Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 35 (1985), No. 4, 632–638

Czechoslovak Mathematical Journal, 35 (110) 1985, Praha

ON THE IDENTITY OF KELDYCH SOLUTIONS

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(Received September 23, 1984)

This paper is motivated by the following question originally posed by J. Lukeš and recently communicated to the author by I. Netuka: Let U be a relatively compact open subset of a "nice" $\mathfrak{P}$-harmonic space $(X, {}^*\mathscr{H})$. Let f be a continuous real function on the boundary U^* of U such that the generalized solution $H^U f$ of the Dirichlet problem and the function $D^U f$ associated to the solution of the weak Dirichlet problem coincide. Does this imply that $Af = H^U f$ for every Keldych operator A for U? Here a Keldych operator for U is a positive linear map A from the space $\mathscr{C}(U^*)$ of all continuous real functions on U^* into the space $\mathscr{H}(U)$ of all harmonic functions on U such that $A(h|_{U^*}) = h|_U$ for every function $h \in \mathscr{C}(\bar{U})$ which is harmonic on U. We recall that H^U and D^U are Keldych operators given by $H^U f(x) = \varepsilon_x^{\complement U}(f)$ and $D^U f(x) = \varepsilon_x^{\beta(\complement U)}(f)$ where $\beta(\complement U)$ is the essential base of $\complement U$.

As we shall see already the heat equation on $\mathbb{R}^2$ provides an example showing that the answer is negative. However, it will turn out that for every harmonic space the answer is positive if we modify the original question allowing only Keldych operators A which satisfy $A(p|_{U^*}) \leqq p|_U$ for every potential p on X.

1. A COUNTEREXAMPLE

If we do not require point separation by harmonic functions, there are of course immediate examples where the answer is negative: Let $(\mathbb{R}, {}^*\mathscr{H})$ be the harmonic space associated to uniform motion to the left on $\mathbb{R}$ and let U be an open interval $]a, b[$. Then $H^U f = D^U f = f(a)$ for every $f \in \mathscr{C}(\{a, b\})$ and the mapping A: $\mathscr{C}(\{a, b\}) \to \mathscr{H}(U)$ defined by $Af = f(b)$ is a Keldych operator for U such that $Af \neq H^U f$ if $f(a) \neq f(b)$.

On the other hand, if $(X, {}^*\mathscr{H})$ is "nice" in the sense that $\mathscr{H}(X)$ is linearly separating, then it is well known that H^U is the only Keldych operator for U if and only if $H^U = D^U$ ([10, p. 123], [1, p. 103]). Moreover, $H^U = D^U$ for every open subset U of X if and only if every semipolar subset of X is polar [1, p. 106]. Hence we may hope to find a nice counterexample only for a harmonic space not satisfying the axiom of polarity and an open subset U such that the operators H^U and D^U are different.

So let us now consider the harmonic space $(\mathbb{R}^2, {}^*\mathscr{H})$ given by the heat equation $\partial^2 u/\partial x^2 = \partial u/\partial t$. Let $V =]0,3[\times]0,3[$, $y = (\frac{3}{2}, \frac{3}{2})$, $B = [1,2] \times \{2\}$ and define

$$U = V \setminus (\{y\} \cup B) .$$

Let $f = 1_{\{y\}}$. Then of course $H^U f = D^U f = 0$, whereas $D^U 1_B = 0$, $H^U 1_B > 0$ on $]0,3[\times]2,3[$. Taking $\alpha > 0$ and defining a linear mapping $A: \mathscr{C}(U^*) \to \mathscr{H}(U)$ by

$$Ag = D^U g + \alpha(g(y) - \varepsilon_y^{\complement U}(g))\, H^U 1_B$$

we have $Af = \alpha H^U 1_B \neq 0$. Moreover, if $h \in \mathscr{C}(\bar{U})$ such that $h|_U \in \mathscr{H}(U)$, then $h(y) = \varepsilon_y^{\complement U}(h)$ and hence $A(h|_{U^*}) = D^U(h|_{U^*}) = h|_U$. In order to obtain the desired counter-example it therefore suffices to show that A is positive if α is sufficiently small. Now there exists $\alpha > 0$ such that $\varepsilon_z^{\complement U} \geqq \alpha \varepsilon_y^{\complement U}$ for every $z \in B$. Then for every $g \in \mathscr{C}^+(U^*)$ end every $x \in U$

$$D^U g(x) = \varepsilon_x^{\beta(\complement U)}(g) = (\varepsilon_x^{\complement U})^{\beta(\complement U)}(g) \geqq \int_B \varepsilon_z^{\beta(\complement U)}(g)\, \varepsilon_x^{\complement U}(\mathrm{d}z) =$$

$$= \int_B \varepsilon_z^{\complement U}(g)\, \varepsilon_x^{\complement U}(\mathrm{d}z) \geqq \alpha\, \varepsilon_y^{\complement U}(g)\, \varepsilon_x^{\complement U}(B) = \alpha\, \varepsilon_y^{\complement U}(g)\, H^U 1_B(x) ,$$

hence $A\, g(x) \geqq 0$, i.e. A is positive.

2. THE POSITIVE RESULT

In the following let $(X, {}^*\mathscr{H})$ be a $\mathfrak{P}$-harmonic space with countable base. Let $\mathscr{P}$ denote the convex cone of all continuous real potentials on X and let $\mathscr{C}_{\mathscr{P}}(X)$ be the space of all continuous real functions on X which are $\mathscr{P}$-bounded.

Given a finely open subset U of X let $S(U)$ ($H(U)$ resp.) be the set of all functions $f \in \mathscr{C}_{\mathscr{P}}(X)$ such that $\varepsilon_x^{\complement V}(f) \leqq f(x)$ ($\varepsilon_x^{\complement V}(f) = f(x)$ resp.) for every $x \in U$ and every fine neighborhood V of x satisfying $\bar{V} \subset U$. We recall that the essential base $\beta(\complement U)$ is the set of all $x \in X$ such that for every fine neighbourhood V of x the set $V \setminus U$ is not semipolar. It is the largest basic subset of $\complement U$, in particular $\beta(\complement U)$ is contained in the base $b(\complement U)$ of $\complement U$. We define kernels H^U and D^U on X by

$$H^U(x, \cdot) = \varepsilon_x^{\complement U} , \quad D^U(x, \cdot) = \varepsilon_x^{\beta(\complement U)}$$

and note that $D^U s \leqq H^U s \leqq s$ for every $s \in S(U)$.

For every $x \in X$ let $\mathscr{M}_x(S(U))$ denote the set of all positive Radon measures μ on X such that $\mu(s) \leqq s(x)$ for every $s \in S(U)$. We know by [3, p. 266] that $\mu(D^U f) = D^U f(x)$ for every $\mu \in \mathscr{M}_x(S(U))$ and that the set of all extreme points of $\mathscr{M}_x(S(U))$ is the set of all measures $\varepsilon_x^{\beta(\complement U) \cup B}$, $B \subset X \setminus \{x\}$.

We intend to show that for every function $f \in \mathscr{C}_{\mathscr{P}}(X)$ satisfying $H^U f = D^U f$ we have the equality $\mu(f) = D^U f(x)$ for every $x \in U$ and every $\mu \in \mathscr{M}_x(S(U))$ which is supported by $\complement U$. The connection to the original problem is resulting from the fact that for every Keldych operator A for an open set U which satisfies $A(p|_{U^*}) \leqq p|_U$ for every $p \in \mathscr{P}$ we have $A(x, \cdot) \in M_x(S(U))$ for every $x \in U$.

2.1. Proposition. *Let U be a finely open subset of X and $f \in \mathscr{C}_{\mathscr{P}}(X)$ such that $H^U f = D^U f$ on U. Then $\varepsilon_x^{\beta(\complement U) \cup A}(f) = D^U f(x)$ for every $x \in X$ and every subset A of $\complement U$.*

Proof. Since the functions $H^U f$ and $D^U f$ are finely continuous, we have $H^U f = D^U f$ on $b(U)$. If $x \in \complement b(U)$, then x is contained in the fine interior of $\complement U$, hence $x \in \beta(\complement U)$, i.e. $\varepsilon_x^{\complement U} = \varepsilon_x^{\beta(\complement U)} = \varepsilon_x$. In particular, $H^U f = D^U f$ on X.

Now fix $x \in U$, let K_0 be a compact subset of $\complement U$, and define $F = \beta(\complement U) \cup K_0$. Fix $\delta > 0$ and a strict potential $p_0 \in \mathscr{P}$. There exist a sequence (p_n) in $\mathscr{P}$ such that, for every $n \in \mathbb{N}$, $p_n = p_0$ in a neighborhood U_n of K_0 and such that the positive function $w = \sum_{n=1}^{\infty} (p_n - R_{p_0}^{K_0})$ satisfies $w(x) < \delta$.

Let $\hat{S}(U)$ denote the set of all limits of increasing sequences in $S(U)$. By [4, p. 38] there exists a function $s \in \hat{S}(U)$ such that $s \geqq f$ on $\beta(\complement U)$ and $s(x) < \varepsilon_x^{\beta(\complement U)}(f) + \delta$. Define a function s' on X by

$$s' = \begin{cases} s + w & \text{on} \quad \complement K_0\,, \\ f & \text{on} \quad K_0\,. \end{cases}$$

Moreover, by [3, p. 264], there exists a function $t \in -\hat{S}(U)$, compact subsets $K_1, \ldots, K_l$ of F and $p_1, \ldots, p_l \in \mathscr{P}$ such that

$$t' := t + \sum_{j=1}^{l} R_{p_j}^{K_j} \leqq f \quad \text{on} \quad F$$

and $\varepsilon_x^F(f) - \delta < t'(x)$. We intend to prove that $s' \geqq t'$. To that end we note first that obviously $s' \geqq f \geqq t'$ on F, i.e. $s' - t' \geqq 0$ on F. Moreover, let $y \in \complement F$. Then there exists an open neighborhood V of $\bigcup_{j=0}^{l} K_j$ such that $y \notin \overline{V}$. Defining $\mu = \varepsilon_y^{\beta(\complement U) \cup V}$ we have $\mu \neq \varepsilon_y$, $\mu(K_0) = 0$, $\mu \in M_y(S(U))$ and $\mu(R_{p_j}^{K_j}) = R_{p_j}^{K_j}(y)$ for every $0 \leqq j \leqq l$. Therefore $\mu(s') = \mu(s + w) \leqq (s + w)(y) = s'(y)$ and $\mu(t') \geqq t'(y)$, hence $\mu(s' - t') \geqq 0$. Next we claim that s' is l.s.c.. Indeed, $s + w$ is l.s.c. and f is continuous. We have $(s + w)(y) \geqq s(y) \geqq \varepsilon_y^{\beta(\complement U)}(f)$ for every $y \in X$ and $\varepsilon_y^{\beta(\complement U)}(f) = f(y)$ for every $y \in \complement b(U)$. Suppose that $z \in K_0$ and that (x_n) is a sequence in $b(U) \setminus K_0$ converging to z. Let (y_n) be a subsequence of (x_n) such that $\lim_{n \to \infty} s'(y_n) = \liminf_{n \to \infty} s'(x_n)$ and the sequence $(R_{p_0}^{K_0}(y_n))$ is convergent. Clearly, $\alpha := \lim_{n \to \infty} R_{p_0}^{K_0}(y_n) \leqq p_0(z)$. If $\alpha = p_0(z)$, then $\lim_{n \to \infty} \hat{R}_{p_0}^{\complement U}(y_n) = p_0(z)$, hence $\lim_{n \to \infty} \varepsilon_{y_n}^{\complement U} = \varepsilon_z$ by [6, p. 440]. Since

$$s'(y_n) \geqq s(y_n) \geqq \varepsilon_{y_n}^{\beta(\complement U)}(f) = \varepsilon_{y_n}^{\complement U}(f)$$

for every $n \in \mathbb{N}$, we conclude in this case that

$$\lim_{n \to \infty} s'(y_n) \geqq \lim_{n \to \infty} \varepsilon_{y_n}^{\complement U}(f) = f(z)\,.$$

Suppose now that $\alpha < p_0(z)$. Then there exists $n_0 \in \mathbb{N}$ such that

$$p_0(y_n) - R_{p_0}^{K_0}(y_n) > \frac{p_0(z) - \alpha}{2}$$

for every $n \geqq n_0$. Given $m \in \mathbb{N}$ there exists $n_1 \geqq n_0$ such that $y_n \in \bigcap_{j=1}^{m} U_j$ for every $n \geqq n_1$. Then for every $n \geqq n_1$

$$w(y_n) \geqq \sum_{j=1}^{m} (p_j - R_{p_0}^{K_0})(y_n) = \sum_{j=1}^{m} (p_0 - R_{p_0}^{K_0})(y_n) \geqq m \frac{p_0(z) - \alpha}{2}.$$

This shows that $\lim_{n\to\infty} w(y_n) = \infty$, hence $\lim_{n\to\infty} s'(y_n) = \infty \geqq f(z)$.

Thus s' is l.s.c.. Moreover, t' is u.s.c.. Applying a general minimum principle to the convex cone $P + \mathbb{R}_+(s' - t')$ we conclude that $s' - t' \geqq 0$. In particular,

$$\varepsilon_x^F(f) - \delta < t'(x) \leqq s'(x) < \varepsilon_x^{\beta(\complement U)}(f) + 2\delta .$$

Since $\delta > 0$ is arbitrary we obtain that $\varepsilon_x^F(f) \leqq \varepsilon_x^{\beta(\complement U)}(f)$. Replacing f by $-f$ the converse inequality follows. Thus $\varepsilon_x^F(f) = \varepsilon_x^{\beta(\complement U)}(f)$.

Finally, let A be a subset of $\complement U$. There exists a set $A' \in \mathscr{B}(X)$ such that $\beta(\complement U) \cup A \subset A' \subset \complement U$ and $\varepsilon_x^{A'} = \varepsilon_x^{\beta(\complement U)\cup A}$, and there exists an increasing sequence (K'_n) of compact subsets of A' such that $\sup \hat{R}_{p_0}^{K'_n}(x) = \hat{R}_{p_0}^{A'}(x)$. Since $K'_n \subset \beta(\complement U) \cup K'_n \subset A'$ for every $n \in \mathbb{N}$, we have $\lim_{n\to\infty} \hat{R}_{p_0}^{\beta(\complement U)\cup K'_n}(x) = \hat{R}_{p_0}^{A'}(x)$, hence $\lim_{n\to\infty} \varepsilon_x^{\beta(\complement U)\cup K'_n} = \varepsilon_x^{A'} = \varepsilon_x^{\beta(\complement U)\cup A}$. Therefore

$$\varepsilon_x^{\beta(\complement U)\cup A}(f) = \lim_{n\to\infty} \varepsilon_x^{\beta(\complement U)\cup K'_n}(f) = \varepsilon_x^{\beta(\complement U)}(f) .$$

An application of the fine continuity of the function $y \mapsto \varepsilon_y^{\beta(\complement U)\cup A}(f)$ and the fact that $\varepsilon_y^{\beta(\complement U)\cup A} = \varepsilon_y$ for every $y \in \beta(\complement U)$ finishes the proof.

The following lemma allows to obtain results for arbitrary finely open sets by proving the statements only for finely open Borel sets. As usual let $T(\complement U)$ denote the smallest closed subset of X supporting the measures $\varepsilon_x^{\complement U}$, $x \in U$ ([2], [6], [8], [12]).

2.2. Lemma. *Let U be a finely open subset of X. Then there exists a finely open Borel subset V of U such that $\beta(\complement V) = \beta(\complement U)$ and $\varepsilon_x^{\complement V} = \varepsilon_x^{\complement U}$ for every $x \in X$, i.e. $D^V = D^U$ and $H^V = H^U$. In particular, $S(V) = S(U)$ and $T(\complement V) = T(\complement U)$.*

Proof. For every $x \in U$ there exists a finely open Borel set U_x such that $x \in U_x \subset U$. Since the fine topology is quasi-Lindelöf, we may choose a sequence (U_n) of finely open Borel sets contained in U such that $U \setminus \bigcup_{n=1}^{\infty} U_n$ is semipolar. Moreover, there exists a finely closed Borel subset F of X containing $\complement U$ such that $\varepsilon_x^F = \varepsilon_x^{\complement U}$ for every $x \in X$. Choosing

$$V := \complement F \cup \bigcup_{n=1}^{\infty} U_n$$

we have a finely open Borel set $V \subset U$ such that $\beta(\complement V) = \beta(\complement U)$ and $\varepsilon_x^{\complement V} = \varepsilon_x^{\complement U}$ for every $x \in X$.

By [7], $S(V) = S(U)$. Moreover, clearly $\varepsilon_x^{\complement V}(\complement T(\complement U)) = \varepsilon_x^{\complement U}(\complement T(\complement U)) = 0$ for every $x \in V$ and hence $T(\complement V) \subset T(\complement U)$. In order to prove the converse inclusion we fix $x \in U$ and a strict potential $p \in \mathcal{P}$. Since the function $\hat{R}_p^{\complement V}$ is finely continuous and $U \setminus V$ is semipolar, there exists a sequence (x_n) in V such that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} \hat{R}_p^{\complement V}(x_n) = \hat{R}_p^{\complement V}(x)$. By [6, p. 440], $\lim_{n \to \infty} \varepsilon_{x_n}^{\complement V} = \varepsilon_x^{\complement V}$ and hence

$$\varepsilon_x^{\complement U}(\complement T(\complement V)) = \varepsilon_x^{\complement V}(\complement T(\complement V)) \leqq \liminf_{n \to \infty} \varepsilon_{x_n}^{\complement V}(\complement T(\complement V)) = 0 .$$

Thus $T(\complement U) \subset T(\complement V)$.

2.3. Theorem. *Let U be a finely open subset of X and $f \in \mathscr{C}_{\mathcal{P}}(X)$ such that $H^U f = D^U f$ on U. Let $x \in X$ and $\mu \in \mathscr{M}_x(S(U))$. Then $\mu^*(\{D^U f \neq f\} \setminus (U \cup \{x\})) = 0$. In particular, $\mu(f) = D^U f(x)$ if $\mu_*(U \cup \{x\}) = 0$. Moreover, $\mu(f) = f(x)$ if $x \in \complement U$ such that $\varepsilon_x^{\complement U}(\{x\}) \neq 0$.*

Proof. By (2.2) we may assume without loss of generality that U is a Borel set. If $x \in \beta(\complement U)$, the statement is trivial since then $\varepsilon_x^{\beta(\complement U)} = \varepsilon_x$ and $\mathscr{M}_x(S(U)) = \{\varepsilon_x\}$. So let us assume that $x \notin \beta(\complement U)$. Let K be a compact subset of $\{D^U f > f\} \setminus (U \cup \{x\})$ and let ν be an extreme point of the compact convex set $\mathscr{M}_x(S(U))$. Then by [3, p. 266] there exists a finely closed Borel subset F of X such that $\beta(\complement U) \subset F$ and $\nu = \varepsilon_x^F$. Let $A = F \cap K$. Since $\beta(\complement U) \cup A \subset F$ and $x \notin \beta(\complement U) \cup A$ we obtain by [2, p. 73] that

$$\nu(K) = \varepsilon_x^F(K) = \varepsilon_x^F(A) \leqq \varepsilon_x^{\beta(\complement U) \cup A}(A) .$$

By (2.1)

$$\varepsilon_x^{\beta(\complement U) \cup A}(f) = D^U f(x) = \varepsilon_x^{\beta(\complement U) \cup A}(D^U f) .$$

Since $f \leqq D^U f$ on the finely closed Borel set $\beta(\complement U) \cup A$ and since $f < D^U f$ on A, we conclude that

$$\varepsilon_x^{\beta(\complement U) \cup A}(A) = 0 .$$

This shows that the u.s.c. affine function $\nu \mapsto \nu(K)$ vanishes at every extreme point of $M_x(S(U))$. Therefore $\mu(K) = 0$, $\mu(\{D^U f > f\} \setminus (U \cup \{x\})) = 0$. Similarly $\mu(\{D^U f < f\} \setminus (U \cup \{x\})) = 0$. If $\mu(U \cup \{x\}) = 0$, this implies that

$$\mu(f) = \mu(D^U f) = D^U f(x) .$$

Finally, let $\alpha = \varepsilon_x^{\complement U}(\{x\})$. By [2, p. 75]

$$\varepsilon_x^{\complement U} = \alpha \varepsilon_x + (1 - \alpha)\, \varepsilon_x^{(\complement U) \setminus \{x\}} .$$

By (2.1), $\varepsilon_x^{\complement U}(f) = \varepsilon_x^{(\complement U) \setminus \{x\}}(f) = D^U f(x)$. Therefore $D^U f(x) = f(x)$ if $\alpha \neq 0$.

It may be interesting to mention the following consequence of (2.3). Note, however, that the results of [6, p. 439 and p. 442] allow a different proof.

2.4. Corollary. *Let U be a finely open subset of X, $x \in \bar{U}$ and $\mu \in \mathscr{M}_x(S(U))$. Then $\mu^*(\complement(U \cup T(\complement U) \cup \{x\})) = 0$.*

Proof. As before we may assume that U is a Borel set. Let K be a compact subset of $\bar{U} \setminus (U \cup T(\complement U) \cup \{x\})$. Then there exists a function $f \in \mathscr{C}^+_{\mathscr{P}}(X)$ such that $f = 0$ on $T(\complement U)$ and $f = 1$ on K. Then $H^U f = D^U f = 0$ on $\bar{U}$, hence $\mu(K) = 0$ by (2.3). Since $\mu(\complement \bar{U}) = 0$ the statement follows.

The next corollary of (2.3) contains the result on Keldych operators we announced at the beginning.

2.5. Corollary. *Let U be an open subset of X and let $f \in \mathscr{C}_{\mathscr{P}}(U^*)$ such that $H^U f = D^U f$ on U.*

Then $Af = D^U f$ on U for every positive linear operator A from $\mathscr{C}_{\mathscr{P}}(U^*)$ into the space of all real functions on U such that $A(h|_{U^*}) = h|_U$ for every $h \in H(U)$ and $A(p|_{U^*}) \leqq p|_U$ for every $p \in \mathscr{P}$.

Proof. The statement is an immediate consequence of (2.3). It suffices to extend f to a function in $\mathscr{C}_{\mathscr{P}}(X)$ and to note that for every $x \in U$ the mapping $g \mapsto A\,g(x)$ from $\mathscr{C}_{\mathscr{P}}(U^*)$ into $\mathbb{R}$ is given by integration with respect to a measure $A(x, \cdot) \in \mathscr{M}_x(S(U))$ which is supported by U^*.

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