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SEMI-PROJECTABLE l-GROUPS

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Introduction. A lattice-ordered group (l-group) is a group having a lattice order compatible with the group operation. If G is an l-group, we call the identity element: $\boldsymbol{e}$, and we designate by G_+, the set of elements of G greater than or equal to $\boldsymbol{e}$ (positive elements). Two elements $\boldsymbol{x}$ and $\boldsymbol{y}$ of G are orthogonal, if $x \wedge y = e$.

An l-group is projectable if, given any positive element $\boldsymbol{a}$ of G, each element $g \in G_+$ can be written in the form

$$g = \alpha\beta$$

where α is orthogonal to $\boldsymbol{a}$, and β, orthogonal to all the elements orthogonal to $\boldsymbol{a}$; it is semi-projectable if, given two orthogonal elements $\boldsymbol{a}$ and $\boldsymbol{b}$, each $g \in G_+$ can be written in the form

$$g = \prod_{i=1}^{n} \alpha_i\beta_i$$

where each α_i is orthogonal to $\boldsymbol{a}$ and each β_i orthogonal to $\boldsymbol{b}$.

Semi-projectable l-groups are defined and their known properties described in ([1], 7.5), where they are used to characterize l-groups that are small direct products of totally ordered groups. An example of a semi-projectable l-group that is not projectable occurs already in ([3], p. 233, first example; see also [2]). It is easy to see that a projectable l-group is representable (sc. as a sub-direct product of totally ordered groups); that this is not necessarily true of semi-projectable l-groups has been proved only recently by an example attributed by Glass to G. M. Bergman ([4], Theorem 11 G).

In the first paragraph of this paper we describe some characteristic properties of semi-projectable l-groups, which imply in particular that direct products, homomorphic images and Archimedean extensions of semi-projectable l-groups are semi-projectable. In the second paragraph, we state some sufficient conditions for a semi-projectable l-group to be representable. The most important perhaps is this: if a semi-projectable l-group is normal-valued, it is representable. Finally, we conclude with some considerations concerning non-representable semi-projectable l-groups.

For terminology and notation we follow ([1]), whose principal results we assume known. We thank M. Giraudet for useful comments, and the referee for valuable suggestions.

1. Characteristic properties of semi-projectable l-groups.

The condition used in ([1]) to define semi-projectable l-groups appears stronger than the one stated here, but it is in fact equivalent. Recall that a subgroup of an l-group G is *solid*, if it is at the same time a convex sublattice of G. The *polar* of an element g of G_+ (notation: $g^\perp$) is the solid subgroup generated by the elements orthogonal to g.

Proposition 1.1. *Let G be a semi-projectable l-group. For any two elements a and b of G_+, $(a \wedge b)^\perp$ is generated (as a solid subgroup) by $a^\perp \cup b^\perp$.*

Proof. It is sufficient to show that every positive element of $(a \wedge b)^\perp$ is a product of elements of $a^\perp$ and elements of $b^\perp$. Suppose that $a \wedge b \wedge g = e$. Then every element of G_+, in particular g, is a product of elements u, orthogonal to a, and elements v, orthogonal to $b \wedge g$. For each such v, $v \leqq g$, hence $g \wedge v = v$, and $e = b \wedge g \wedge v = b \wedge v$. Thus g is a product of elements orthogonal to a and elements orthogonal to b.

Recall that a prime subgroup of an l-group G is a solid subgroup P, such that the solid subgroups of G that contain P, ordered by inclusion, form a chain. In this paper, to simplify statements, a prime subgroup of G is always different from G. A solid subgroup M of G is regular, if there exists an element g of G, such that M is maximal among the solid subgroups of G that do not contain g. We also say in this case that M is a *value* of g. Regular subgroups are known to be prime.

Proposition 1.2. *An l-group G is semi-projectable if and only if it satisfies one of the following equivalent conditions:*

i) *Every prime subgroup of G contains a unique minimal prime subgroup;*

ii) *The set of prime subgroups of G, ordered by inclusion, is the union of a set of disjoint chains;*

iii) *Every regular subgroup of G contains a unique minimal prime subgroup;*

iv) *The set of regular subgroups of G, ordered by inclusion, is the union of a set of disjoint chains.*

Proof. The equivalence of the conclusion of the Proposition 1.1 (given as definition) with the condition i) above is proved in ([1], 7.5.1).

Since every prime subgroup contains a minimal prime, i) and ii) are equivalent; and, since regular subgroups are prime, i) or ii) implies iii), and iii) implies iv). It remains to show that iv) implies i). This is straightforward by Zorn's Lemma. Let P be a prime subgroup that contains two distinct minimal prime subgroups, M and N. Choose $x \in G \setminus P$, $y \in M \setminus N$, $z \in N \setminus M$. P is contained in some value X of x, M in some value Y of y, and N in some value Z of z. The regular subgroup X contains each of the regular subgroups Y and Z, and clearly neither of these is contained in the other, so that the condition iv) is not satisfied.

Let G be a subgroup and sub-lattice (l-subgroup) of the l-group H. Recall that H is an Archimedean extension of G, if the mapping $:C \mapsto C \cap G$, is a bijection of the set of solid subgroups of H on that of G. Let us call H a *quasi-Archimedean extension* of G, if the same mapping, restricted to the set of regular subgroups of H, defines a bijection on the set of regular subgroups of G. It is easy to see that an Archimedean extension is quasi-Archimedean, but the converse is an open question. (It is known to be true if H is normal-valued.)

Proposition 1.3. *The property of l-groups of being semi-projectable is preserved by*:

a) *direct products*;

b) *homomorphic images*;

c) *quasi-Archimedean extension.*

Proof. a) If G is the direct product of the l-groups G_i, then every prime subgroup of G is of the form: $P_j \times \prod_{i \neq j} G_i$ where P_j is a prime subgroup of G_j, for a certain value j of i. Hence if the condition ii) holds for each G_i, it clearly holds for G.

b) If G is an l-group and K the kernel of a homomorphism $G \to H$, of l-groups, then the prime subgroups of G/K are in order-preserving bijection with the prime subgroups of G that contain K. Hence, if G satisfies the condition ii), so does G/K.

c) is obvious by the condition iv).

2. Representability of semi-projectable l-groups.

Recall that an l-group G is normal-valued, if each regular subgroup of G is normal in the solid subgroup that covers it in the lattice of all solid subgroups of G.

Proposition 2.1. *If a semi-projectable l-group is normal-valued, it is representable.*

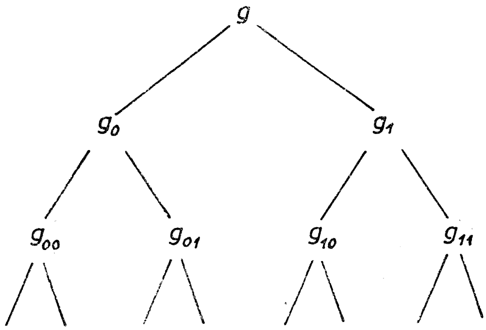
Proof. It is known that an l-group is representable if and only if all its minimal prime subgroups are normal. Suppose that G is semiprojectable and normal-valued, let M be a minimal prime subgroup of G. Let g be any element of G, we have to show that $g^{-1}Mg = M$. This is obvious if $g \in M$, so suppose that $g \notin M$. Then M is contained in some value V of g, and $g^{-1}Vg = V$. But conjugation by g (or any automorphism of G) permutes the chains of prime subgroups; since it preserves V, the chain that contains V is transformed into itself. Hence $g^{-1}Mg = M$.

To prove our next two propositions we need three lemmas, the first two of which have nothing in particular to do with semi-projectable groups.

Recall that an element $g > e$ of an l-group G is *basic*, if the interval $[e, g]$ is totally ordered; and G is said to have a *basis*, if every positive element is basic or exceeds a basic element.

Lemma 1. *Let g be a positive element of an l-group G. If g belongs to all but a countable subset of the set of minimal prime subgroups of G, then g is basic or exceeds a basic element. Hence if the set of minimal prime subgroups of G is countable, G has a basis.* (Here "countable" includes finite.)

Proof. Suppose that g is not basic and that it exceeds no basic element. Then the interval $[e, g]$ contains two non-comparable elements x and y. If we define: $xy^{-1} \vee e = g_0$, $yx^{-1} \vee e = g_1$, then we have: $g > g_0 > e$, $g > g_1 > e$ and $g_0 \wedge g_1 = e$. Clearly neither g_0 nor g_1 is basic or exceeds a basic element. So we a situation like this.



with the elements in each line mutually orthogonal. Hence g belongs to each of an uncountable set of chains, any two of which contain mutually orthogonal elements.

Each of these chains can be extended to an utrafilter (see [1], 3.4), and these ultrafilters are all distinct, because no ultrafilter contains orthogonal elements. Finally, since the mapping $U \mapsto G_+ \setminus U$, defines a bijection of the set of ultrafilters of G_+ on the set of positive cones of the minimal prime subgroups of G, we have an uncountable set of minimal primes none of which contains g.

Lemma 2. *For an l-group G with a basis, the following conditions are equivalent*:

i) *G is Archimedean*;

ii) *G is a subdirect product of reals (i.e. subgroups of* $\mathbb{R}$*)*;

iii) *The intersection of the maximal prime subgroups of G is* $\{e\}$.

Proof. That i) implies ii) follows from ([1], 14.4.1) and ii) clearly implies i), whether G has a basis or not.

ii) likewise implies iii), whether G has a basis or not. Let R_i be a family of real groups, P, their direct product, and, for each i, p_i the canonical projection of P on R_i. Then, if G is a subdirect product of the R_i, the restriction of p_i to G defines a homomorphism of G onto R_i, and the kernel of that homomorphism is a (normal) maximal prime subgroup of G. Clearly the intersection of these maximal prime subgroups (a fortiori of all the maximal prime subgroups) of G is equal to $\{e\}$.

Finally, suppose that the condition iii) is satisfied. Then, for every basic element b, there is a maximal prime subgroup M_b of G that does not contain b. M_b is clearly a value of b, in fact it is the only value of b, and we know that in this case the regular subgroup in question is normal in the solid subgroup that covers it, here G. So $M_b \lhd G$, and G/M_b is a real group. The intersection of all the M_b, for b basic, is a solid subgroup of G that contains no basic element; since G has a basis, this intersection is equal to $\{e\}$. Hence G is a subdirect product of the real groups G/M_b.

Lemma 3. *Let G be a semi-projectable l-group, and let L be the intersection of all the maximal prime subgroups of G (we suppose that G has maximal prime subgroups). G/L is representable if and only if G is representable.*

Proof. Since we know that the homomorphic image of a representable l-group is representable, it is the "only if" part that is interesting. Suppose that G is not representable, and let M be a minimal prime subgroup of G, g an element of G, such that $g^{-1}Mg \neq M$. Conjugation by g transforms the chain of prime subgroups containing M into a second chain. The union of all the elements of the first chain is a maximal prime subgroup that does not contain g; and it is not normal. But clearly, if G/L is representable, every maximal prime subgroup of G is normal.

Proposition 2.2. *Let G be a semi-projectable l-group. If the set of minimal prime subgroups of G is countable, G is representable.*

Proof. Let L be as in the preceding lemmas (for the case when G has no maximal prime subgroup, cf. Proposition 2.3 below). The inverse images of distinct minimal prime subgroups of G/L belong to distinct chains of prime subgroups of G, hence the set of minimal prime subgroups of G/L is countable. It follows by lemma 1 that G/L has a basis. But the intersection of the maximal prime subgroups of G/L is equal to its identity element. Thus, by lemma 2, G/L is Archimedean, and so representable.

Proposition 2.3. *A semi-projectable l-group without maximal prime subgroups is representable.*

Proof. By the proof of lemma 3.

Bergman's proof of the existence of non-representable semi-projectable l-groups restes on a lemma due to Holland and Glass ([4], Lemma 11.7). Proof of this lemma can be simplified and the result strengthened by direct use of the definition of semi-projectability adopted here.

Recall that, if T is a chain, the group $A(T)$ of order-preserving permutations of T, under pointwise ordering, is an l-group. The support of an element $g \in A(T)$ is the set: $\{t \in T \mid g(t) \neq t\}$. A *bump* of g is a convex component of its support. For more details see ([4]).

Proposition 2.4. (cf. [4], Lemma 11.7). *Let T be a chain, G a doubly transitive l-subgroup of $A(T)$. If the support of each element of G is bounded above and consists of a finite number of non-adjacent bumps, then G is semi-projectable.*

Proof. Let S be the support of g, $B_1, \ldots, B_n$, the bumps of a or of b that intersect S. We can suppose that B_n is a bump of a and that it is the last in order. Let t_0 and t_1 be, in order, two points of T betweeen B_{n-1} and B_n, t_2 a point beyond S. There is a positive element $x \in G$ that sends t_1 to t_2 and whose support lies beyond t_0 ([4], 1.10.6).

$\boldsymbol{x}$ is orthogonal to $\boldsymbol{b}$, and the support of $x^{-1}gx$ intersects only $n-1$ bumps of $\boldsymbol{a}$ or of $\boldsymbol{b}$. Proceeding step by step by conjugation with elements orthogonal to $\boldsymbol{a}$ or to $\boldsymbol{b}$, we get an element whose support intersects at most one of the B_i. Clearly this element is orthogonal either to $\boldsymbol{a}$ or to $\boldsymbol{b}$.

The chain T in Bergman's example is quite complicated. However no such examples are to be found in $A(\mathbb{R})$. More generally:

Proposition 2.5. *Let T be a (conditionally) complete chain. If G is a semi-projectable transitive l-subgroup of A(T), no element of G has bounded support.*

Proof. Let $\boldsymbol{a}$ be such an element. Without restriction of generality, $a \geqq e$. Let t_0 and t_1 be the g.l.b. and the l.u.b. of the support of $\boldsymbol{a}$ (these exist because T is complete). Since G is transitive, there is a positive element $\boldsymbol{g}$ in G, such that $g(t_0) = t_1$. Let $gag^{-1} = b$. Then t_1 is the g.l.b. of the support of $\boldsymbol{b}$, and $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal. Any permutation of T, orthogonal to $\boldsymbol{a}$, fixes t_1, and the same is true of any permutation orthogonal to $\boldsymbol{b}$. Hence $\boldsymbol{g}$ cannot be written as a product of elements orthogonal to $\boldsymbol{a}$ and elements orthogonal to $\boldsymbol{b}$, and G is not semi-projectable.

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