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TWO PROOFS OF TRANSCENDENCY OF $\pi$ AND e

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In 1873 Hermite [2] succeeded in proving the transcendency of $e$ while in 1882 Lindemann [3] proved the transcendency of $\pi$. The present paper follows the papers [4] and [5]. The transcendency of $\pi$ and $e^a$, where $a$ is a nonzero real algebraic number, is proved here by two similar methods. In order to understand the proofs it is not necessary to have the knowledge of the theory of complex functions but it suffices to know Rolle’s theorem, the Euler function and some properties of algebraic integers.

Lemma 1. Let $V$, $p_0, p_1, \ldots, p_V$ be natural numbers and let $x_0, \ldots, x_V$ be real numbers ($x_i = x_j$ if and only if $i = j$). Put

$$N = \sum_{i=0}^{V} (p_i + 1) - 1,$$

$$Q_j(z) = \prod_{k=0}^{V} (z - x_k)^{p_k+1} \quad \text{for} \quad j = 0, 1, \ldots, V.$$

Let us assume that $f$ has $N$ continuous derivatives on

$$\left\langle \min_{0 \leq k \leq V} x_k, \max_{0 \leq k \leq V} x_k \right\rangle.$$

Then there is

$$y \in \left\langle \min_{0 \leq k \leq V} x_k, \max_{0 \leq k \leq V} x_k \right\rangle$$

such that

$$J(p_0, \ldots, p_V, f(x), V) = \sum_{k=0}^{V} \sum_{m=0}^{p_k} \frac{f^{(p_k-m)}(x_k)}{m! (p_k - m)!} \left[ \frac{1}{Q_k(z)} \right]_{z=x_k}^{(m)} = \frac{f^{(N)}(y)}{N!}.$$

Proof. Put

$$P(x) = \sum_{k=0}^{V} \sum_{m=0}^{p_k} \frac{f^{(p_k-m)}(x_k)}{(p_k - m)!} \sum_{s=0}^{m} \frac{1}{(m-s)!} \left[ \frac{1}{Q_k(z)} \right]_{z=x_k}^{(m-s)} Q_k(x)(x-x_k)^{p_k-s}.$$

For $j = 0, \ldots, V$, $q = 0, \ldots, p_j$ we have

$$P^{(q)}(x_j) = \sum_{k=0}^{V} \sum_{m=0}^{p_k} \frac{f^{(p_k-m)}(x_k)}{(p_k - m)!} \sum_{s=0}^{m} \frac{1}{(m-s)!} \left[ \frac{1}{Q_k(z)} \right]_{z=x_k}^{(m-s)} \left[ Q_k(x)(x-x_k)^{p_k-s} \right]_{x=x_j}^{(q)} = \frac{f^{(N)}(y)}{N!}.$$
By Rolle's theorem there is
\[ y \in \left\{ \min_{0 \leq k \leq V} x_k, \max_{0 \leq k \leq V} x_k \right\} \]

such that
\[ f^{(N)}(y) - P^{(N)}(y) = 0. \]

Hence
\[ f^{(N)}(y) = P^{(N)}(y) = \]
\[ = \sum_{k=0}^{V} \sum_{m=0}^{p_k} f^{p_k-m}(x_k) \sum_{s=0}^{m} \frac{1}{(p_k - m)!} \left[ \frac{1}{Q_k(z)} \right]_{z=x_k}^{(m-s)} \left[ Q_k(x) (x - x_k)_{p_k-s} \right]^{(N)}_{x=y} = \]
\[ = \sum_{k=0}^{V} \sum_{m=0}^{p_k} f^{p_k-m}(x_k) \left[ \frac{1}{Q_k(z)} \right]_{z=x_k}^{(m)} N!, \]

which implies (1).

**Lemma 2.** Let \( V, p_0, \ldots, p_V, Q_k(z) \) be the same as in Lemma 1. Put \( p_0 = p_1 = \ldots = p_V = n, x_k = k, k = 0, \ldots, V. \) Then there is \( c \) depending only on \( V \) and such that
\[ (2) \quad c^n \geq \frac{(V!)^{2n+V}}{m!} \left[ \frac{1}{Q_k(z)} \right]_{z=k}^{(m)} \in Z \text{ for } m = 0, \ldots, n, \]

where \( Z \) denotes the set of all integers.

**Proof.**
\[ \frac{(V!)^{2n+V}}{m!} \left[ \frac{1}{Q_k(z)} \right]_{z=k}^{(m)} = \]
\[ = \sum_{\sum_{m_i = m} \sum_{i \neq k} (n + m_i)!} \prod_{s \neq k} \frac{(-1)^{m_s}}{n! m_s!} \frac{(V!)^{2n+1}}{(k-s)^{n+1+m_s}} = \]
\[ = \sum_{\sum_{m_i = m} \sum_{i \neq k} (n + m_i)!} \prod_{s \neq k} \frac{(-1)^{m_s}}{n! m_s!} \frac{(V!)^{2n+1}}{(k-s)^{n+1+m_s}}. \]

Hence (2) follows.
Lemma 3. Let $V, n, V \neq 0$ be natural numbers and let $f(x)$ have $(V + 1) n + V$ continuous derivatives. Put

$$I_n(f(x), V) = \frac{(V!)^{2n+V}}{(n!)^{V+1}} \int_0^1 \cdots \int_0^1 f^{(V+1)n+V}(Q(x_1, \ldots, x_V)) \prod_{s=1}^V (x_s^{2n+s-1}(1 - x_s)^n \, dx_s)$$

where

$$Q(x_1, \ldots, x_V) = \sum_{i=1}^V \prod_{j=i}^V x_i.$$

Then the identity

$$I_n(f(x), V) = \sum_{j=0}^n \sum_{k=0}^n \frac{1}{n!} A_{kj} f^{(k)}(j)$$

holds, where

$$A_{kj}$$

are integers and $|A_{kj}| \leq c^n$

and $c$ depends only on $V$.

Proof. The proof proceeds by induction on $V$.

1. For $V = 1$ we have

$$I_n(f(x), 1) = \int_0^1 \frac{x^2(1 - x)^n}{(n!)^2} f^{(2n+1)}(x_1) \, dx_1 =$$

$$= \sum_{k=0}^n \frac{(2n - k)!}{(n!)^2} \binom{n}{k} \left( (-1)^{n+k} f^{(k)}(1) + (-1)^{n+1} f^{(k)}(0) \right) =$$

$$= \sum_{j=0}^1 \sum_{k=0}^n \frac{1}{k!} A_{kj} f^{(j)}(j),$$

where $A_{kj}$ satisfy the condition (4).

2. Suppose (3), (4) are valid for $(V - 1)$, we will prove that they hold for $V$. We have

$$I_n(f(x), V) = \frac{(V!)^{2n+1}}{n!} \int_0^1 (1 - x_V)^n \left( 2n(V-1) + V - 1 \right) I_n(f^{(n+1)}(x_V(x + 1)), V - 1) \, dx_V =$$

$$= \sum_{j=0}^V \sum_{k=0}^n \frac{A_{kj} (V!)^{2n+1}}{n! \, k!} \int_0^1 x_V^k (1 - x_V)^n f^{(n+k+1)}(x_V(j + 1)) \, dx_V,$$

where $A_{kj}$ satisfy the condition (4). Put

$$P(x) = x^k(1 - x)^n.$$

Then the repeated integration by parts yields

$$I_n(f(x), V) = \sum_{j=0}^{V-1} \sum_{k=0}^{n} \frac{A_{kj} (V!)^{2n+1}}{n! \, k!} \left( \frac{V!}{(j + 1)} \right)^{2n+1} B_{kj},$$

where

$$B_{kj} = \sum_{s=0}^{n+k} (-1)^s (j + 1)^{n+k-s} \left( P^{(s)}(1) f^{(n+k-s)}(j + 1) - P^{(s)}(0) f^{(n+k-s)}(0) \right).$$

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Now it is easy to see that

\[
P^{(s)}(1) = s! (-1)^n \binom{k}{s} f_{0k} \quad n \leq s \leq n + k,
\]

\[
P^{(s)}(0) = 0 \quad \text{otherwise},
\]

\[
P^{(s)}(0) = s! (-1)^{s-n+k} \binom{n}{k} f_{0r} \quad k \leq s \leq n + k,
\]

\[
P^{(s)}(0) = 0 \quad \text{otherwise}.
\]

The substitution \( s = n + k - s \) yields

\[
(6) \quad \frac{A_{kl}}{n! k! \left(1 + j\right)^{2n+1}} B_{kj} = \sum_{s=0}^{n} \frac{1}{s!} \left( A_{ks} f^{(s)}(j + 1) + B_{ks} f^{(s)}(0) \right),
\]

where \( A_{ks}, B_{ks} \) satisfy the condition (4). By (5), (6) we obtain (3) together with the condition (4).

**Lemma 4.** Introduce \( N, V, n, f(x), I_n(f(x), V) \) as in Lemma 3. Let

\[
\max_{x \in (0, V)} \left| f^{(N)}(x) \right| \leq c^n
\]

Then the inequality

\[
\left| I_n(f(x), V) \right| \leq \frac{c^n}{(n!)^{V+1}}
\]

holds with \( c_1 \) depending only on \( V \).

**Proof.** Lemma 3 implies

\[
\left| I_n(f(x), V) \right| \leq \left( \frac{V}{(n!)^{V+1}} \right)^{2n+1} \max_{x \in (0, V)} \left| f^{(N)}(x) \right| \leq \frac{c^n}{(n!)^{V+1}}.
\]

**Lemma 5.** Let

\[
(7) \quad B_n = \sum_{j=1}^{V} \sum_{r=0}^{n} A_{js} A_{rs}^j A_{rs}^j,
\]

where \( A_1, A_2, A_3 \) are algebraic numbers depending only on \( V \), \( B_n \) is also an algebraic number of degree \( V \) and

\[
(8) \quad \left| B_n \right| < \frac{c^n}{(n!)^{V}},
\]

\( A_{js} \) are integers and

\[
\left| A_{js} \right| \leq c_2 n!,
\]

where \( c_2 \) depends only on \( V \). Then there is \( n_0 \) such that \( B_n = 0 \) for every \( n > n_0 \).

**Proof.** Put

\[
(9) \quad D_n = K^{2n+1} B_n = \sum_{j=1}^{V} \sum_{r=0}^{n} A_{js}^j (KA_1)^j (KA_2)^j (KA_3)^j,
\]

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where $K$ is a nonzero integer such that the numbers $KA_1, KA_2, KA_3$ are algebraic integers. Then the conjugates of $D_n$ satisfy

$$D_n^\prime = \sum_{j=1}^{V} \sum_{r=0}^{n} \sum_{s=0}^{n} A_{jr}E_1E_2^2E_3,$$

where $E_i$ is the conjugate of $KA_i$ for $i = 1, 2, 3$. Consequently,

$$|D_n| \leq \sum_{j=1}^{V} \sum_{r=0}^{n} \sum_{s=0}^{n} A_{jr}|E_1|^j |E_2|^r |E_3|^s \leq c_2 n!,$$

where $c_2$ depends only on $V$.

In virtue of (7), (8), (9) and (10), for the norm of $D_n$ we obtain the inequalities

$$N(D_n) \leq (c_2 n!)^{V-1} |K|^{2n+V} \frac{c_3^n}{(n!)^V} \leq \frac{c_3^n}{n!}.$$

Hence $N(D_n) = 0$ for every $n > n_0$, which implies $B_n = 0$.

**Theorem 1.** Let $a$ be a nonzero real algebraic number. Then $e^a$ is transcendental.

**Proof.** Suppose $a$ and $e^a$ are algebraic numbers. Denote by $V$ the degree of the field $Q(a, e^a)$. In Lemma 1 put $p_0 = p_1 = \ldots = p_V = n$, $x_k = k$ for $k = 0, \ldots, V$, $f(x) = e^{ax}$ and in Lemma 5 put

$$B_n = J(n, \ldots, n, e^{ax}, V) n! (V!)^{2n+V}.$$

By Lemmas 2, 5 it is easy to see that $B_n = 0$ for every $n > n_0$. However, this is impossible because by Lemma 1

$$B_n = \frac{e^{ax} a^{N(V!)(2n+V)} N!}{N!} \neq 0 \quad y \in \langle 0, V \rangle.$$

**Proof.** Suppose $a$ and $e^a$ are algebraic numbers. Denote by $V$ the degree of the field $Q(a, e^a)$. In Lemma 3 put $f(x) = e^{ax}$ and in Lemma 5 put $B_n = n! I_n(e^{ax}, V)$. Lemmas 3, 4, 5 imply that $B_n = 0$ for every $n > n_0$. However, this is impossible, because the function which is integrated in Lemma 3 is almost everywhere positive (negative).

**Theorem 2.** The number $\pi$ is transcendental.

**Proof.** Suppose $\pi$ is an algebraic number. Denote by $M$ the degree of the field $Q(\pi)$. Let $V$ be a number for which the inequality

$$\frac{4\phi(2V)}{V} \leq M^{-1}$$

holds. ($\phi$ is the Euler function and the inequality evidently holds if $V$ has a large
enough number of different prime divisors). Hence the degree of the field

$$Q \left( \pi, \cos \frac{\pi}{4V}, \sin \frac{\pi}{4V} \right)$$

is less than or equal to $V$.

Now the proof can be completed in two different ways.

The first way:
In Lemma 1 put $p_0 = p_1 = \ldots = p_V = n$, $x_k = k + 1$ for $k = 0, 1, \ldots, V$,

$$f(x) = \sin \frac{\pi}{4V} x,$$

and in Lemma 5 put

$$B_n = J(n, \ldots, n, f(x), V) n! (V!)^{2nV + V}.$$

By Lemmas 1, 2 it is easy to see that $B_n$ satisfies the conditions of Lemma 5. Hence by Lemma 5, $B_n = 0$ for $n > n_0$. However, this is impossible because by Lemma 1

$$B_n = \frac{\left( \sin \left( \frac{\pi}{4V} x \right) \right)^{(N)}}{N!} (V!)^{2nV + V} \neq 0, \quad x \in \langle 1, V + 1 \rangle.$$

The second way:
In Lemma 3 put

$$f(x) = \sin \frac{\pi}{4V} x$$

and in Lemma 5 put

$$B_n = n! I_n(f(x), V).$$

Now it is easy to see that $B_n$ satisfies the conditions of Lemma 5. Hence by Lemma 5, $B_n = 0$ for $n > n_0$. However, this is impossible, because the function which is integrated in Lemma 3 is almost everywhere positive (negative).

Remark 1. We could prove Theorem 2 by Lemma 3 if we put

$$f(x) = e^{(nV)ix}$$

(it is necessary to prove Lemma 3 for complex functions) and then continue as in the proof of Theorem 1.

Remark 2. Ju. V. Nesterenko presented similar proofs in [4]. He integrated the function over a simplex and obtained an identity similar to (3).

Remark 3. Identity (1) is a consequence of the Hermite identity which can be found e.g. in [1].

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References


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