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Czechoslovak Mathematical Journal, Vol. 35 (1985), No. 2, 333–337

Persistent URL: `http://dml.cz/dmlcz/102021`

Czechoslovak Mathematical Journal, 35 (110) 1985, Praha

MODULARITY AND DISTRIBUTIVITY OF TOLERANCE LATTICES OF COMMUTATIVE SEPARATIVE SEMIGROUPS

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(Received May 7, 1984)

By a tolerance on a semigroup S we mean a reflexive and symmetric subsemigroup of the direct product $S \times S$. The set $\mathscr{L}(S)$ of all tolerances on S forms a complete algebraic lattice with respect to set inclusion (see [1] and [2]). The aim of this paper is to consider modularity and distributivity of $\mathscr{L}(S)$ when S is a commutative separative semigroup.

For any tolerance T on a semigroup S we have

$$(1) \qquad (au, bv) = (a, b)(u, v) \in T$$

whenever $(a, b) \in T$ and $(u, v) \in T$. This implies that for every positive integer m we have

$$(2) \qquad (a, b)^m = (a^m, b^m) \in T$$

whenever $(a, b) \in T$. For all $a, b, z \in S$ we shall use the following notation: $(a, b)z = (az, bz)$.

Let $\emptyset \neq A \subseteqq S \times S$. By $T(A)$ we denote the least tolerance on S containing A. The symbol S^1 stands for S if S has an identity, otherwise it stands for S with an identity adjoined.

Lemma 1. *Let S be a commutative semigroup. For $x, y \in S$, $x \neq y$, we have $(x, y) \in T(A)$ if and only if $x = x_1x_2 \ldots x_nz$ and $y = y_1y_2 \ldots y_nz$, where $z \in S^1$ and either $(x_i, y_i) \in A$ or $(y_i, x_i) \in A$.*

Proof. Apply (1).

This implies the following:

Lemma 2. *Let S be a commutative semigroup and $a, b \in S$, $a \neq b$. For $x, y \in S$, $x \neq y$, we have $(x, y) \in T(a, b)$ if and only if there exist $z \in S^1$ and a positive integer m such that either $(x, y) = (a, b)^m z$ or $(x, y) = (b, a)^m z$.*

By $\vee$ and $\wedge$ we denote the join or meet in the lattice $\mathscr{L}(S)$. Clearly we have $A \vee B = T(A \cup B)$ and $A \wedge B = A \cap B$ for all $A, B \in \mathscr{L}(S)$.

First, we shall consider commutative regular semigroups. Recall that every commutative regular semigroup S is a semilattice of commutative groups (see [3]). The set of all idempotents of S is denoted by $E(S)$ and is partially ordeted by: $e \leqq f$ if $ef = e$. We write $e < f$ for $e \leqq f$ and $e \neq f$. By $e \parallel f$ we denote the fact that idempotents e, f are incomparable. For any integer k, by x^k we denote the k-power of an element x of S in the maximal subgroup G_e of S containing an idempotent $e = x^0$. It is known that for all $x, y \in S$ and all integers k we have

$$(3) \qquad (xy)^k = x^k y^k .$$

By $J(S)$ we denote the set of all tolerances I on a commutative regular semigroup S satisfying the following condition:

$$(4) \qquad \text{If} \quad (a, b) \in I, \quad \text{then} \quad (a^{-1}, b^{-1}) \in I .$$

Using (3) and Lemma 1 we can show that $J(S)$ is a sublattice of $\mathscr{L}(S)$.

Theorem 1. *Let S be a commutative regular semigroup. Then the lattice $\mathscr{L}(S)$ is modular if and only if S satisfies the following conditions:*

(M1) *If e, f are two idempotents of S such that $e \parallel f$, then at least one of them is maximal with respect to the order in $E(S)$ and there exists no idempotent g of S such that $g \parallel ef$.*

(M2) *If e, f are two idempotents of S such that $e < f$, then $ze = e$ for every element z of the maximal subgroup G_f of S.*

(M3) *If e, f, g are three idempotents of S such that $e < f$ and $e \parallel g$, then the maximal subgroup G_g of S contains exactly one element.*

(M4) *S is either periodic or $E(S)$ contains the greatest element i and the maximal subgroup G_e of S is periodic for each $e < i$.*

Proof. I. Suppose that the lattice $\mathscr{L}(S)$ is modular. Then the sublattice $J(S)$ of $\mathscr{L}(S)$ is modular and so according to Theorem 1.1 of [4], S satisfies the conditions (M1), (M2) and (M3). Now, we shall show that S has the property (M4).

Let a be a non periodic element of S. By way of contradiction, assume that there exists an idempotent e of S such that either $a^0 < e$ or $a^0 \parallel e$. Put $A = T(a^2, e)$, $B = T(a^3, e)$ and $C = T((a^2, e), (a^5, e))$. It is clear that $A \subseteq C$. Since the lattice $\mathscr{L}(S)$ is modular, it follows from Lemma 1 that $(a^5, e) \in (A \vee B) \wedge C = A \vee \vee (B \wedge C)$.

Suppose that $(a^5, e) \in A$. Then, by Lemma 2 and (3), we have $(a^5, e) = (a^2, e)^m z$ for a positive integer m and $z \in S^1$. Assume that $z \in S$, then $e \leqq z^0$. If $a^0 < e$, then $a^0 < z^0$ and so, by (M2), we have $a^0 = a^0 z$. This implies that $a^5 = a^{2m} z = a^{2m}$, which is a contradiction. If $a^0 \parallel e$, then according to (M3), we obtain that $e = z^0$. Then, by (3), we have $a^0 = (a^5)^0 = (a^{2m} z)^0 = a^0 z^0 \leqq z^0 = e$, a contradiction. If $z \notin S$, then we have $a^5 = a^{2m}$, a contradiction.

In an analogous manner it can be shown that $(a^5, e) \notin B$. According to Lemma 1 and (1), we have $(a^5, e) = (u, x)(v, y)$ for some $(u, x) \in A \setminus B$ and $(v, y) \in (B \cap C) \setminus A$.

From (3) it follows that $e \leqq x^0$ and so $x \notin Sa^0$. Thus, by Lemma 2, we have $(u, x) = (a^2, e)^m z$ for some $z \in S^1$ and a positive integer m. Analogously it can be shown that $(v, y) = (a^3, e)^n w$ for some $w \in S^1$ and a positive integer n. We shall prove that $v = a^3$. We have $(a^5, e) = (a^{2m+3n}, e) zw$. Assume that $zw \in S$, then $e \leqq (zw)^0$. If $a^0 < e$, then $a^0 < (zw)^0$. It follows from (M2) that $a^0 = a^0 zw = a^0 w$ and so $a^5 = a^{2m+3n}$. This implies that $n = 1$ and so $v = a^3 w = a^3$. If $a^0 \parallel e$, then, by (M3), we have $e = (zw)^0$. Using (3) we get $a^0 = (a^5)^0 = a^0(zw)^0 \leqq e$, which is a contradiction. If $zw \notin S$, then we have $a^5 = a^{2m+3n}$ and so $v = a^3$. Since $(v, y) \in C \setminus A$ and $y \notin Sa^0$, we have according to Lemma 1, $(v, y) = (a^5, e)^r t$ or $(v, y) = (a^5, e)^r (a^2, e)^s t$ for some $t \in S^1$ and some positive integers r, s. This implies $a^3 = v = a^{5r}t$ or $a^3 = a^{5r+2s}t$ and so $t \in S$. It follows from (3) that $e \leqq y^0 \leqq t^0$. If $a^0 < e$, then $a^0 < t^0$ and so, by (M2), we have $a^0 = a^0 t$. Therefore $a^3 = a^{5r}$ or $a^3 = a^{5r+2s}$, which is a contradiction. If $a^0 \parallel e$, then according to (M3) we have $e = t^0$. Using (3) we get $a^0 \leqq v^0 \leqq t^0 = e$, a contradiction. Therefore S satisfies the condition (M4).

II. Suppose that S satisfies the conditions (M1), (M2), (M3) and (M4). According to Theorem 1.1 of [4], the lattice $J(S)$ is modular. It follows from (M4) and Theorem 7 of [5] that $\mathscr{L}(S) = J(S)$ and so the lattice $\mathscr{L}(S)$ is modular.

Corollary 1. *Let S be a commutative regular semigroup. If the lattice $\mathscr{L}(S)$ is modular, then $\mathscr{L}(S) = J(S)$.*

The following result is a generalization of the well known Ores' theorem (see [6]) that for every commutative group G the lattice $\mathscr{L}(G)$ (which coincides with the lattice of all congruences on G, see [7]) is distributive if and only if G is locally cyclic, i.e. every its subgroup generated by a finite set of generators is cyclic. Let x be a periodic element of a commutative regular semigroup S. By ord x we denote the order of x in the maximal subgroup G_{x^0} of S.

Theorem 2. *Let S be a commutative regular semigroup. Then the lattice $\mathscr{L}(S)$ is distributive if and only if S satisfies:*

(M1), (M2), (M3) *and* (M4).

(D1) *Every maximal subgroup of S is locally cyclic.*

(D2) *Let G_e, G_f be two maximal subgroups of S such that $e \parallel f$, $e, f \in E(S)$. If $x \in G_e$ and $y \in G_f$, then* ord x, ord y *are relatively prime.*

The proof follows from Theorem 1, Theorem 7 of [5] and Theorem 1.1 of [4].

A semigroup S is said to be *separative* if $a^2 = ab = b^2$ imply $a = b$ $(a, b \in S)$.

Lemma 3. *Let S be a commutative separative semigroup, $a \in S$. If $a^2 \in a^3 S^1$, then $a \in a^2 S^1$.*

Proof. Suppose that $a^2 \in a^3 S^1$. Then $a^2 = a^2(ab)$ for some $b \in S^1$ and so $a^2(ab) = a^2(ab)^2$. Hence we have $a = a(ab) \in a^2 S^1$.

Theorem 3. *Let S be a commutative semigroup whose lattice $\mathscr{L}(S)$ is modular. Then S is regular if and only if it is separative.*

Proof. Suppose that the lattice $\mathscr{L}(S)$ of a commutative semigroup is modular. It follows from (3) that every commutative regular semigroup is separative. Now, we shall assume that S is separative and not regular. Then there exists $a \in S$ such that $a \notin a^2S^1$.

We shall distinguish two cases.

Case 1: a is periodic. Then there exist positive integers k and m such that $a^k = e = e^2$, $a^m e \neq a^m$ and $a^{2m}e = a^{2m}$. We have $(a^m)^2 = a^m(a^m e) = (a^m e)^2$ and so $a^m = a^m e$, which is a contradiction.

Case 2: a is not periodic. Put $A = T(a^2, a)$, $B = T(a^3, a)$ and $C = T((a^2, a), (a^5, a^2))$. It is clear that $A \subseteqq C$ and so, by Lemma 1, $(a^5, a^2) \in (A \vee B) \wedge C = A \vee (B \wedge C)$.

Suppose that $(a^5, a^2) \in A$. Then, by Lemma 2, we have $(a^5, a^2) = (a^2, a)^m z$ or $(a^5, a^2) = (a, a^2)^m z$ for a positive integer m and $z \in S^1$. Assume that $(a^5, a^2) = (a^2, a)^m z$. Then $a^5 = a^{2m}z = a^m(a^m z) = a^{m+2}$ and so $m = 3$. Therefore $a^2 \in a^3S^1$. Lemma 3 implies that $a \in a^2S^1$, which is a contradiction. Assume that $(a^5, a^2) = (a, a^2)^m z$. Then $a^2 = a^{2m}z = a^m(a^m z) = a^{m+5}$, a contradiction.

Analogously we can show that $(a^5, a^2) \notin B$. According to Lemma 1 and (1), we have $(a^5, a^2) = (u, x)(v, y)$ for some $(u, x) \in A \setminus B$ and $(v, y) \in (B \cap C) \setminus A$. Then, by Lemma 2, we obtain that $(u, x) = (a^2, a)^m z$ or $(u, x) = (a, a^2)^m z$ for a positive integer m and $z \in S^1$. Further, there exist a positive integer n and $w \in S^1$ such that $(v, y) = (a^3, a)^n w$ or $(v, y) = (a, a^3)^n w$. We have the following two possibilities:

Case a: $(u, x) = (a^2, a)^m z$ and $(v, y) = (a^3, a)^n w$. Then $a^5 = uv = a^{2m+3n}zw = a^{m+2n}(a^m z)(a^n w) = a^{m+2n}xy = a^{m+2n+2}$ and so $m = n = 1$. Hence we have $a^2 = xy = a^2zw = a^2(zw)^2$ and so $a = azw$. Since $(v, y) \in C \setminus A$, it follows from Lemma 1 that $y = a^2c$ for some $c \in S^1$. Thus we have $a = awz = yz = a^2cz$, which is a contradiction.

Case b: $(u, x) = (a, a^2)^m z$ or $(v, y) = (a, a^3)^n w$. In both cases we have $a^2 = xy \in a^3S^1$. Lemma 3 implies $a \in a^2S^1$, which is a contradiction.

Corollary 2. *Let S be a commutative semigroup. Then S is separative with the modular (or distributive) lattice $\mathscr{L}(S)$ if and only if S is regular and satisfies the conditions* (Mi) *(respectively* (Mi), (D1) *and* (D2)) *for* i = 1, 2, 3 *and* 4.

The proof follows from Theorems 1, 2 and 3.

Compare with [8].

It is easy to show that every commutative cancellative semigroup is separative and every commutative regular cancellative semigroup is a group.

Corollary 3. *Let S be a commutative semigroup. Then S is cancellative with the modular (or distributive) lattice $\mathscr{L}(S)$ if and only if S is a group (respectively a locally cyclic group).*

References

[1] *Chajda I.:* Lattices of compatible relations. Arch. Math. (Brno) *13* (1977), 89—96.

[2] *Chajda I.* and *Zelinka B.:* Lattices of tolerances. Čas. pěst. mat. *102* (1977), 10—24.

[3] *Clifford A. H.* and *Preston G. B.:* The algebraic theory of semigroups. Amer. Math. Soc., Providence, R. I. Vol. *I* (1961); Vol. *II* (1967).

[4] *Pondělíček B.:* Modularity and distributivity of tolerances lattices of commutative inverse semigroups. Czech. Math. J. (to appear).

[5] *Pondělíček B.:* Atomicity of tolerance lattices of commutative semigroups. Czech. Math. J. *33* (*108*) 1983, 485—498.

[6] *Ore O.:* Structures and group theory II. Duke Math. J. *4* (1938), 247—269.

[7] *Zelinka B.:* Tolerance in algebraic structures II. Czech. Math. J. *25* (*100*) 1975, 175—178.

[8] *Hamilton H.:* Modularity and distributivity of the congruence lattice of a commutative separative semigroup. Math. Japonica *27*, No. 5 (1982), 581—589.

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