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ON UNIFORM APPROXIMATION OF BOUNDED APPROXIMATELY CONTINUOUS FUNCTIONS BY DIFFERENCES OF LOWER SEMICONTINUOUS AND APPROXIMATELY CONTINUOUS ONES

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It is known [2] that a real function f is in Baire class one if and only if f is a uniform limit of differences of lower semicontinuous functions. Our aim is to show that an analogous result holds in the class of all bounded approximately continuous functions.

The proof of our assertion is based on a simple generalization of the Stone-Weierstrass theorem which we restate as a lemma:

Lemma. *Let $\mathscr{F}$ be a linear space of bounded real functions on a set X containing with each $f \in \mathscr{F}$ and each $\alpha \in R$ the function* $\max(f, \alpha)$. *If $\mathscr{G}$ is a linear subspace of $\mathscr{F}$ containing the constants and having the property that for every pair of disjoint $\mathscr{F}$-zero sets $\mathscr{Z}_1, \mathscr{Z}_2$ there is a function $g \in \mathscr{G}$ such that*

$$0 \leqq g \leqq 1\,, \quad g = 0 \quad on \quad \mathscr{Z}_1\,, \quad g = 1 \quad on \quad \mathscr{Z}_2\,,$$

then $\mathscr{G}$ is dense in $\mathscr{F}$ ($\mathscr{F}$ is supposed to be equipped with the uniform norm $\|\ldots\|$).

Remark. A set $\mathscr{Z}$ is said to be an *$\mathscr{F}$-zero set* if there is an $f \in \mathscr{F}$ such that $\mathscr{Z} = f^{-1}(0)$. If $\mathscr{F}$ satisfies the conditions of our lemma, then $\mathscr{Z}$ is an $\mathscr{F}$-zero set if and only if there is $F \in \mathscr{F}$ and $\alpha \in R$ such that $\mathscr{Z} = \{x \in X;\ F(x) \leqq \alpha\}$.

Proof. Since $\mathscr{F}$ and $\mathscr{G}$ are linear spaces and $\mathscr{G}$ contains the constant functions, the proof will be completed by showing that for each function $f \in \mathscr{F}$, $\|f\| \leqq 1$, there is a $g_1 \in \mathscr{G}$ such that $\|f - g_1\| \leqq 2/3$. Indeed, if this assertion is proved, then $\|f\,.\,3/2 - g_1\,.\,3/2\| \leqq 1$ and therefore there is $h_1 \in \mathscr{G}$ such that $\|f\,.\,3/2 - g_1\,.\,3/2 - h_1\| \leqq 2/3$. Put $g_2 = g_1 + h_1\,.\,2/3$. Then we have $\|f - g_2\| \leqq (2/3)^2$. Proceeding in this way we obtain for each n a function $g_n \in \mathscr{G}$ such that $\|f - g_n\| \leqq$

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$\leqq (2/3)^n$. Let $f \in \mathscr{F}$ be a function with $\|f\| \leqq 1$. In virtue of our assumptions there is $g \in \mathscr{G}$ such that

$$0 \leqq g \leqq 1\,, \quad g = 0 \quad \text{on} \quad [f \leqq -1/3] \quad \text{and} \quad g = 1 \quad \text{on} \quad [f \leqq 1/3]\,.$$

The function $g_1 = g \,.\, 2/3 - 1/3$ satisfies $\|f - g_1\| \leqq 2/3$.

Now we are able to present our main approximation theorem. First, however, let us agree on some terminology. Let (X, ϱ) be a topological space, and let τ be another topology on X finer than ϱ. Following [3] we say that τ has the *Lusin-Menchoff property* (*with respect to* ϱ) if for each pair of disjoint subsets F, F_τ of X, F closed, F_τ τ-closed there are $G, G_\tau \subset X$, G open, G_τ τ-open such that $F_\tau \subset G$, $F \subset G_\tau$, $G \cap G_\tau = 0$ (topological notions referring to the "fine" topology τ will be indicated by the prefix τ to distinguish them from those pertaining to the "initial" topology ϱ).

In [3] it is proved that any topology τ having the Lusin-Menchoff property with respect to ϱ satisfies the following *Zahorski property*: Any τ-closed G_δ-set is a zero set of a τ-continuous and upper semicontinuous function on X.

Approximation theorem. *Let τ have the Lusin-Menchoff property with respect to a metric space (X, ϱ). Then any bounded τ-continuous Baire one function on X is a uniform limit of differences of nonnegative τ-continuous and lower semicontinuous functions on X.*

Proof. We shall use our Lemma. Let $\mathscr{F}$ be set of all bounded τ-continuous Baire one functions, and let $\mathscr{G}$ be the set of all differences $f - g \in \mathscr{F}$, where f, g are nonnegative, τ-continuous and lower semicontinuous functions. As (X, ϱ) is a metric space, the inclusion $\mathscr{G} \subset \mathscr{F}$ holds. Assume that $\mathscr{Z}_1, \mathscr{Z}_2$ are disjoint $\mathscr{F}$-zero sets. In view of the Zahorski property there are τ-continuous and upper semicontinuous functions h_1, h_2 such that $0 \leqq h_i \leqq 1$ on X, $\mathscr{Z}_i = h^{-1}(0)$ for $i = 1, 2$. Put

$$f = 1/(h_1 + h_2)\,, \quad g = (1 - h_1)/(h_1 + h_2)\,.$$

Then $f - g \in \mathscr{G}$ and $0 \leqq f - g \leqq 1$, $f - g = 0$ on $\mathscr{Z}_1$ and $f - g = 1$ on $\mathscr{Z}_2$. Now, we can complete the proof by applying Lemma.

Remark. Since in the above proof we need any lower semicontinuous function to be in Baire class one, Theorem remains true if we suppose that τ has the Lusin-Menchoff property with respect to a perfectly normal space (X, ϱ) only.

Corollary 1. *Any bounded approximately continuous function on the real line is a uniform limit of a sequence $\{f_n - g_n\}$, where f_n, g_n are nonnegative approximately continuous and lower semicontinuous functions.*

Proof. The family of all approximately continuous functions is the set of all continuous functions in the so-called *density* topology. Since this topology has the Lusin-Menchoff property [3] and since any approximately continuous function is in Baire class one, the proof follows from our Approximation Theorem.

Remark. In general, the functions f_n, g_n from Corollary 1 need not be bounded.

Proposition. *There is a bounded approximately continuous function h on the real line such that: if f, g are nonnegative lower semicontinuous functions and $\|h - (f - g)\| < 1$ then f, g are unbounded.*

Proof. Let $A \subset R$. Set $A^{(0)} = A$ and denote by $A^{(n)}$ the set of all accumulation points of the set $A^{(n-1)}$, for all $n \in N$. By a standard process we construct sets K_k such that

(i) $K_k \subset (k, k + 1)$,
(ii) the sets K_k are countable,
(iii) $K_k^{(j+1)} \subset K_k^{(j)}$ for any $j \in N$,
(iv) $K_k^{(k)} \neq \emptyset$ and $K_k^{(k+1)} = \emptyset$.

Denote $A_{i,j} = K_i^{(j)} \setminus K_i^{(j+1)}$. Evidently $A_{i,j} \neq \emptyset$ for $j \leqq i$. Define $\mathscr{A} = \bigcup_{i=1}^{\infty} \bigcup_{j=0}^{i} A_{i,j}$. As K_k are G_δ sets [1 p. 109] the set $\mathscr{A}$ is countable and G_δ. Since $A_{i,j} \cap A_{n,k} = \emptyset$ for $j \neq k$, there is [4] a bounded approximately continuous function h on the real line such that $h(x) = 0$ for $x \in A_{i,2j}$ and $h(y) = 8$ for $y \in A_{i,2j-1}$ for any $i, j \in N$ and $2j \leqq i + 1$. This function h satisfies the conditions from Proposition.

First we prove the following property.

(P) Let $x \in A_{i,2j}$ if $j \geqq 2$ and let f, g be nonnegative lower semicontinuous functions and $\|h - (f - g)\| < 1$. Then there is $x' \in A_{i,2j-2}$ such that $f(x') \geqq f(x) + 4$.

We have $h(x) = 0$ and therefore $g(x) \geqq f(x) - 1$. Since g is lower semicontinuous and $A_{i,2j} = A_{i,2j-1}^{(1)} \neq \emptyset$ there is $y \in A_{i,2j-1}$ such that $g(y) \geqq g(x) - 1 \geqq f(x) - 2$. Since $h(y) = 8$ we have $f(y) \geqq g(y) + 7 \geqq f(x) + 5$. And because f is lower semicontinuous there is $x' \in A_{i,2j-2}$ such that $f(x') \geqq f(y) - 1$ and therefore $f(x') \geqq$ $\geqq f(x) + 4$. Property (P) implies that $\|f|_{(2k,2k+1)}\| \geqq 4k$ and therefore f is unbounded.

Corollary 2. *Any bounded Baire one function on a metric space (X, ϱ) is a uniform limit of a sequence of differences of lower semicontinuous functions.*

Proof. Use the discrete topology τ on X. This topology evidently has the Lusin-Menchoff property with respect to ϱ.

Remark. Theorem and Corollaries are valid even in the case of unbounded functions. This assertion as well as many other related topics can be found in [3].

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