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HALF-AUTOMORPHISMS OF TRANSFORMATION SEMIGROUPS

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In [4] we first surveyed the literature on two generalisations of "automorphism" and their application to groups and rings, and then we considered their interpretation for transformation semigroups. In each case the aim has been to provide conditions on the underlying algebraic structure to ensure that these generalised automorphisms are "disjunctive": that is, each is either an automorphism or an anti-automorphism. In this paper we first note two important omissions from our survey and then prove a conjecture (mentioned at the end of [4]) that concerns "half-automorphisms" of transformation semigroups.

We recall from [4] that if R, S are rings then by a semihomomorphism $\phi: R \to S$ we mean an additive mapping satisfying $a^2\phi = (a\phi)^2$ and $(aba)\,\phi = a\phi b\phi a\phi$ for all $a, b \in R$. As usual, S is a *prime* ring if in S, $xSy = 0$ implies $x = 0$ or $y = 0$. After a long sequence of simple, but most ingenious, steps Herstein arrived in [1] at the following generalisation of earlier work by Kapansky, Hua, and Jacobson and Rickart (see [4] for detailed references).

Theorem 1. *Every semi-homomorphism from a ring R onto a prime ring S with characteristic different from 2 and 3 is disjunctive.*

As noted in [4], Herstein and Ruchte proved in [3] that every semi-automorphism of a non-abelian simple group containing an element of order 4 is disjunctive. However, a decade later, Herstein generalised this to

Theorem 2. *Every semi-automorphism of a non-abelian simple group with an element of order 2 is disjunctive.*

This was achieved by considering the more general situation of semi-homomorphisms between two groups satisfying certain elementary conditions. Of course, in view of the Feit-Thompson Theorem, the above result implies that any finite non-abelian simple group has disjunctive semi-automorphisms.

In [4] we defined a semi-automorphism ϕ of a semigroup S to be a permutation of S satisfying $(aba)\,\phi = a\phi b\phi a\phi$ for all $a, b \in S$. We also said that if S is a transformation semigroup defined on a set X then S extremally covers X if it contains all the total constants X_a $(a \in X)$ and all the injective constants a_b $(a, b \in X)$. The following result generalises Theorem 3 in [4].

Theorem 3. *If S extremally covers X then every semi-automorphism of S is an automorphism.*

Proof. By Lemma 2 [4] we know that $X_a\phi = Y_b$ for some $b \in Y$ and $Y \subseteq X$; we aim to show $Y = X$.

If $a_a\phi = Z_c$ and there exists $y \in Z \setminus c$, we put $A_x\phi = c_y$ (using Lemma 2 [4] again) and note that $x \notin A$. Then $(A_x \,.\, a_a \,.\, A_x)\,\phi = c_y \,.\, Z_c \,.\, c_y \neq \square$ implies $x = a \in A$, a contradiction. Hence ϕ maps the set of all injective idempotent constants onto itself, and it follows that $X_a\phi = Y_b$ for some $Y \subseteq X$ if and only if $a_a\phi = b_b$.

Hence, if there exists $z \in X \setminus Y$ and $y_y\phi = z_z$ then $X_y\phi = C_z$ for some $C \subseteq X$ and we have $(X_aX_yX_a)\,\phi = Y_bC_zY_b = \square$, a contradiction. Thus, ϕ maps the set of all total constants onto itself, and the proof can now proceed as in that of Theorem 3 [4].

Unfortunately the above result does not include Symons' original theorem: namely, every semi-automorphism of a total transformation semigroup covering X is an automorphism. However if we attempt to change the assumption in Theorem 3 in order to accommodate Symons' result, the conclusion may not follow.

For example, take S to be the set of all constants in $\mathscr{I}_X$, let $g \in \mathrm{Sym}(X)$, and define $\phi\colon S \to S$ by $x_x\phi = x_x$ and $\square\phi = \square$, and $x_y\phi = xg_{yg}$ if $x \neq y$. Then S is transitive and covers X but ϕ is a semi-automorphism. Suppose that $g \neq \iota_X$ and $\operatorname{card} X > 2$. Then ϕ is neither an automorphism nor an anti-automorphism (note that this example also shows Theorem 4 [4] is best possible). Suppose instead we take S to be the set of all total constants defined on X, together with all maps α_{ab} with $\operatorname{dom} \alpha_{ab} = \{a, a\theta\}$ and $\operatorname{ran} \alpha_{ab} = \{b\}$ where $a, b \in X$ and θ is a fixed permutation of X satisfying $\theta^2 = \iota_X$ and $x\theta \neq x$ for all $x \in X$. Define $\phi\colon S \to S$ by putting $\alpha_{aa}\phi = \alpha_{a,a\theta}$ and $\alpha_{a,a\theta}\phi = \alpha_{aa}$, and allowing ϕ to fix all other elements of S. Then ϕ is a semi-automorphism of S which is neither an automorphism nor an anti-automorphism: in other words, Symons' result for semi-automorphisms cannot be directly generalised to partial transformation semigroups.

A half-automorphism ϕ of a semigroup S was defined in [4] to be a bijection $\phi\colon S \to S$ such that for all $a, b \in S$, $(ab)\,\phi$ equals $a\phi b\phi$ or $b\phi a\phi$. We now aim to prove a conjecture enunciated at the end of [4]: namely, every half-automorphism of a 2-transitive transformation semigroup extremally covering X is an automorphism. In doing this we shall use Lemma 4 of [4] without explicit mention.

Lemma 1. *If S is 2- transitive and covers X, $\phi\colon S \to S$ is a half-automorphism of S and α is a total constant in S then $\alpha\phi$ is also.*

Proof. Assume $X_a\phi = Y_b$ where $b \in Y$. Suppose there exists $z \in X \setminus Y$ and choose $\lambda, \mu \in S$ such that $b\lambda = z$, $z\lambda = b$ and $\mu\phi = \lambda$. Then $\mu\phi \,.\, X_a\phi = \lambda \,.\, Y_b$ and $X_a\phi \,.\, \mu\phi = Y_b \,.\, \lambda$, where both these elements are non-zero and satisfy $\gamma^2 = \square$. This contradicts the fact that $X_a\mu$ is a (non-zero) idempotent.

As shown by Symons in [5], the inverse of a half-automorphism need not be a half-automorphism. However, if $S = S^1$ and ϕ is a half-automorphism of S then $1\phi = 1$. We therefore use this to prove:

Lemma 2. *If S is 2-transitive and contains the set $TK(X)$ of all total constants on X then any half-automorphism ϕ of S maps $TK(X)$ onto itself.*

Proof. Suppose $b \in X$ and $\lambda\phi = X_b$. Then $\operatorname{dom} \lambda = X$ since if there exists $s \notin \operatorname{dom} \lambda$ we obtain a contradiction by considering $(X_s\lambda)\,\phi$. Now let $a \in X$ and $X_a\phi = X_z$. Then $X_{a\lambda}\phi$ equals X_zX_b or X_bX_z: since, in the first case, we conclude λ is a constant, we may assume that $a\lambda = a$ for all $a \in X$; that is, $\lambda = \iota_X$, a contradiction.

Theorem 4. *If S is 2-transitive and contains $TK(X)$ then every half-automorphism ϕ of S is an automorphism.*

Proof. As usual we start by defining $g \in \operatorname{Sym}(X)$ such that $ag = b$ if and only if $X_a\phi = X_b$, and then define $\psi\colon S \to S$, $\alpha \to g(\alpha\phi)\,g^{-1}$. Note that ψ is a half-automorphism of S fixing each total constant: we intend to show that $\psi = \iota_S$.

Let $\alpha \in S$ and $a \in \operatorname{dom} \alpha$. Then $X_{a\alpha} = X_{a\alpha}\psi$ and this in turn equals $X_a \,.\, \alpha\psi$ or $\alpha\psi \,.\, X_a$: since in either case $a \in \operatorname{dom}(\alpha\psi)$ we have shown $\operatorname{dom} \alpha \subseteq \operatorname{dom}(\alpha\psi)$; a converse argument establishes equality, and moreover $\alpha = \alpha\psi$ for all $\alpha \in S$.

We showed in [4] that any semi-automorphism of a 2-transitive inverse subsemigroup S of $\mathscr{I}_X$ covering X is either an automorphism or an anti-automorphism (in which case it is the composition of an automorphism and $\theta\colon S \to S$, $\alpha \to \alpha^{-1}$). Rather surprisingly, for half-automorphisms we can show:

Theorem 5. *If S is a transitive inverse subsemigroup of $\mathscr{I}_X$ covering X then a half-automorphism ϕ of S is either an automorphism or an anti-automorphism (in which case it is the composition of θ and an automorphism).*

Proof. By Lemma 4 [4] we know $x_x\phi = y_y$ for some $y \in X$. Suppose $x \neq a$ and $(x_xx_a)\,\phi = x_a\phi = \lambda$. Then λ equals $y_y\lambda$ or λy_y, both of which are constants. Hence $x_a\phi = c_d$ for some $c \neq d$, and either $y = c$ or $y = d$. Suppose $y = c$ and $a_a\phi = b_b$; then $(x_aa_a)\,\phi$ equals y_db_b or b_by_d: since $b \neq y$, we must have $b = d$ and $x_a\phi = y_b$. On the other hand, if $y = d$ then we obtain $x_a\phi = b_y$. Hence if we define $g \in \operatorname{Sym}(X)$ by

$$xg = y \quad \text{if and only if} \quad x_x\phi = y_y$$

then $x_a\phi$ equals xg_{ag} or ag_{xg} for all $x, a \in X$, and so $x_a\phi = xg_{ag}$ if and only if $a_x\phi = ag_{xg}$.

Now assume $x_a\phi = xg_{ag}$ for fixed $a \in X \setminus x$, let $z \in X \setminus x$ and suppose $x_z\phi = zg_{xg}$. Since, by considering $(a_xx_z)\,\phi$, this produces a contradiction, we conclude that $x_z\phi = xg_{zg}$ for all $z \in X$. Then a similar argument using distinct $r, s \in X \setminus x$ and $(r_xx_s)\,\phi$ shows that $r_s\phi = rg_{sg}$ for all $r, s \in X$. Hence either $r_s\phi = rg_{sg}$ for all $r, s \in X$ or $r_s\phi = sg_{rg}$ for all $r, s \in X$.

Suppose the first case occurs: we will show that then for each $\alpha \in S$, $(\operatorname{dom} \alpha) g = \operatorname{dom}(\alpha\phi)$ and $\alpha\phi = g^{-1}\alpha g$. Let $a \in \operatorname{dom} \alpha$ and suppose $(a_a\alpha)\phi = \alpha\phi \,.\, ag_{ag}$. This means $ag_{a\alpha g} = \alpha\phi \,.\, ag_{ag}$ and so $(ag)\alpha\phi = ag = a\alpha g$. Since $(a_a\alpha)\phi = ag_{ag} \cdot \alpha\phi$ also leads to $a\alpha g = (ag)\alpha\phi$, we have shown that $(\operatorname{dom} \alpha) g \subseteq \operatorname{dom}(\alpha\phi)$. Conversely, let $a \in \operatorname{dom}(\alpha\phi)$, $b \in \operatorname{ran}(\alpha\phi)$, $c = ag^{-1}$, $d = bg^{-1}$. Then, using transitivity, $(d_c\alpha)\phi \neq \square$ and $c \in \operatorname{dom} \alpha$.

In the second case, put $\alpha\psi = g(\alpha\phi) g^{-1}$ for each $\alpha \in S$ and note that $x_y\psi = y_x$ for all $x, y \in X$: we will show that $\psi = \theta$. To do this, let $a \in \operatorname{ran} \alpha$ and $b\alpha = a$. Then $(a_b\alpha)\psi$ equals $b_a \,.\, \alpha\psi$ or $\alpha\psi \,.\, b_a$: if the former occurs, we have $a = b = a(\alpha\psi)$ and if the latter occurs, we have $a(\alpha\psi) = b = a\alpha^{-1}$. In this way we can show $\operatorname{dom} \alpha^{-1} = \operatorname{dom}(\alpha\psi)$, and the result follows.

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