.*

Proof. Let $\langle \theta_1, \theta_2 \rangle, \langle \psi_1, \psi_2 \rangle \in \mathrm{Con}_2(L)$. It is routine to show that $\langle \theta_1 \wedge \psi_1, \theta_2 \wedge \psi_2 \rangle \in \mathrm{Con}_2(L)$. In order to show that $\langle \theta_1 \vee \psi_1, \theta_2 \vee \psi_2 \rangle \in \mathrm{Con}_2(L)$, let $a \equiv b(\theta_1 \vee \psi_1)$ and $c \equiv d(\theta_2 \vee \psi_2)$. Then there are sequences

$$a = a_0, a_1, \ldots, a_m = b \text{ in } L^{\circ\circ} \quad \text{and} \quad c = c_0, c_1, \ldots, c_n = d \text{ in } L^\vee$$

such that $a_{i-1} \equiv a_i(\theta_1 \cup \psi_1)$ and $c_{j-1} \equiv c_j(\theta_2 \cup \psi_2)$, whenever $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$. Observe that

$$c^\circ_{j-1} \equiv c^\circ_j(\theta_1 \cup \psi_1) \quad \text{and} \quad a_{i-1} \vee c \equiv a_i \vee c(\theta_2 \cup \psi_2)\,,$$

by (CP_1) and (CP_2). Thus, the sequences

$$c^\circ = c^\circ_0, c^\circ_1, \ldots, c^\circ_n = d^\circ \text{ in } L^{\circ\circ}$$

$$\text{and} \quad a \vee c = a_0 \vee c,\ a_1 \vee c, \ldots, a_m \vee c = b \vee c \text{ in } L^\vee$$

ensure that $c^\circ \equiv d^\circ(\theta_1 \vee \psi_1)$ and $a \vee c \equiv b \vee c(\theta_2 \vee \psi_2)$, respectively. Consequently, $\langle \theta_1 \vee \psi_1, \theta_2 \vee \psi_2 \rangle \in \mathrm{Con}_2(L)$ and we conclude that $\mathrm{Con}_2(L)$ is a sublattice of $\mathrm{Con}(L^{\circ\circ}) \times \mathrm{Con}(L^\vee)$. That $\theta \mapsto \langle \theta \mid L^{\circ\circ}, \theta \mid L^\vee \rangle$ is an (order) isomorphism is easily verified using Theorem 6.

Recall that if $\langle L, \vee, \wedge, {}^\circ, 0, 1 \rangle \in \boldsymbol{S}$ then $L^{\circ\circ}$ is commonly called the *skeleton of* L, usually denoted by $B(L)$, and is a Boolean sublattice of L. In addition, $L^\vee$ coincides with the *dense filter* $D(L) := \{x \in L;\ x^{\circ\circ} = 1\}$ and a pair $\langle \theta_1, \theta_2 \rangle \in \mathrm{Con}(B(L) \times \times \mathrm{Con}(D(L))$ is called a *congruence pair* if it satisfies the condition:

$$a \equiv 1(\theta_1)\ \&\ a \leqq d \in D(L) \Rightarrow d \equiv 1(\theta_2)\,.$$

T. Katriňák [10] and H. Lakser [13] (see also [9]) have shown that the statement of Theorem 6, in which "K_2-congruence pair" is replaced by "congruence pair", holds for the class of distributive p-algebras and so, in particular, for the class $\boldsymbol{S}$ of Stone algebras. In fact, T. Katriňák [12] has recently shown that exactly the same result holds in a much wider variety of p-algebras which properly contains all modular p-algebras. With this in mind, we prove

Corollary 8. *Let L be a Stone algebra and let* $\langle \theta_1, \theta_2 \rangle \in \mathrm{Con}(B(L)) \times \mathrm{Con}(D(L))$. *Then* $\langle \theta_1, \theta_2 \rangle \in \mathrm{Con}_2(L)$ *if and only if it is a congruence pair.*

Proof. Let $\langle \theta_1, \theta_2 \rangle$ be a congruence pair. Since $d^\circ = 0$ whenever $d \in D(L)$, property (CP_1) trivially holds when $L \in \boldsymbol{S}$. Now, suppose that $a \equiv b(\theta_1)$ and $c \in D(L)$. Let $\alpha = (a \vee b^\circ) \wedge (b \vee a^\circ)$. Then $\alpha \in B(L)$, $a \wedge \alpha = b \wedge \alpha = a \wedge b$ and $\alpha \equiv \equiv 1(\theta_1)$. Since $\alpha \leqq c \vee \alpha \in D(L)$, we have $c \vee \alpha \equiv 1(\theta_2)$ which implies that

$$c \vee a \equiv (c \vee a) \wedge (c \vee \alpha)(\theta_2) = c \vee (a \wedge \alpha) = c \vee (a \wedge b)\,.$$

Similarly, $c \vee b \equiv c \vee (a \wedge b)(\theta_2)$. Therefore, $a \vee c \equiv b \vee c(\theta_2)$ and we conclude that (CP_2) holds. Thus, any congruence pair belongs to $\mathrm{Con}_2(L)$. Finally, if $\langle \theta_1, \theta_2 \rangle \in \in \mathrm{Con}_2(L)$, $a \in B(L)$, $a \leqq d \in D(L)$ and $a \equiv 1(\theta_1)$ then $d = d \vee a \equiv d \vee 1(\theta_2)$, by (CP_2), and so $d \equiv 1(\theta_2)$. Thus, any member of $\mathrm{Con}_2(L)$ is a congruence pair.

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