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ON LATTICES WITH ISOMORPHIC INTERVAL LATTICES

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Let A be a lattice. Denote by $\mathrm{Int}(A)$ the lattice of all intervals of A (including the empty set). Let B be a lattice and f an isomorphism of $\mathrm{Int}(A)$ onto $\mathrm{Int}(B)$. The one-element intervals of A are just atoms of $\mathrm{Int}(A)$ and so the isomorphism f induces a bijective mapping f' of A onto B defined by: $f'(a) = b$ iff $f([a, a]) = [b, b]$. The aim of this paper is to give answers to the following two questions (see [1], Problem I.10):

1) Under what conditions is f' an isomorphism or a dual isomorphism?

2) Under what conditions does $\mathrm{Int}(A)$ determine A up to isomorphism or dual isomorphism?

Recall that if $I = [a, b]$ and $J = [c, d]$ are intervals of A, then $I \vee J = [a \wedge c, b \vee d]$ and $I \wedge J = I \cap J = [a \vee c, b \wedge d]$ or the empty set if $a \vee c \nleq b \wedge d$.

Lemma 1. *For all* $x, y \in A$,

$$f([x \wedge y, x \vee y]) = [f'(x) \wedge f'(y), f'(x) \vee f'(y)] .$$

Proof. $f([x \wedge y, x \vee y]) = f([x, x] \vee [y, y]) = f([x, x]) \vee f([y, y]) = [f'(x), f'(x)] \vee [f'(y), f'(y)] = [f'(x) \wedge f'(y), f'(x) \vee f'(y)]$.

Lemma 2. *If* $f'(x) \leqq f'(y)$ *for all* $x, y \in A$ *such that* $x \leqq y$, *then* f' *is an isomorphism. If* $f'(x) \geqq f'(y)$ *for all* $x, y \in A$ *such that* $x \leqq y$, *then* f' *is a dual isomorphism.*

Proof. It is enough to show that if $f'(x) \leqq f'(y)$ then $x \leqq y$. Suppose that $f'(x) \leqq f'(y)$. By Lemma 1, $f([x \wedge y, x \vee y]) = [f'(x), f'(y)]$. Since $x \wedge y \in [x \wedge y, x \vee y]$, $f'(x \wedge y) \geqq f'(x)$. By assumption, $f'(x) \geqq f'(x \wedge y)$ and so $f'(x) = f'(x \wedge y)$. Thus $x = x \wedge y$, i.e. $x \leqq y$. Dually one can obtain the rest.

1. BOUNDED LATTICES

In this section let A be a bounded lattice with the least element 0_A and the greatest element 1_A. Let B be a lattice and f an isomorphism of $\mathrm{Int}(A)$ onto $\mathrm{Int}(B)$. Then, evidently, $B = f([0_A, 1_A])$ is a bounded lattice with the extreme elements 0_B and 1_B.

If $f'(0_A) < f'(1_A)$, then $f'(0_A) = 0_B$, $f'(1_A) = 1_B$ and $f([0_A, x]) = [0_B, f'(x)]$ for all $x \in A$. Thus f' is an isomorphism. Dually, if $f'(0_A) > f'(1_A)$, the mapping f' is a dual isomorphism.

Now suppose that f' is neither an isomorphism nor a dual isomorphism. Then the elements $f'(0_A)$ and $f'(1_A)$ are incomparable and, by Lemma 1, $f'(0_A) \wedge f'(1_A) = 0_B$ and $f'(0_A) \vee f'(1_A) = 1_B$.

Lemma 3. *The elements $r, s \in A$ such that $f'(r) = 0_B$ and $f'(s) = 1_B$ are incomparable and $r \vee s = 1_A$ and $r \wedge s = 0_A$.*

Proof. By Lemma 1, $f([r \wedge s,\ r \vee s]) = [f'(r) \wedge f'(s),\ f'(r) \vee f'(s)] = [0_B, 1_B] = f([0_A, 1_A])$. So $r \wedge s = 0_A$ and $r \vee s = 1_A$.

Lemma 4. *For all $x \in A$, $x = (x \wedge r) \vee (x \wedge s)$ and $x = (x \vee r) \wedge (x \vee s)$.*

Proof. By Lemma 1, $f([x \wedge r, x \vee r]) = [f'(r), f'(x)]$ and $f([x \wedge s, x \vee s]) = [f'(x), f'(s)]$. Since the interval $[(x \wedge r) \vee (x \wedge s), x]$ is a subinterval of both the intervals $[x \wedge r, x \vee r]$ and $[x \wedge s, x \vee s]$, we get that $f([(x \wedge r) \vee (x \wedge s), x]) \subseteq [f'(r), f'(x)] \cap [f'(x), f'(s)] = [f'(x), f'(x)]$. Thus $(x \wedge r) \vee (x \wedge s) = x$. Dually, $(x \vee r) \wedge (x \vee s) = x$.

Lemma 5. *Let x, y be elements of A. If $x \wedge r = y \wedge r$ and $x \wedge s = y \wedge s$, then $x = y$. If $x \vee r = y \vee r$ and $x \vee s = y \vee s$, then $x = y$.*

Proof. Let $x \wedge r = y \wedge r$ and $x \wedge s = y \wedge s$. Then, by Lemma 4, $x = (x \wedge r) \vee (x \wedge s) = (y \wedge r) \vee (y \wedge s) = y$. Dually we get the rest.

Lemma 6. *Let x, y be elements of A. The following equalities hold:*

$$r \wedge (x \vee y) = (r \wedge x) \vee (r \wedge y), \quad r \vee (x \wedge y) = (r \vee x) \wedge (r \vee y),$$

$$s \wedge (x \vee y) = (s \wedge x) \vee (s \wedge y), \quad s \vee (x \wedge y) = (s \vee x) \wedge (s \vee y).$$

Proof. Denote $a = r \wedge (x \vee y)$ and $b = (r \wedge x) \vee (r \wedge y)$. It is clear that $r \vee a = r \vee b = r$. By Lemma 4, $s \vee a = s \vee (s \wedge (x \vee y)) \vee (r \wedge (x \vee y)) = s \vee x \vee y$ and $s \vee b = s \vee (s \wedge x) \vee (s \wedge y) \vee (r \wedge x) \vee (r \wedge y) = s \vee x \vee y$. Thus, by Lemma 5, $a = b$. The rest can be shown similarly.

2. CONGRUENCE RELATIONS

In this section let A and B be lattices and let f be an isomorphism of $\text{Int}(A)$ onto $\text{Int}(B)$. Suppose that the induced mapping f' is neither an isomorphism nor a dual isomorphism. By Lemma 2, there exist elements $a, b, c, d \in A$ such that $a < b$, $c < d$ and $f'(a) < f'(b)$, $f'(c) > f'(d)$. Let M be the set of all intervals of A containing elements a, b, c, d. For any interval $I \in M$, the mapping $f/\text{Int}(I)$ is an isomorphism of $\text{Int}(I)$ onto $\text{Int}(f(I))$ such that the induced mapping f'/I is neither an isomorphism nor a dual isomorphism. Let r_I and s_I be elements of I such that $f(I) = [f'(r_I), f'(s_I)]$.

By Lemma 3, $I = [r_I \wedge s_I, r_I \vee s_I]$. It is evident that any interval of A is a subinterval of an interval from M.

Lemma 7. *Let $I, J \in M$ and $I \subseteq J$. Let x be an element of I. The following equalities hold*:

$$x \wedge r_J = x \wedge r_I \wedge r_J\,, \quad x \vee r_J = x \vee r_I \vee r_J\,,$$

$$x \wedge s_J = x \wedge s_I \wedge s_J\,, \quad x \vee s_J = x \vee s_I \vee s_J\,.$$

Proof. By Lemma 1, $f([r_I \wedge x, r_I \vee x]) = [f'(r_I), f'(x)]$ and $f([r_J \wedge x, r_J \vee x]) = [f'(r_J), f'(x)]$. If $I \subseteq J$, then $f'(r_J) \leqq f'(r_I)$ and so $[r_I \wedge x, r_I \vee x] \subseteq \subseteq [r_J \wedge x, r_J \vee x]$. Thus we have $r_J \wedge x \leqq r_I \wedge x$ and $r_I \vee x \leqq r_J \vee x$. Hence $x \wedge r_J = x \wedge r_I \wedge r_J$ and $x \vee r_J = x \vee r_I \vee r_J$. The rest can be proved in the same way.

Lemma 8. *Let x, y be elements of an interval $I \in M$ such that $x \wedge r_I = y \wedge r_I$ (or $x \vee r_I = y \vee r_I$, $x \wedge s_I = y \wedge s_I$, $x \vee s_I = y \vee s_I$). Then $x \wedge r_J = y \wedge r_J$ ($x \vee r_J = y \vee r_J$, $x \wedge s_J = y \wedge s_J$, $x \vee s_J = y \vee s_J$, respectively) for any interval $J \in M$ such that $I \subseteq J$.*

Proof follows from Lemma 7.

Now define relations α, β on A by the following rules:

$$x\alpha y \quad \text{iff} \quad x \wedge r_I = y \wedge r_I \quad \text{for some} \quad I \in M\,,$$

$$x\beta y \quad \text{iff} \quad x \wedge s_I = y \wedge s_I \quad \text{for some} \quad I \in M\,.$$

Lemma 9. *The relations α, β are congruence relations on A and $\alpha \cap \beta = \mathrm{id}_A$ (the identity relation on A).*

Proof. The relation α is clearly reflexive and symmetric. Let $x\alpha y$ and $y\alpha z$, i.e. $x \wedge r_I = y \wedge r_I$ and $y \wedge r_J = z \wedge r_J$ for some $I, J \in M$. Let $K \in M$ be an interval such that $I \subseteq K$ and $J \subseteq K$. Then, by Lemma 8, $x \wedge r_K = y \wedge r_K = z \wedge r_K$. One can easily show that $x\alpha y$ implies $(x \wedge c)\,\alpha(y \wedge c)$ and, by Lemmas 6 and 8, $(x \vee c)\,\alpha(y \vee c)$ for all $c \in A$. Thus α is a congruence relation on A. In the same way we can prove that β is a congruence relation on A. It follows from Lemma 5 that $\alpha \cap \beta = \mathrm{id}_A$.

Lemma 10. *Let u, v be elements of A and let $I \in M$ be an interval containing u, v. Then $((u \wedge r_I) \vee (v \wedge s_I))\,\alpha u$ and $((u \wedge r_I) \vee (v \wedge s_I))\,\beta v$.*

Proof. Using Lemma 6 we get $((u \wedge r_I) \vee (v \wedge s_I)) \wedge r_I = u \wedge r_I$ and $((u \wedge r_I) \vee (v \wedge s_I)) \wedge s_I = v \wedge s_I$.

For $x \in A$, denote by $\alpha(x)$ and $\beta(x)$ the congruence classes of the congruence relations α and β containing x.

Lemma 11. *Let x, y be elements of A. Then $f'(x) \leqq f'(y)$ if and only if $\alpha(x) \geqq \alpha(y)$ in the lattice A/α and $\beta(x) \leqq \beta(y)$ in the lattice A/β.*

Proof. Let $I \in M$ be an interval containing x, y. Using Lemma 1 we get that

$f'(x) \leqq f'(y)$ iff $[x \wedge r_I, x \vee r_I] \subseteq [y \wedge r_I, y \vee r_I]$, i.e. $x \wedge r_I \geqq y \wedge r_I$ and $x \vee r_I \leqq y \vee r_I$. Evidently, $\alpha(x) = \alpha(x \wedge r_I)$ and $\alpha(y) = \alpha(y \wedge r_I)$. By Lemma 6, $\beta(x) = \beta(x \vee r_I)$ and $\beta(y) = \beta(y \vee r_I)$.

Lemma 12. *The lattice A is isomorphic to $A/\alpha \times A/\beta$ and the lattice B is isomorphic to $(A/\alpha)^* \times A/\beta$, where $(A/\alpha)^*$ denotes the lattice dual to A/α.*

Proof follows from Lemmas 9, 10 and 11.

Lemma 13. *Let L, M be lattices. The lattice* Int $(L \times M)$ *is isomorphic to* Int $(L \times M^*)$, *where M^* is the dual of M.*

Proof is easy.

3. THE RESULTS

Let A be a lattice. Denote by CSub (A) the lattice of all convex sublattices of A (including the empty set). The lattice Int (A) is a sublattice of CSub (A) and any nonempty interval of A is either an atom or a join of two atoms in CSub (A). Thus whenever CSub (A) is isomorphic to CSub (B), then Int (A) is isomorphic to Int (B), too. On the other hand, any isomorphism of Int (A) onto Int (B) can be extended to an isomorphism of CSub (A) onto CSub (B) in a natural way (any convex sublattice of A is the join of all intervals of this sublattice).

Using Lemmas 12 and 13 one can easily prove the following theorems and corollaries.

Theorem 1. *Let A and B be lattices. The following assertions are equivalent:*

(i) Int (A) *is isomorphic to* Int (B).

(ii) CSub (A) *is isomorphic to* CSub (B).

(iii) *There exist lattices A_1, A_2, B_1, B_2 such that $A = A_1 \times A_2$, $B = B_1 \times B_2$, A_1 is isomorphic to B_1 and A_2 is dually isomorphic to B_2.*

Theorem 2. *Let A be a lattice. The following two assertions are equivalent:*

(i) *Whenever B is a lattice and f an isomorphism of* Int (A) *onto* Int (B), *then the induced mapping f' is either an isomorphism or a dual isomorphism.*

(ii) *A is directly irreducible.*

Theorem 3. *For a lattice A, the following two conditions are equivalent:*

(i) *Whenever B is a lattice such that* Int (A) *is isomorphic to* Int (B), *then A is either isomorphic or dual isomorphic to B.*

(ii) *Either A is directly irreducible or whenever $A = A_1 \times A_2$ (both A_1 and A_2 have more than one element), then both A_1 and A_2 are self dual.*

Corollary 1. *The lattice* Int (A) *determines A up to isomorphism if and only if whenever $A = A_1 \times A_2$, then both A_1 and A_2 are self dual.*

Corollary 2 ([2]). *Let V be a variety of lattices. With any lattice A, the variety V contains all lattices B such that* Int (A) *is isomorphic to* Int (B) *if and only if V is self dual.*

Remark. V. I. Igošin proved in [3] that any finite lattice A having just one atom is determined by Int (A) up to isomorphism or dual isomorphism. This result follows immediately from Theorem 3.

References

[1] *G. Grätzer:* General Lattice Theory, Birkhäuser Verlag, 1978.

[2] *V. Slavík:* A note on the lattice of intervals of a lattice, to appear.

[3] *V. I. Igošin:* The lattices of intervals and convex sublattices of lattices (Russian), Uporjadočennyje množestva i rešetki, Saratov 1980, 69—76.

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