Joachim Machner

T-algebras of the monad L-Fuzz


Persistent URL: http://dml.cz/dmlcz/102045

Terms of use:

© Institute of Mathematics AS CR, 1985

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://dml.cz
The main purpose of this article is the characterization of $T$-algebras $(A, h)$ of the monad $L$-Fuzz, which is connected with the fuzzification of mathematical objects, especially automata, applying a brouwerian lattice $L$ [1]. The power set monad is a submonad of $L$-Fuzz and, as is well known, its $T$-algebras are precisely the complete sup-semi-lattices [2]. In the case of $T$-algebras of $L$-Fuzz the set $A$ will also have the complete lattice structure making it possible to construct a Galois correspondence $(h, g)$ between $TA$ and the dual $A'$ of $A$. The above mentioned characterization will be performed by the statement of four independent conditions on $g$ to be the residuated map of the morphism $h$ of a $T$-algebra for $L$-Fuzz.

In Section 1, basic facts on the Kleisli and Eilenberg-Moore constructions are summarized and the monad $L$-Fuzz is constituted. Notations from the category theory not defined here may be found in [3]. For lattice theoretical facts see [4]. Section 2 starts with a partial order on the underlying set $A$ of the $T$-algebra $(A, h)$, which is shown to be a complete lattice order. Having introduced the Galois correspondence $(h, g)$ additional properties of $g$ are established, a suitable selection of which will be characteristic, as pointed out in the main result 2.13, 2.14. Section 3 studies the independence of the characteristic conditions obtained in the preceding section, while the last section is supplementary and contains some applications.

1. MONADS, $T$-ALGEBRAS AND THE MONAD $L$-FUZZ

1.0. There are several equivalent notions of a monad over a category $K$ [5]. A monad $(T, \eta, \mu)$ in the monoid form consists of an endofunctor $T : K \to K$, and two natural transformations $\eta : \text{Id} \to T$ and $\mu : T^2 \to T$, such that — composition left before right —

$$\eta_{TA} \mu_A = 1_{TA} = T \eta_A \mu_A, \quad T \mu_A \mu_A = \mu_{TA} \mu_A$$

for every object $A$. The Kleisli category $K_T$ of $(T, \eta, \mu)$ has the same object class as $K$ and the morphism classes $K_T(A, B) = K(A, TB)$ with the morphism composition

$$\alpha \circ \beta = \alpha T \beta \mu_C$$
where $\alpha \in K_T(A, B)$, $\beta \in K_T(B, C)$. There exists a pair of adjoint functors $(\Delta, \#)$ between $K$ and $K_T$, given by

$$A^\Delta = A, \quad f^\Delta = f\eta_B,$$

$$A^\# = TA, \quad \alpha^\# = T\alpha\mu_B$$

if $f \in K(A, B)$ and $\alpha \in K_T(A, B)$. $(\Delta, \#)$ generates the given monad for $TA = A^{\Delta^\#}$, $Tf = f^{\Delta^\#}$, $\eta$ being the unit of the adjunction and $\mu$ the natural transformation associated with the counit $\varepsilon$ [6], [3]. The Kleisli construction gives rise to the definition of a monad $(T, \eta, \circ)$ in a clone form: here $T$ is an object map of $K$, $\eta = (\eta_A)_A \in K$ a family of object maps $\eta_A: A \to TA$ and $\circ$ a family $(\circ_{ABC})_{A, B, C \in K}$ of mappings

$$\circ_{ABC}: K(A, TB) \times K(B, TC) \to K(A, TC)$$

such that (object indices in the composition sign will be omitted)

$$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma),$$

$$\alpha \circ \eta_B = \alpha,$$

$$(f^\Delta) \circ \beta = f\beta$$

for all composable morphisms $\alpha, \beta, \gamma, f$. By the functor properties of $\Delta, \#$ we have the identities

$$(fg)^\Delta = f^\Delta \circ g^\Delta, \quad 1^\Delta_A = \eta_A,$$

$$(\alpha \circ \beta)^\# = \alpha^\# \beta^\#, \quad \eta_A^\# = 1_{TA}.$$  

Replacing $\circ$ in the clone form by a family $\# = (\#_{AB})_{A, B \in K}$ of mappings

$$\#_{AB}: K(A, TB) \to K(TA, TB)$$

satisfying (object indices omitted)

$$\eta_A \alpha^\# = \alpha,$$

$$\eta_A^\# = 1_{TA},$$

$$\alpha \beta^\# = \alpha^\# \beta^\#$$

one gets the notion of a monad in the extension form $(T, \eta, \#)$, connected with the monoid form and the clone form by the relations

$$\alpha^\# = T\alpha\mu_B = 1_{TA} \circ \alpha,$$

$$\mu_A = 1_{TA} \circ 1_{TA} = 1_{TA}^\#$$

for $\alpha \in K(A, TB)$.

Eilenberg and Moore [7], [3] generated the monad $(T, \eta, \mu)$ by a pair of adjoint functors $(F^T, U^T)$ between $K$ and the category $K_T$ of $T$-algebras. Such a $T$-algebra $(A, h)$ consists of an object $A$ in $K$ and a $K$-morphism $h: TA \to A$ satisfying

$$\eta_A h = 1_A, \quad Th = \mu_A h.$$
A morphism \( f \in K(A, A') \) is a \( K^T \)-morphism from \((A, h)\) to \((A', h')\) iff \( T f h = h' f \). \( K^T \)-morphisms are combined by the composition law of \( K \). Evidently \((TA, \mu_A)\) is a \( T \)-algebra and the adjunction is given by
\[
F^T A = (TA, \mu_A), \quad F^T f = T f, \quad G^T(A, h) = A, \quad G^T g = g
\]
where \( f \in K(A, A'), \ g \in K^T((A, h), (A', h')) \).

1.1. In Zadeh's classical paper on fuzzy sets [8] characteristic functions are ranging over the interval \([0, 1]\) of real numbers. With respect to inf, sup this is a special case of a brouwerian lattice [9] (or complete JID-lattice [4]). Precisely, a lattice \( L \) is called brouwerian iff it is complete and the intersection distributive over the suprema
\[
\bigwedge_{i \in I} y_i = \bigvee_{i \in I} (\bigwedge_{i \in I} y_i).
\]
We establish a fuzzification making use of a fixed brouwerian lattice \( L \). Let \( A \) be a set and
\[
TA = L^A.
\]
p \in TA may be interpreted as a fuzzy set on \( A \), \((a)p\) is the grade of membership of \( a \) in \( p \). By \( \alpha: A \to TB \) to every \( a \in A \) we attribute a fuzzy set \((a)\alpha\) on \( B \) and adopting notation similar to that of conditional probability we set
\[
\alpha(b|a) := (b)((a)\alpha),
\]
a \in A, b \in B. In particular, \( \eta_A: A \to TA \) is defined by
\[
\eta_A(a'|a) := \begin{cases} 1, & a' = a, \\ 0, & a' \neq a, \end{cases}
\]
where 1 denotes the greatest and 0 the smallest element of the lattice \( L \).
\( \alpha: A \to TB \) and \( \beta: B \to TC \) are composed to \( \alpha \circ \beta: A \to TC \) by
\[
(\alpha \circ \beta)(c|a) := \bigvee_{b \in B} (\alpha(b|a) \wedge \beta(c|b)),
\]
a \in A, c \in C, with \( \bigvee \) the supremum and \( \wedge \) the intersection in \( L \). One verifies without difficulty (see also [1]) that \((T, \eta, \circ)\) is a monad in the clone form over the category \( \text{Set} \), which will be denoted \( L \)-Fuzz in what follows. Consequently, \( T \) is an endofunctor \( \text{Set} \to \text{Set} \) and \( \eta \) must be the natural transformation \( \text{Id} \to T \), which can be tested also immediately. An easy computation using 1.0 (1), (2) gives the essential parts of \( L \)-Fuzz in the other monad forms
\[
a^*(b|p) = \bigvee_{a \in A} ((a)p \wedge \alpha(b|a)),
\]
b \in B, p \in TA, \( \alpha: A \to TB \), or
\[
(1) \quad \mu_A(a|\Phi) = \bigvee_{p \in TA} ((p)\Phi \wedge (a)p),
\]
\( \Phi \in T^2A, \ a \in A \).
The functor $T$ transforms $f: A \to B$ into $Tf: TA \to TB$ so that

(2) \[ Tf(b/p) = \bigvee \{ (a) p/a \in A, (a) f = b \} , \]

$b \in B$, $p \in TA$.

2. CHARACTERIZATION OF $T$-ALGEBRAS OF $L$-FUZZ

2.0. In the sequel $(A, h)$ always means a $T$-algebra of $L$-Fuzz with the additional assumption $A \neq \emptyset$. If no confusion arises $\eta_A$ and $\mu_A$ often will be written without subscripts. It will be of advantage to distinguish typographically the elements of $L$-small greek letters with the exception of the bounds 0, 1 — from those of $A$ — small Roman characters.

2.1. Together with $(L; \wedge, \vee, 0, 1)$ also $TA = L^A$ is a brouwerian lattice with respect to $\inf$, $\sup$, $c_0$, $c_1$ defined by

\[
(x) (\inf p_i) := \bigwedge_{i \in I} (x) p_i ,
\]

\[
(x) (\sup p_i) := \bigvee_{i \in I} (x) p_i ,
\]

\[
(x) c_0 := 0 , \quad (x) c_1 := 1
\]

for every $x \in A$ and every family $(p_i/i \in I)$ of fuzzy sets $p_i \in TA$.

2.2. Definition. If $(A, h)$ is a $T$-Algebra and $a, b \in A$, set

\[ a \leq b \quad \text{iff} \quad (\sup \{ (a) \eta_A, (b) \eta_B \}) h = b , \]

where $\sup$ denotes the operation from 2.1.

2.3. Lemma. The relational system $(A, \leq)$ has the following properties:

1. $(A, \leq)$ is a partial order;
2. $h$ is order-preserving;
3. $\forall x \in A \forall M \subseteq TA$;
   \[ \text{if } M \subseteq h^{-1}(x) \text{ then } (\sup M) h = x ; \]
4. $\forall X \subseteq A$:
   \[ (\sup \{ (x) \eta | x \in X \}) h \text{ is the supremum of } X \text{ in the partial order } (A, \leq) ; \]
5. $h$ is $\sigma$-preserving (that is, $h$ preserves suprema).

Proof. (1): The antisymmetry of $\leq$ obviously holds. Reflexivity is an immediate consequence of the identity $\eta_A h = 1_A$ for $T$-algebras. Transitivity will be established by choosing suitable maps $\Phi \in T^2 A$ and applying the second identity for $T$-algebras $\mu h = Th h$:

Supposing $a \leq b$, $b \leq c$ let for $p \in TA$

\[
(p) \Phi := \begin{cases} 1 & \text{if } p = (c) \eta \text{ or } p = \sup \{ (b) \eta, (a) \eta \} , \\ 0 & \text{else} , \end{cases}
\]
\((p) \Phi' := \begin{cases} 1 & \text{if } p = (a) \eta \text{ or } p = \sup \{(b) \eta, (c) \eta\}, \\ 0 & \text{else} \end{cases} \)

By 1.1 (1),
\[(\Phi) \mu = \sup \{(a) \eta, (b) \eta, (c) \eta\} = (\Phi') \mu,\]
while 1.1 (2) yields
\[Th(z/\Phi) = \begin{cases} 1 & \text{if } z = (c) \eta \eta = c \text{ or } z = (\sup \{(a) \eta, (b) \eta\}) h = b, \\ 0 & \text{else}, \end{cases} \]
thus \((\Phi) Th h = (\sup \{(c) \eta, (b) \eta\}) h = c.\)

A similar computation with \(\Phi'\) shows that
\[(\Phi') Th h = (\sup \{(a) \eta, (c) \eta\}) h,\]
therefore \(c = (\sup \{(a) \eta, (c) \eta\}) h\), which was to be shown.

(2): The lattice ordering in \(TA\) is given by components: if \(p, q \in TA\),
\[p \leq q \text{ iff } \forall a \in A: (a) p \leq (a) q.\]
(We use \(\leq\) for the ordering of \(L, TA\) and \(A\), the particular meaning being clear from the context).

If \(p, q \in TA\), \(p \leq q\), set for an arbitrary \(r \in TA\)
\[(r) \Phi := \begin{cases} 1 & \text{if } r = p \text{ or } r = q, \\ 0 & \text{else} \end{cases} \]

Then we obtain for every \(z \in A\)
\[\mu(z/\Phi) = (1 \land (z) p) \lor (1 \land (z) q) = (z) q,\]
\[Th(z/\Phi) = \eta(z/(p) h) \lor \eta(z/(q) h),\]
therefore \((\Phi) \mu = q, (\Phi) Th = \sup \{(p) h, (q) h\}\) and finally
\[(q) h = (\Phi) \mu h = (\Phi) Th h = (\sup \{(p) h, (q) h\}) h,\]
that is \((p) h \leq (q) h\), in accordance with 2.2.

(3): Since \(h\) preserves the order,
\[x \leq (p) h \leq (\sup M) h\]
and because of \(\sup M \leq \sup h^{-1}(x)\) we only have to prove the equality \((\sup h^{-1}(x)) h = x\). Taking
\[(p) \Phi := \begin{cases} 1 & \text{if } p \in h^{-1}(x), \\ 0 & \text{else} \end{cases} \]
we have \((\Phi) \mu = \sup h^{-1}(x)\) and \((\Phi) Th = (x) \eta\), and from the both \(T\)-algebra identities we get
\[x = (x) \eta h = (\Phi) Th h = (\Phi) \mu h = (\sup h^{-1}(x)) h.\]
(4): If $X = \emptyset$ then

$$\sup \{(x) \eta/x \in X\} = c_0 \in L^a,$$

$c_0$ being the constant 0-map. Since $\eta, h = 1_A$, $h$ is surjective and therefore $(c_0) h$ must be the smallest element of $A$. Taking $x \in X \neq \emptyset$ and

$$a := (\sup \{(x) \eta/x \in X\}) h$$

we have $x \leq a$ by (2). If $b$ is any upper bound of $X$ in $A$, then for every $x \in X$

$$b = (\sup \{(x) \eta, (b) \eta\}) h$$

and taking into account (3), 2.2 we find

$$b = (\sup \{(x) \eta/x \in X\}, (b) \eta\}) h.$$

Setting

$$(p) \Phi = \begin{cases} 1 & \text{if } p = \sup \{(x) \eta/x \in X\} \text{ or } p = (b) \eta, \\ 0 & \text{else} \end{cases}$$

one gets

$$(\Phi) \mu = \sup \{(x) \eta/x \in X\}, (b) \eta\},$$

$$(\Phi) Th = \sup \{(a) \eta, (b) \eta\}$$

and therefore $a \leq b$.

(5): If $M \subseteq TA$ and $X := \{(p) h/p \in M\}$, then (4) implies

$$\sup \{(p) h\eta/p \in M\} h = \sup \{(p) h/p \in M\},$$

Sup denoting the supremum operation in $(A, \leq)$. Defining $(p) \Phi = 1$ if $p \in M$, 0 else, 1.1 (1), (2), immediately yield $(\Phi) \mu = \sup M$ and $Th (z/\Phi) = 1$ if $M \cap \cap h^{-1}(z) \neq \emptyset$, 0 else. Therefore

$$(\Phi) Th = \sup \{(p) h\eta/p \in M\},$$

$$(\Phi) Th h = (\Phi) \mu h = (\sup M) h.$$

**2.4. We list the conclusions of 2.3, some of them involving merely the basic facts of the lattice or set theory:**

(1) $A$ is sup-complete.
(2) With respect to Sup and Inf defined by

$$\inf X := \sup \{a/a \in A, \forall x \in X: a \leq x\},$$

$A$ is a complete lattice.
(3) ker $h$ (the kernel of $h$) is an equivalence relation in $TA$, which is sup-compatible and separates the set $\{(a) \eta/a \in A\}$.
(4) Every equivalence class of ker $h$ contains its supremum.

These suprema of equivalence classes are the object of the forthcoming considerations.

**2.5. Definition.** If $(A, h)$ is a $T$-algebra, $a \in A$, then $g(a) := \sup \{p/p \in TA, (p) h \leq a\}$. 

520
Remark. $g$ maps $a$ to $\sup h^{-1}(a)$, because $h$ is surjective and 2.3 (5) has been proved.

2.6. Lemma. Supposing 2.5, we have

$$\forall p \in TA \quad \forall a \in A: (p) h \leq a \iff p \leq (a) p.$$ 

The assertion is a direct consequence of 2.5 and, as is well known, it is equivalent to $(h, g)$ being a Galois connection between $TA$ and the dual $A^d$ of the lattice $A$. $g$ is called residuated to $h$. (This differs from the terminology in [10], where $h$ would be called residuated. Further properties of the Galois connections are listed in 2.7 below, see e.g. the article just quoted.)

2.7. For every $X \subseteq A$, $a \in A$, $p \in TA$,

1. $(\inf X) g = \inf \{(x) g | x \in X\}$,
2. $1_{TA} \leq hg$,
3. $gh \leq 1_A$, even $gh = 1_A$,
4. $(a) g = \sup \{p | p \in TA, (p) h = a\}$,
5. $(p) h = \inf \{a/a \in A, p \leq (a) g\}$.

2.8. As we have seen the underlying set $A$ of a $T$-algebra $(A, h)$ is a complete lattice. Simultaneously, the map $h$ must have a residuated map $g$. Therefore it seems reasonable to ask for those maps $g$ from a complete lattice $A$ to the brouwerian lattice $TA$, for which the map $h$, being now defined by equation 2.7 (5), produces a $T$-algebra $(A, h)$ of the monad $L$-Fuzz. In other words, our aim is the characterization of a $T$-algebra by residuated maps. To this end some further properties of $g$ must be investigated. We start with an auxiliary notion which will be useful in the sequel.

2.9. Definition. Let $\Phi \in T^2 A$.

$\Phi$ is concentrated iff $\exists p \in TA \exists x \in L \forall q \in TA$:

$$(q) \Phi = \begin{cases} x & \text{if} \quad q = p, \\ 0 & \text{else}. \end{cases}$$

2.10. Lemma. Suppose $h$ is a $\sigma$-preserving map $TA \to A$ and $\eta_A h = 1_A$. Then $(A, h)$ is a $T$-algebra iff for every concentrated $\Phi$,

$$(\Phi) \mu_A h = (\Phi) Th h.$$ 

Proof. The implication from the left to the right is obvious. For the opposite direction take into account that together with $TA$ also $T^2 A$ must be a brouwerian lattice. Denoting here for simplicity the supremum operation in $T^2 A$, $TA$ and $A$ with the same symbol sup, an arbitrary $\Phi \in T^2 A$ can be represented in the form

$$\Phi = \sup \{\Phi_p | p \in TA\},$$

$$(q) \Phi_p := \begin{cases} (p) \Phi & \text{if} \quad q = p, \\ 0 & \text{else}. \end{cases}$$

by a set of concentrated $\Phi_p$'s.

521
$Tf$ is always $\sigma$-preserving without any other supposition of $f$ than that of $f$ being a map. For, if $f: B \to C$, $c \in C$, $p_i \in TB$ for $i \in I$, then

$$(c) \left( \sup \{p_i / i \in I\} Tf \right) = Tf (c / \sup \{p_i / i \in I\})$$

$$= \bigvee \{ (b) \left( \sup \{p_i / i \in I\} / b \in B, \ (b) f = c \right) \}$$

$$= \bigvee \left\{ \bigvee_{i \in I} \left( b / p_i / b \in B, \ (b) f = c \right) \right\}$$

$$= \bigvee \left\{ \bigvee_{i \in I} \left( b / p_i / b \in B, \ (b) f = c \right) \right\}$$

$$= \bigvee \left\{ \bigvee_{i \in I} \left( b / p_i / b \in B, \ (b) f = c \right) \right\}$$

Also $\mu_A$ is $\sigma$-preserving since $(TA, \mu_A)$ is a $T$-algebra and 2.3 (5) holds.

$h$ is supposed to be $\sigma$-preserving, therefore for every $\Phi \in T^2 A$,

$$(\Phi) \mu h = (\sup \Phi_p) \mu h = \sup \left( (\Phi_p) \mu h \right)$$

$$= \sup \left( (\Phi_p) Th h \right) = (\sup \Phi_p) Th h$$

$$= (\Phi) Th h.$$

2.11. Let $\to$ denote the implication operation in the brouwerian lattice $L$, that is

$$\alpha \to \beta := \bigvee \{ \gamma / \gamma \in L, \ \alpha \land \gamma \leq \beta \},$$

$a, \beta \in L$. By components it can be carried over to $TA$:

$$(a) (p \to q) := (a) p \to (a) q,$$

$a \in A; \ p, q \in TA$. For every $\alpha \in L$ let $c_\alpha$ be the constant map $A \to L$ with value $\alpha$. If $g: A \to TA$, the Kleisli composition $g \circ g$ makes sense.

2.12. Lemma. Suppose $(A, h)$ is a $T$-algebra and $g$ is defined by 2.5. Then

(1) $\eta_A \leq g$,

(2) $g \circ g = g$,

(3) $\forall a \in A \ \forall x \in L$:

$$\alpha \leq g(\inf \{ x / x \in A, \ c_\alpha \to (a) g \leq (x) g \}) a.$$

Proof. (1): By 2.7 (2) and $\eta_A h = 1_A$ we have for every $a \in A$:

$$(a) \eta \leq (a) \eta h g = (a) g.$$  

(2): Both $\mu_A$ and $Tg$ are $\sigma$-preserving (see the proof in 2.10), therefore order preserving and (1) implies for every $a \in A$

$$(a) g = (a) (\eta \circ g) = (a) \eta Tg \mu \leq (a) g Tg \mu$$

$$= (a) (g \circ g).$$

On the other hand,

$$(a) (g \circ g) h = (a) g Tg \mu h = (a) g Tg Th h,$$
and with regard to 2.7 (3) the last expression equals

\[(a) \ g \ T_{A} \ h = (a) \ g \ h = a,\]

showing \((a) \ (g \circ g) \ h \leq (a) \ g\) by definition of \(g\) in 2.5.

(3): For \(p \in TA, \ x \in L,\)

\[\alpha p := \inf \{c_{x}, p\}\]

denotes the "\(\alpha\)-cut" of \(p\). The concentrated \(\Phi \in T^{2} A\) defined by \((q) \ \Phi = x\) if \(q = p,\)

0 else, fulfils

\[(\Phi) \ Th = x(p(h)) \eta, \ (\Phi) \ mu = \alpha p.\]

Since \((\alpha p) h = (x(p(h)) \eta) h\), for every \(a \in A\) we have the implications:

- if \(\alpha p \leq (a) g\), then \(x(p(h)) \eta \leq (a) g\), or equivalently, \(p \leq c_{x} \rightarrow (a) g\) implies \(\alpha \leq g((p(h)/a)\). The assertion is now verified by 2.7 (5).

2.13. Now we are able to find the characteristic conditions on \(g\) to be a residuated map of a \(T\)-algebra map \(h\). Their necessity is formulated in the following

**Theorem.** If \((A, h)\) is a \(T\)-algebra of the monad \(L\)-Fuzz, then \(A\) is a complete lattice with respect to the partial order defined in 2.2. The map \(g\) introduced by 2.5 is injective, \(\delta\)-preserving and satisfies 2.12 (2), (3).

**Proof.** 2.4 (2), 2.7 (3), 2.7 (1), 2.12 (2), (3).

The selected properties are also sufficient to get a \(T\)-algebra \((A, h)\) defining its map by means of 2.7 (5). The exact formulation is given in the next point:

2.14. **Theorem.** Suppose \((A, \leq)\) is a complete lattice, \((T, \eta, \circ)\) the monad \(L\)-Fuzz and \(g: A \rightarrow TA\) satisfies

1. \(g\) injective,
2. \(g\) \(\delta\)-preserving, (2.7 (1)),
3. \(g \circ g = g,\)
4. \(\forall a \in A \ \forall x \in L:\)

\[\alpha \leq g(\text{Inf} \{x/\ x \in A, \ c_{x} \rightarrow (a) g \leq (x) g\}/a).\]

If \(h: TA \rightarrow A\) is defined by

\[(p) h = \text{Inf} \{x/\ x \in A, \ p \leq (x) g\},\]

then \((A, h)\) is a \(T\)-algebra and for every \(a \in A\)

\[(a) g = \sup \{p/p \in TA, \ (p) h \leq a\}.\]

**Proof.** First of all, \((h, g)\) is shown to be a Galois connection between \(TA\) and \(A^{d}\), the dual of \(A\). Evidently \(h\) is monotone and also \(g\) is monotone by (2). If \(a \in A\) then, by (2),

\[(a) ghg = (\text{Inf} \{x/(a) g \leq (x) g\}) g = \inf \{(x) g/(a) g \leq (x) g\} = (a) g,\]

523
and the injectivity of \( g \) yields \( gh = 1_A \). For every \( p \in TA \),

\[
(p) \quad hg = (\text{Inf} \{ x/p \leq (x) g \}) g = \inf \{ (x) g/p \leq (x) g \} \geq p ,
\]

therefore \( hg \geq 1_{TA} \), leading to the desired result on \((h, g)\). But then, as is well known from the theory of Galois connection, also the last statement of the theorem concerning \( g \) is true.

The next preparatory step is to demonstrate

(i) \( (a) \eta \leq (a) g \),

(ii) \( a \leq b \iff g(a/b) = 1 \)

for all \( a, b \in A \).

The brouwerian implication has the property \( c_1 \to (a) g = (a) g \), and (4), (2) imply

\[
1 \leq g \left( \text{Inf} \{ x/c_1 \to (a) g \leq (x) g \} \right) a = \text{Inf} \{ x/(a) g \leq (x) g \} .
\]

Therefore \( 1 \leq g(a/a) \) and in virtue of the implication \( \eta(a'/a) = 0 \) if \( a' \neq a \), (i) has been shown.

If \( a \leq b \), then (2) and, further, (i) imply

\[
(a) g = \inf \{ (a) g, (b) g \} ,
\]

\[
1 = g(a/a) = g(a/a) \land g(a/b) ,
\]

therefore \( g(a/b) = 1 \).

Supposing \( g(a/b) = 1 \), (3) gives for every \( c \in A \)

\[
g(c/b) = \bigvee_{x \in A} (g(x/b) \land g(c/x))
\]

\[
\geq g(a/b) \land g(c/a) = g(c/a) ,
\]

consequently \( (a) g \leq (b) g \), and as previously shown \( gh = 1_A \), therefore \( a \leq b \).

Now we can verify the \( T \)-algebra identities. Because of (i), monotony of \( h \) and \( gh = 1_A \), for every \( a \in A \) we have

\[
(a) \eta h \leq (a) gh = a .
\]

From the properties of the Galois connection \((h, g)\) we obtain \( (a) \eta hg \geq (a) \eta \), which implies \( g(a/(a) \eta h) \geq \eta(a/a) = 1 \). By (ii) we conclude that

\[
a \leq (a) \eta h ,
\]

completing the proof of \( \eta_A h = 1_A \).

The more complicated second identity

\[
(\Phi) \quad Th = (\Phi) \mu_A h ,
\]

\( \Phi \in T^2 A \), will be verified only for concentrated \( \Phi \).

This will do, since \( h \) is \( \sigma \)-preserving because of the Galois properties, and 2.10 completes the proof.
Defining $\Phi$ by

$$(q) \Phi = \alpha \text{ if } q = p, \ 0 \text{ else},$$

$\alpha \in L, p \in TA$ being arbitrary but fixed, we assert the following:

(iii) $\forall a \in A$: if $(\Phi) Th h \leq a$ then $(\Phi) \mu h \leq a$.

Setting $z := (p) h$ we get $(\Phi) Th = \alpha(z) \eta$ and $(\Phi) \mu = \alpha p$ (see 2.12 for notation).

Supposing now the premise of (iii):

$$(\alpha(z) \eta) h = \text{Inf} \{x | x \in A, \alpha(z) \eta \leq (x) g\} \leq a,$$

one concludes by (2)

$$\inf \{ (x) g | x \in A, \alpha(z) \eta \leq (x) g \} \leq (a) g,$$

in particular $\alpha \leq g(z/a)$.

Since

$$(\Phi) \mu h = (\alpha p) h \leq a \iff \alpha p \leq (a) g,$$

it suffices to prove the validity of the relation on the right. This relation is a consequence of $p \leq (z) g$ and

$$\alpha \land (y) p \leq \alpha \land g(y/z) \leq g(z/a) \land g(y/z) \leq \bigvee_{x \in A} (g(x/a) \land g(y/x)) = g(y/a),$$

$y \in A$, again with help of (3) in the last step. Consequently, (iii) is valid.

Taking $a = (\Phi) Th h$ we get

$$(\Phi) \mu h \leq (\Phi) Th h.$$

Supposing now the validity of the other relation

$$(\Phi) \mu h = (\alpha p) h \leq a,$$

we immediately see that $\alpha p \leq (a) g$ and since for every $x \in A$

$$\alpha p \leq (x) g \iff p \leq c_x \to (x) g,$$

we have

$$\{x | c_x \to (a) g \leq (x) g\} \subseteq \{x | p \leq (x) g\},$$

$$\text{Inf} \{x | c_x \to (a) g \leq (x) g\} \subseteq \text{Inf} \{x | p \leq (x) g\} = (p) h.$$

By (ii) and (3), if $a, x, y \in A$ and $x \leq y$ then

$$g(x/a) = \bigvee_{z \in A} (g(z/a) \land g(x/z)) \geq g(y/a) \land g(x/y) = g(y/a) \land 1 = g(y/a).$$

This together with the last estimate of Inf and (4) finally yields

$$g(\text{Inf} \{x | c_x \to (a) g \leq (x) g\}/a) \leq g((p) h/a),$$

$$\alpha \leq g((p) h/a) = g(z/a),$$

$$\alpha'(z) \eta \leq (a) g,$$

$$(\Phi) Th h = (\alpha(z) \eta) h \leq (a) gh = a.$$

Therefore $(\Phi) Th h \leq (\Phi) \mu h.$
3. INDEPENDENCE OF THE CHARACTERIZING CONDITIONS

3.0. The characterizing conditions (1)–(4) of Theorem 2.14 will be shown to be independent. The corresponding counterexamples are constructed with \( L = \{0, 1\} \), \( A = L \times L \). The individual maps \( g \) will be given in the form of a matrix \( (g_{ij}) \), \( 1 \leq i, j \leq 4 \), where \( g_{ij} = g^i_j(l) \) and the elements of \( A \) \( (0, 0), (0, 1), (1, 0), (1, 1) \) are assigned to the rows and columns 1, 2, 3, 4, respectively. The Kleisli composition is performed by the max-min-product of matrices (that is \( \vee, \wedge \) instead of \( +, \cdot \)).

3.1. \( g_{ij} = 1, \ 1 \leq i, j \leq 4 \).
Obviously \( g \) is \( \delta \)-preserving and idempotent with respect to \( \circ \). As for condition (4), it is sufficient to take \( x = 1 \):

\[
1 \leq g(\text{Inf} \{x/c_1 \rightarrow (y) \ g \leq (x)g/y\}) = g(\text{Inf} \{x/(y) \ g \leq (x)g/y\}) = g(y/y).
\]

But evidently, \( g \) is not injective.

3.2. \( g_{ij} = \delta_{ij} \) (Kronecker symbol).
\( g \) is injective and idempotent. (4) can be shown as in 3.1.
\( g \) is not \( \delta \)-preserving.

3.3.

\[
g = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}
\]

\( g \) is injective and \( \delta \)-preserving. Computation of \( g \circ g \) results in the matrix from 3.1, hence (2) is violated. (4) is valid as in the preceding examples.

3.4.

\[
g = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1
\end{pmatrix}
\]

\( g \) is injective, \( \delta \)-preserving and idempotent.
Since \( g(1/1) \neq 1 \), (4) does not hold.

4. SPECIAL LATTICES AS UNDERLYING OBJECTS OF \( T \)-ALGEBRAS

4.1. Example. The brouwerian lattice \( L \) is itself the underlying object of a \( T \)-algebra \( (L, h) \) with the morphism

\[
(p)h := \vee_{a \in L} (a \wedge (a)p).
\]

\( p \in TL \). This can be verified without difficulty by directly testing the monad identities.

526
The residuated map $g$ is proved just to be the implication of $L$-interpreting $g: L \to TL$ as a binary operation $L \times L \to L$: for every $\alpha, \beta \in L$,

$$g(\beta/\alpha) = (\beta) (\sup \{ p/\forall \gamma(\gamma \in L, \gamma \land (\gamma) \ p \leq \alpha) \})$$

$$= (\beta) (\sup \{ p/\forall \gamma(\gamma \in L, \gamma \leq \gamma \rightarrow \alpha) \})$$

$$= (\beta) (\sup \{ p/p \leq 1_L \rightarrow c_\gamma \}) = (\beta) (1_L \rightarrow c_\alpha) =$$

$$= \beta \rightarrow \alpha .$$

The validity of conditions (1)–(4) from 2.14 can be restated using the well known identities concerning implication (see [11]).

$1 \rightarrow \alpha = \alpha$, therefore $(\alpha) \ g = (\beta) \ g$ if $\alpha = \beta$, and $g$ must be injective. $\delta$-preservation is expressed by

$$\beta \rightarrow \bigwedge_{i \in I} \alpha_i = \bigwedge_{i \in I} (\beta \rightarrow \alpha_i) .$$

The identity $\alpha \rightarrow \alpha = 1$ together with $\eta_L \leq g$ gives $g \leq g \circ g$. On the other hand,

$$\gamma \rightarrow (\gamma \rightarrow (\beta \rightarrow \gamma)) \leq (\beta \rightarrow \gamma) ,$$

therefore

$$(g \circ g)(\beta/\alpha) = \bigvee_{\gamma \in L} ((\gamma \rightarrow \alpha) \land (\beta \rightarrow \gamma)) \leq g(\beta/\alpha) ,$$

so that $\circ$ is idempotent. Condition (4) amounts to

$$(4') \quad \alpha \leq \bigwedge \{ \xi/c_\alpha \rightarrow (\beta) \ g \leq (\xi) \ g \} \rightarrow \beta .$$

By the series of equivalences

$$c_\alpha \rightarrow (\beta) \ g \leq (\xi) \ g ,$$

$$\forall \gamma \in L: \alpha \rightarrow (\gamma \rightarrow \beta) \leq (\gamma \rightarrow \xi) ,$$

$$\forall \gamma \in L: (\gamma \rightarrow \alpha) \rightarrow \beta) \leq (\gamma \rightarrow \xi) ,$$

$$\forall \gamma \in L: \gamma \land (\alpha \rightarrow \beta) \leq (\gamma \land (\alpha \rightarrow \beta) \leq (\alpha \rightarrow \beta) \land 1 \leq \xi ,$$

$$(4')$$ is reduced to $\alpha \leq (\alpha \rightarrow \beta) \rightarrow \beta$ or equivalently, $\alpha \land (\alpha \rightarrow \beta) = \alpha \land \beta \leq \beta$.

4.2. Application of 2.14

**Theorem.** Let $(A, \leq)$ be a complete lattice and $\delta$ a meet-irreducible element of $L$, $\delta \neq 1$.

If mappings $g: A \to TA$, $h: TA \to A$ are defined by

$$g(b/a) = 1 \text{ if } b \leq a , \text{ else } \delta ; \ a, b \in A ,$$

$$p/h = \text{Inf} \{ a/a \in A, p \leq (a) g \} ; \ p \in TA ,$$

then $(A, h)$ is a $T$-algebra.
Proof. By 2.14 it is sufficient to prove conditions (1)–(4). If \( a, c \in A \), \( a \leq c \), then \( g(a/c) = \delta \neq 1 = g(a/a) \), which yields injectivity of \( g \).

For every \( X \subseteq A \), \( b \in A \)

\[
g(b/\text{Inf } X) = 1 \quad \text{iff} \quad b \leq \text{Inf } X \quad \text{iff} \quad \forall x \in X: b \leq x \quad \text{iff} \quad \bigwedge_{x \in X} g(b/x) = 1.
\]

\( g \) being two-valued, (2) follows. Having again the codomain of \( g \) in mind, (3) follows from

\[
(g \circ g)(b/a) = \bigvee_{x \in A} (g(x/a) \land g(b/x)) = 1
\]

iff \( \exists x \in A: x \leq a \), \( b \leq x \) iff \( g(b/a) = 1 \).

Evidently, (4) holds for \( x \leq \delta \). If \( x \not\leq \delta \) then \( x \to \delta \leq \delta \), because

\[
x \land (x \to \delta) \leq \delta , \quad (x \lor \delta) \land ((x \to \delta) \lor \delta) = \delta
\]

and \( \delta \) is supposed to be meet-irreducible.

Now, for every \( a, y \in A \)

\[
x \to g(y/a) \begin{cases} = 1 & \text{if } y \leq a , \\ \leq \delta & \text{else} . \end{cases}
\]

Consequently \( \text{Inf } \{ x/c_x \to (a) g \leq (x) g \} = a \) and \( x \leq 1 = g(a/a) \).

References


Author's address: Bergakademie Freiberg, Sektion Mathematik, 9200 Freiberg, Bernh.-v.-Cotta-Str. 2, DDR.