Czechoslovak Mathematical Journal

Ján Andres

Boundedness of solutions of the third order differential equation with oscillatory restoring and forcing terms

Czechoslovak Mathematical Journal, Vol. 36 (1986), No. 1, 1–6

Persistent URL: http://dml.cz/dmlcz/102058

CZECHOSLOVAK MATHEMATICAL JOURNAL
Mathematical Institute of the Czechoslovak Academy of Sciences
V. *36* (*111*), PRAHA 28. 3. 1986, No 1

BOUNDEDNESS OF SOLUTIONS OF THE THIRD ORDER DIFFERENTIAL EQUATION WITH OSCILLATORY RESTORING AND FORCING TERMS

JAN ANDRES, Olomouc

(Received December 2, 1983)

1. In this paper we study the behaviour of solutions of the equation

$$x''' + ax'' + bx' + h(x) = p(t)\,, \tag{1}$$

where $a > 0$, $b > 0$ are constants with $a^2 > 4b$, the functions $h(x)$, $p(t)$ have their first derivatives continuous for all real values of their arguments and are oscillatory in the following sense:

for each argument u there exist such numbers $\beta_1 > \alpha_1 > u > \alpha_{-1} > \beta_{-1}$ that

$$f(\alpha_1) < 0\,, \quad f(\beta_1) > 0\,, \quad f(\alpha_{-1}) < 0\,, \quad f(\beta_{-1}) > 0\,,$$

where f is either $h(x)$ or $p(t)$, u is either x or t and all roots of the restoring term $h(x)$ are isolated.

2. Our main tool for attacking the equation (1) will be the well-known *Cauchy formula* for the particular solution of nonhomogeneous linear differential equations with constant coefficients.

Lemma 1. *If there exist such positive constants H, P that for all $x \in \mathcal{R}^1$ and $t \geqq 0$ the inequalities*

$$1)\ |h(x)| \leqq H\,, \quad 2)\ |p(t)| \leqq P$$

hold, then each solution $x(t)$ of the equation (1) *satisfies the inequalities*

$$\limsup_{t\to\infty} |x'(t)| \leqq (H + P)/b := D'\,, \tag{2}$$

$$\limsup_{t\to\infty} |x''(t)| \leqq 2(H + P)/a := D''\,.$$

Proof. Substituting $y := x'$, we get from (1) the equation

$$y'' + ay' + by = p(t) - h(x(t)) \tag{3}$$

with solutions of the form

$$/x'(t) = /y(t) = C_1 e^{\varrho_1 t} + C_2 e^{\varrho_2 t} + \int_0^t \frac{e^{\varrho_1(t-\tau)} - e^{\varrho_2(t-\tau)}}{\varrho_1 - \varrho_2} [p(\tau) - h(x(\tau))] \, d\tau ,$$

where $\varrho_{1,2} = (-a \pm \sqrt{(a^2 - 4b)})/2$ and C_1, C_2 are arbitrary constants.

Hence by virtue of 1), 2), for $t \geqq 0$ we have not only

$$\left| \int_0^t \frac{e^{\varrho_1(t-\tau)} - e^{\varrho_2(t-\tau)}}{\varrho_1 - \varrho_2} [p(\tau) - h(x(\tau))] \, d\tau \right| \leqq \frac{H + P}{b} \left(1 + \frac{\varrho_2 e^{\varrho_1 t} - \varrho_1 e^{\varrho_2 t}}{\varrho_1 - \varrho_2}\right),$$

but also

(4) $$\limsup_{t \to \infty} |x'(t)| \leqq (H + P)/b .$$

Furthermore, putting $z := y'$, we get from (3) the equation

$$z' + az = p(t) - b\,x'(t) - h(x(t))$$

with solutions of the form

$$/x''(t) = /z(t) = Ce^{-at} + \int_{T_x}^t e^{-a(t-\tau)} [p(\tau) - b\,x'(\tau) - h(x(\tau))] \, d\tau ,$$

where C is an arbitrary constant and T_x a great enough number.

Thus by virtue of 1), 2) and (4), for $t \geqq T_x$ we have not only

$$\left| \int_{T_x}^t e^{-a(t-\tau)} [p(\tau) - b\,x'(\tau) - h(x(\tau))] \, d\tau \right| \leqq 2(H + P + |o(T_x)|) \int_{T_x}^t e^{-a(t-\tau)} \, d\tau \leqq$$

$$\leqq \frac{2}{a} (H + P + |o(T_x)|)(1 - e^{-a(t - T_x)}) ,$$

but also

$$\limsup_{t \to \infty} |x''(t)| \leqq 2(H + P)/a , \quad \text{q.e.d.}$$

Lemma 2. *Under the assumptions of Lemma 1, if*

1') $$|h'(x)| \leqq H' \text{ for all } x \in \mathscr{R}^1 , \quad 3) \left| \int_0^\infty p(t) \, dt \right| < \infty ,$$

where H' is a suitable constant, then every bounded solution $x(t)$ of the equation (1) either satisfies the relation

(5) $$\lim_{t \to \infty} x(t) = \bar{x} , \quad \lim_{t \to \infty} x'(t) = \lim_{t \to \infty} x''(t) = 0 \quad (h(\bar{x}) = 0)$$

or there exists such a root $\bar{x}$ of $h(x)$ that $(x(t) - \bar{x})$ oscillates.

Proof. Substituting a fixed bounded solution $x(t)$ of (1) into (1) and integrating the result from T_x to t (T_x — a great enough number, whose magnitude will be specified later in (9)), we get the identity

$$(6)\qquad \int_{T_x}^{t} h(x(\tau))\,d\tau = -\{b[x(t) - x(T_x)] + a[x'(t) - x'(T_x)] + x''(t) - x''(T_x)\} + \\ + \int_{T_x}^{t} p(\tau)\,d\tau \ (:\equiv I(t))\,.$$

Therefore, by virtue of the condition 3), the assertion of Lemma 1 and the boundedness of $x(t)$, there exists such a constant M_x that for $t \geqq T_x$ the relation

$$(7)\qquad |I(t)| \leqq M_x \quad \text{i.e.} \quad \left|\int_{T_x}^{t} h(x(\tau))\,d\tau\right| \leqq M_x$$

is satisfied.

Now let us assume that $x(t)$ does not converge to any root $\bar{x}$ of $h(x)$: i.e.,

$$(8)\qquad \limsup_{t\to\infty} |x(t) - \bar{x}| > 0$$

and simultaneously, for $t \geqq T_x$,

$$(9)\qquad h(x(t)) \geqq 0 \quad \text{or} \quad h(x(t)) \leqq 0\,.$$

Then

$$H(t) :\equiv \int_{T_x}^{t} h(x(\tau))\,d\tau \quad (\text{for } t \geqq T_x)$$

evidently is a composed monotone function with a finite or infinite limit for $t \to \infty$. Since (7) implies that the "divergent case" can be disregarded, it follows from (9) that not only

$$(7')\qquad \lim_{t\to\infty} \int_{T_x}^{t} |h(x(\tau))|\,d\tau = \lim_{t\to\infty} \left|\int_{T_x}^{t} h(x(\tau))\,d\tau\right| \leqq M_x$$

but also

$$(8')\qquad \liminf_{t\to\infty} |x(t) - \bar{x}| = 0$$

holds, because otherwise (i.e. if

$$\liminf_{t\to\infty} |x(t) - \bar{x}| > 0)$$

(9) together with the fact that the roots of $h(x)$ are isolated would yield

$$\liminf_{t\to\infty} |h(x(t))| = \liminf_{t\to\infty} |h(x(t)) - h(\bar{x})| > 0\,,$$

a contradiction to (7').

Thus (8) and (8') imply

$$\limsup_{t\to\infty} |h(x(t))| = \limsup_{t\to\infty} |h(x(t)) - h(\bar{x})| > 0 = \liminf_{t\to\infty} |h(x(t))|$$

and consequently there exists such a sequence $\{t_i\} \geqq T_x$ and such a constant $\tilde{H} > 0$

that (in what follows, $\mathrm{d}(x, y)$ denotes the distance between x and y)

$$\alpha) \quad \liminf_{i\to\infty/\Rightarrow t_i\to\infty/} \mathrm{d}(t_i, t_{i-1}) > 0\,, \quad \beta) \; |h(x(t_i))| \geqq \tilde{H}$$

hold. Hence

$$M_x \geqq \lim_{t\to\infty} \int_{t_1}^{t} |h(x(\tau))| \, \mathrm{d}\tau = \sum_{i=2}^{\infty} \int_{t_{i-1}}^{t_i} |h(x(\tau))| \, \mathrm{d}\tau \Rightarrow \limsup_{i\to\infty/\Rightarrow t_i\to\infty/} \int_{t_{i-1}}^{t_i} |h(x(t))| \, \mathrm{d}t = 0$$

or (cf. α), β))

$$H' \limsup_{t\to\infty} |x'(t)| \geqq \limsup_{t\to\infty} \left| \frac{\mathrm{d}h(x(t))}{\mathrm{d}x(t)} \, x'(t) \right| = \limsup_{t\to\infty} \left| \frac{\mathrm{d}h(x(t))}{\mathrm{d}t} \right| = \infty\,.$$

But according to the assertion of Lemma 1, this is impossible and that is why $(x(t) - \bar{x})$ necessarily oscillates.

The remaining part of our lemma follows immediately from the assertion

$$(10) \qquad x(t) \in \mathfrak{C}^{(n)}\langle 0, \infty)\,, \quad \limsup_{t\to\infty} |x^{(n)}(t)| < \infty\,,$$

$$\lim_{t\to\infty} |x(t)| < \infty \Rightarrow \lim_{t\to\infty} x^{(k)}(t) = 0\,,$$

(where $n \geqq 2$ is a natural number and $k = 1, \ldots, (n - 1)$),

whose proof can be found e.g. in [1, p. 161]. This completes the proof.

Lemma 3. *Under the assumptions of Lemma 2 and if*

$$2') \quad |p'(t)| \leqq P' \quad \text{for all} \quad t \geqq 0\,, \qquad 2'') \; \limsup_{t\to\infty} |p(t)| > 0$$

hold, where P' is a suitable constant, then for every bounded solution $x(t)$ of the equation (1) there exists such a root $\bar{x}$ of $h(x)$ that $(x(t) - \bar{x})$ oscillates.

Proof. If Lemma 3 does not hold, then according to Lemma 2 (5) holds and the fourth derivative of $x(t)$ satisfies

$$x''''(t) = p'(t) - ax'''(t) - bx''(t) - h'(x)\,x'(t)\,.$$

But it can be readily checked that, by the ultimate boundedness of $x'(t)$, $x''(t)$, $x'''(t)$ (see (2)) and 1'), 2'), there exists such a constant D_4 that

$$\limsup_{t\to\infty} |x''''(t)| \leqq D_4\,,$$

which according to (10) gives the relations

$$\lim_{t\to\infty} x(t) = \bar{x}/ \Rightarrow \lim_{t\to\infty} h(x(t)) = h(\bar{x}) = 0/\,, \quad \lim_{t\to\infty} x^{(j)}(t) = 0 \quad j = 1, 2, 3$$

or

$$\limsup_{t\to\infty} |p(t)| = \limsup_{t\to\infty} |x'''(t) + a\,x''(t) + b\,x'(t) + h(x(t))| = 0\,,$$

a contradiction to $\limsup_{t\to\infty} |p(t)| > 0$ (cf. 2'')), q.e.d.

3. Now we can give the principal result of our paper.

Theorem. *If there exist such positive constants H, H', P, P', P_0, R that for $|x| > R$ and $t \geqq 0$ the following conditions are satisfied:*

1) $|h(x)| \leqq H, \ |h'(x)| \leqq H'$,

2) $|p(t)| \leqq P, \ |p'(t)| \leqq P', \ \left|\int_0^t p(\tau)\, d\tau\right| \leqq P_0, \ \limsup_{t\to\infty} |p(t)| > 0$,

3) $\min\left[d(\bar{x}_k, \bar{x}_{k+1}), d(\bar{x}_k, \bar{x}_{k-1})\right] > \dfrac{2(H + P)}{b}\left(\dfrac{2}{a} + \dfrac{a}{b}\right) + \dfrac{P_0}{b}$,

where $\bar{x}_k$ are roots of $h(x)$ with $h'(\bar{x}_k) > 0$ and $\bar{x}_{k-1}, \bar{x}_{k+1}$ denote the couple of adjacent roots of $\bar{x}_k$ $(k = 0, \pm 2, \pm 4, \ldots)$, then all solutions $x(t)$ of the equation (1) are bounded and for each of them there exists such a root $\bar{x}$ of $h(x)$ that $(x(t) - \bar{x})$ oscillates.

Proof. Let us assume, on the contrary, that $x(t)$ is an unbounded solution of (1); i.e., for example, $\limsup_{t\to\infty} x(t) = \infty$.

Lemma 1 implies the existence of such a number $T_0 \geqq 0$ great enough that for $t \geqq T_0$

$$|x'(t)| \leqq D' + \varepsilon_1 , \quad |x''(t)| \leqq D'' + \varepsilon_2 ,$$

with $\varepsilon_1 > 0, \varepsilon_2 > 0$ small enough constants.

Let $T_1 \geqq T_0$ be the last point with $x(T_1) = \bar{x}_k$ (k-even) and $T_2 > T_1$ be the first point with $x(T_2) = \bar{x}_{k+1}$. If we integrate (1) from T_1 to t, $T_1 \leqq t \leqq T_2$, we come to

$$[x'(t) - x''(T_1)] + a[x'(t) - x'(T_1)] + b[x(t) - x(T_1)] + \int_{T_1}^t h(x(\tau))\, d\tau = \int_{T_1}^t p(\tau)\, d\tau . \tag{11}$$

However, for $T_1 \leqq t \leqq T_2$ we have $h(x(t)) \operatorname{sgn} x(t) \geqq 0$, whence we can obtain (multiplying (11) by sgn x)

$$|x(t)| \leqq |x(T_1)| + \frac{2}{b}\left[D'' + aD' + \tfrac{1}{2}P_0\right] + \varepsilon ,$$

where $\varepsilon > 0$ is an arbitrarily small constant, a contradiction to $x(T_2) = \bar{x}_{k+1}$ with respect to 3).

Since the remaining part of our theorem immediately follows from Lemma 3, the proof is complete.

4. In the end, let us note that in [2] we have dealt also with the case

$$\int_0^\infty |p(t)|\, dt < \infty .$$

References

[1] *W. A. Coppel:* Stability and Asymptotic Behavior of Differential Equations, D. C. Heath, Boston, 1975.

[2] *J. Andres:* Asymptotic properties of solutions of a certain third order differential equation with the oscillatory restoring term, to appear.

Author's address: 771 46 Olomouc, Gottwaldova 15, Czechoslovakia (Joint Laboratory of Optics of Czech. Acad. Sci. and Palacký University).