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_Czechoslovak Mathematical Journal_, Vol. 36 (1986), No. 1, 35–43

Persistent URL: [http://dml.cz/dmlcz/102063](http://dml.cz/dmlcz/102063)

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ISOMETRIES IN RIESZ GROUPS

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(Received June 8, 1984)

Isometries in the lattice ordered groups have been studied by K. L. Swamy [8], [9] and W. B. Powell [6] for the abelian case and by J. Jakubík in [3], [4] for the general case. Isometries in the 2-isolated abelian Riesz groups have been investigated by J. Rachůnek [7].

In this paper isometries in abelian Riesz groups are studied and some of Rachůnek's results on isometries from [7] are generalised. It is also shown that the results on the relations between isometries and direct decompositions of lattice ordered groups [3], which J. Jakubík and M. Kolibiar extended to abelian distributive multilattice groups [5], can be also extended to abelian Riesz groups. Note that a Riesz group need not be a multilattice group and conversely, a multilattice group need not be a Riesz group.

First we recall some notions and notations used in the paper.

Let $G$ be a partially ordered group. The group operation will be written additively. We denote $G^+ = \{x \in G; x \geq 0\}$, $G^- = \{x \in G; x \leq 0\}$. If $a_1, \ldots, a_n$ are elements of $G$, then we denote by $U(a_1, \ldots, a_n)$ and $L(a_1, \ldots, a_n)$ the set of all upper bounds and the set of all lower bounds of the set $\{a_1, \ldots, a_n\}$, respectively. For each $a \in G$, $|a| = U(a, -a)$.

The following notion of isometry in partially ordered groups was introduced by J. Rachůnek [7].

If $G$ is a partially ordered group, then a bijection $f: G \rightarrow G$ is called an isometry in $G$ if $|a - b| = |f(a) - f(b)|$ for each $a, b \in G$. An isometry $f$ in an ordered group $G$ is called a 0-isometry if $f(0) = 0$.

A Riesz group is any partially ordered group which is directed and satisfies the Riesz interpolation property, i.e., for each $a_i, b_j \in G$ $(i, j = 1, 2)$ such that $a_i \leq b_j$ $(i, j = 1, 2)$ there exists $c \in G$ such that $a_i \leq c \leq b_j$ $(i, j = 1, 2)$. See [1].

Throughout the paper we assume that $G$ is an abelian Riesz group and $f$ is a 0-isometry in $G$.

1. Lemma. a) If $x \in G^+$, then there exist $x_1, x_2 \in G^+$ such that $x = x_1 + x_2$, $f(x_1) \geq 0$, $f(x_2) \leq 0$, $f(x) \leq x_1 \leq x + f(x)$.
   b) If $x \in G^+$, $t \in G$, $t \in [0; x] \cap [f(x); x + f(x)]$, then $x + f(x) = 2t$. 
Proof. If \( x \in G^+ \), \( x' = f(x) \), then \( U(x) = |x| = |x'| \). Thus \( x \geq x' \), \( x \geq -x' \), hence \( x + x' \geq 0 \). Because of \( x \geq 0 \), \( x + x' \geq x' \). Since \( G \) is a Riesz group, there exists \( b' \) in \( G \) such that

\[
0 \leq b' \leq x, \quad x' \leq b' \leq x + x'.
\]

(Cf. Fig. 1.) Let \( b = f^{-1}(b') \). From \( b' \geq 0 \), \( x \geq b' \) we get \( x \in U(b') = |b'| = |b| \).

Thus \( x \geq b \). Because of \( x - b \geq 0 \), \( x' - b' \leq 0 \), from \( |x - b| = |x' - b'| \) it follows that \( x - b = b' - x' \). Let \( d' = x' - b' \), then \( d' \leq 0 \), \( d' \leq x' \), \( -d' = x - b \).

Denote \( d = f^{-1}(d') \). Then we obtain \( x \geq x - d \), since

\[
x \in |b'| = |x' - d'| = |x - d|.
\]

Hence \( d \geq 0 \). From \( |d'| = |d| \) we get \( d = -d' = x - b \). Thus \( x = b + d \). Because of \( x + x' \geq b' \), \( b' \geq 0 \) we get \( x + x' \in U(b') = |b'| = |x' - d'| \). Thus \( x \geq -d' = x - b \), hence \( b \geq 0 \).

From the relations \( b \geq 0 \), \( f(b) \geq 0 \) and \( |b| = |f(b)| \) we obtain \( f(b) = b \). If we put \( x_1 = b \) and \( x_2 = d \) we obtain the required elements. We have proved that \( f(x_1) = x_1 \) and also \( f(x_2) = -x_2 \). Thus \( x' = b' + d' = b - d = x_1 - x_2 \) and clearly \( x + + x' = 2x_1, \ x - x' = 2x_2 \).

It is clear that for each \( t \in G \) such that \( t \in [0, x] \cap [f(x), x + f(x)] \) the relation \( x + f(x) = 2t \) is valid.

Hence the following assertion is valid.

2. Lemma. Let \( x, x_1, x_2 \) be as in Lemma 1a) and let \( x' = f(x) \). Then \( f(x_1) = x_1 \), \( f(x_2) = -x_2 \), \( x' = x_1 - x_2 \), \( x + x' = 2x_1, \ x - x' = 2x_2 \), \( x \geq x' \).

The following assertion can be verified analogously:

3. Lemma. If \( x \in G^- \), then there exist elements \( x_1, x_2 \in G^- \) such that \( x = x_1 + + x_2 \), \( f(x_1) = x_1 \), \( f(x_2) = -x_2 \).

4. Lemma. Let \( x, x_1, x_2 \) be as in 1a) and \( x' = f(x) \). If \( 0 \leq y \leq x, \ x' \leq y \leq x + + x' \) holds for some \( y \in G \), then \( y = x_1 \).

Proof. Let \( y \in G \) such that \( 0 \leq y \leq x, \ x' \leq y \leq x + x' \). Since \( x_1 \leq x, \ x_1 \leq x' \), \( x' \leq y \leq x + + x' \).
x + x', there exists $y_1 \in G$ such that
\[ y \leq y_1 \leq x, \quad x_1 \leq y_1 \leq x + x'. \]
From Lemma 1b) and Lemma 2 we obtain $x + x' = 2y, x + x' = 2y_1, x + x' = 2x_1$. Thus we get $2(y_1 - y) = 0; 2(y_1 - x_1) = 0$. Since $y_1 - y \geq 0, y_1 - x_1 \geq 0$, we have $y = y_1 = x_1$.

4'. Lemma. Let $x, x_1, x_2$ be as in 1a) and let $x' = f(x)$. If $0 \leq y \leq x, -x' \leq y \leq x - x'$ hold for some $y \in G$, then $y = x_2$.

Proof. From the assumptions we have $x' \leq y + x' \leq x + x', 0 \leq y + x' \leq x$. In view of 4 we obtain $y + x' = x_1$. Then 2 implies that $y = x_2$.

5. Lemma. Let $x \in G^+, x = u + v, u, v \in G^+, f(u) \geq 0, f(v) \leq 0$ and let $x_1, x_2$ be as in 1a). Then $u = x_1, v = x_2$.

Proof. Clearly $f(u) = u, f(v) = -v$. Let $x' = f(x)$. Because of $x - u \geq 0$, from $|x - u| = |f(x) - f(u)| = |x' - u|$ we infer that $x - u \geq -x' + u$. Since $2u \geq u$ we obtain $x + x' \geq u$. Thus $u \leq x, u \leq x + x', x_1 \leq x, x_1 \leq x + x'$. Then there exists an element $t \in G$ such that $u \leq t \leq x, x_1 \leq t \leq x + x'$. In view of 4 we have $t = x_1$. Thus $u \leq x_1$. Since $x = x_1 + x_2 = u + v$, then $v - x_2 = x_1 - u \geq 0$.

Because of $x - v \geq 0, f(v) = -v$ we obtain $x - v \in |x - v| = |x' - f(v)| = = |x' + v|$.

Thus $x - v \geq x' + v$. In view of 2 we infer that $2(x_2 - v) \geq 0$. In view of $x_2 - v \leq 0$ we have $x_2 = v$. Then clearly $x_1 = u$.

6. Lemma. Let $x, y \in G^+$ such that $x = x_1 + x_2, y = y_1 + y_2, f(x_1) \geq 0, f(y_2) \leq 0$ where $x_1, x_2, y_1, y_2 \in G^+$.

Then the following conditions are equivalent:
(i) $y \leq x$;
(ii) $x_1 \succeq y_1$ and $x_2 \succeq y_2$.

Proof. The implication (ii) $\Rightarrow$ (i) is obvious. Let $y \leq x$ be valid, and let $x' = f(x), y' = f(y)$.

Because of $x - y = x_1 + x_2 - y_1 - y_2 \geq 0$, from $|x - y| = |x' - y'|$ we obtain
\[ x - y \geq x' - y', \quad x - y \geq y' - x'. \]
Thus $x - x' \geq y - y', x + x' \geq y + y'$. In view of 2 and 5 we have $x + x' \geq 2y_1 \geq y_1, x - x' \geq 2y_2 \geq y_2$.

Clearly $y_1 \leq x, y_2 \leq x$. Since $G$ is a Riesz group, there exist $u, v \in G$ such that $y_1 \leq u \leq x, x' \leq u \leq x + x', -x' \leq v \leq x - x', y_2 \leq v \leq x$. From 4,4' it follows that $x_1 = u, x_2 = v$. Thus $y_1 \leq x_1, y_2 \leq x_2$.

We denote $A_1 = \{x \in G^+; f(x) \geq 0\}, B_1 = \{x \in G^+; f(x) \leq 0\}$.

7. Lemma. The set $A_1$ is closed with respect to the operation $+$.

Proof. Let $x, y \in A_1, x = x_1 + x_2, y = y_1 + y_2$, where $x_1, x_2, y_1, y_2 \in G^+$,
\[ f(x_1) \geq 0, f(x_2) \leq 0, f(y_1) \geq 0, f(y_2) \leq 0. \] Then from 5 we obtain \( x_1 = x, y_1 = y, x_2 = 0, y_2 = 0. \) Using analogous notation for \( x + y \) we infer from 6 that \( x_1 \leq \langle x + y \rangle_1; y_1 \leq \langle x + y \rangle_1 \) is valid.

From the above inequalities and 2 we infer that \( x_1 + y_1 \leq x + y + f(x + y). \) Since \( x + y = x_1 + y_1, \) we obtain \( f(x + y) \geq 0. \)

Analogously we can verify

8. Lemma. The set \( B_1 \) is closed with respect to the operation \(+.\)

9. Lemma. Let \( x, y \in G^+ \) and let the elements \( x_1, x_2, y_1, y_2, (x + y)_1, (x + y)_2 \) be determined according to 1a). Then \( (x + y)_1 = x_1 + y_1, (x + y)_2 = x_2 + y_2. \)

Proof. This is a consequence of 5, 7, 8.

Summarizing, we have

10. Lemma. The partially ordered semigroup \( G^+ \) is a direct product of partially ordered semigroups \( A_1 \) and \( B_1. \)

Put \( A = A_1 - A_1, B = B_1 - B_1. \) Then from 10 and Thm. 2.3 [2] we infer

11. Lemma. The partially ordered group \( G \) is a direct product of partially ordered groups \( A \) and \( B. \)

Remark. For \( g \in G \) we denote by \( g_A \) and \( g_B \) the components of \( g \) in the direct factor \( A \) and \( B, \) respectively. If \( x \in G^+ \) and elements \( x_1, x_2 \) are as in 1a), then according to the definition of \( A_1 \) and \( B_1 \) we have \( x_1 = x_A, x_2 = x_B. \)

The following two lemmas generalize Theorems 2.3 and 2.4 of Rachůnek [7] (in [7] it was assumed that \( G \) is a 2-isolated abelian Riesz group).

12. Lemma. If \( g \) is an isometry in a partially ordered group \( H, a, c \in H, a \leq c, \)

\[ g(a) \leq g(c), \] then \( g([a, c]) = [g(a); g(c)]. \)

Proof. Because of \( c - a \geq 0, g(c) - g(a) \geq 0; \) then from \( |c - a| = |g(c) - g(a)| \)
we obtain \( c - a = g(c) - g(a), \) hence \( -g(c) + c = -g(a) + a. \) Let \( b \in [a, c]. \)

Since \( b - a \geq 0, \) from \( |b - a| = |g(b) - g(a)| \) we get \( -g(b) + b \geq -g(a) + a. \) Thus \( g(c) - g(b) \geq c - b \geq 0, \) hence \( g(c) \geq g(b). \) Because of \( c - b \geq 0, \) from \( |c - b| = |g(c) - g(b)| \) we obtain \( -g(c) + c \geq -g(b) + b. \) Thus \( g(b) - g(a) \geq b - a \geq 0, \) hence \( g(b) \geq g(a). \) We obtain \( g([a, c]) \subseteq [g(a); g(c)]. \) If we consider the isometry \( g^{-1} \) instead of \( g \) we get \( g^{-1}([g(a); g(c)]) \subseteq [a, c]. \) Thus \( [g(a), g(c)] \subseteq g([a, c]). \)

Analogously we can verify

13. Lemma. If \( g \) is an isometry in a partially ordered group \( H, a, c \in H, a \leq c, g(a) \geq g(c), \) then \( g([a, c]) = [g(c), g(a)]. \)

If \( H \) is a partially ordered group, then a quadruple \( \{a, b, u, v\} \) of elements of \( H \)

is said to be elementary if \( u \in L(a, b), v \in U(a, b) \) and \( v - a = b - u. \)
14. Lemma. Let \( \{a, b, u, v\} \) be an elementary quadruple in an abelian partially ordered group \( H \) and let \( g \) be an isometry in \( H \). Assume that \( g(a) \leq g(u), g(a) \leq g(v) \). Then \( \{g(u), g(v), g(a), g(b)\} \) is an elementary quadruple.

Proof. Let \( v_1' = g(v) - g(a) + g(u) \). Then the quadruple \( \{g(u), g(v), g(a), v_1'\} \) must be elementary. Let \( v_1 = g^{-1}(v_1') \). Because of \( u - a = b - v \) we get

\[
|v_1 - v| = |g(v_1) - g(v)| = |g(u) - g(a)| = |u - a| = |b - v| = |v - b|.
\]

Since \( v - b \geq 0 \), we obtain \( v - b \geq v - v_1 \). Thus \( v_1 \geq b \). Analogously we have

\[
|v_1 - u| = |g(v_1) - g(u)| = |g(v) - g(a)| = |v - a| = |b - u|.
\]

Then \( b - u \geq 0 \) implies \( b - u \geq v_1 - u \). Thus \( v_1 \leq b \), hence \( b = v_1 \).

The following assertion can be verified similarly.

15. Lemma. Let \( \{a, b, u, v\} \) be an elementary quadruple in an abelian partially ordered group \( H \) and let \( g \) be an isometry in \( H \). Assume that \( g(b) \geq g(u), g(b) \geq g(v) \). Then \( \{g(u), g(v), g(a), g(b)\} \) is an elementary quadruple.

16. Lemma. For each \( x \in G \) we have \( f(x) = x_A - x_B \).

Proof. Let \( x \in G \). Then there exists \( v \in U(0, x) \). If we put \( u = x - v \), then \( \{0, x, u, v\} \) is an elementary quadruple. Because of \( v \geq 0 \), in view of 1a), 2 there exist elements \( v_1, v_2 \in G^+ \) such that \( v = v_1 + v_2, f(v_1) = v_1, f(v_2) = -v_2, f(v) = v_1 - v_2 \). Since \( u \leq 0 \), it follows from 3 that there exist elements \( u_1, u_2 \in G^- \) such that \( u = u_1 + u_2, f(u_1) = u_1, f(u_2) = -u_2, f(u) = u_1 - u_2 \).

Let \( z' = v_1 - u_2 \). Because of \( z' \geq 0 \), we obtain from 2 and 10 (by considering the isometry \( f^{-1} \)) that \( f^{-1}(z') = v_1 + u_2 \). If we put \( z = f^{-1}(z'), t = v + u_2 \) then \( \{0, z, u_2, v_1\}, \{v_1, t, z, v\} \) are elementary quadruples. Since \( z' = v_1 - u_2, f(v) = v_1 - v_2 \) we have \( z' \geq f(v) \). Because of \( z \leq t \leq v \), 13 implies that

\[
f(v) \leq f(i) \leq f(z) = z'.
\]
Next we put \( t_1 = u + v_1 \). (Cf. Fig. 2.) Then we obtain \( u \leq t_1 \leq z \), \( t_1 \leq x \leq t \). Thus the quadruples \( \{u_2, t_1, u, z\}, \{z, x, t_1, t\} \) are elementary. From \( f(z) = v_1 - u_2 \), \( f(u) = u_1 - u_3 \) it follows that \( f(z) \geq f(u) \). Because of \( u \leq t_1 \leq z \), by using 12 we get \( f(u) \leq f(t_1) \leq f(z) \). Then according to 15 we obtain that \( \{f(t_1), f(t), f(x), f(z)\} \) is an elementary quadruple. Since in each Riesz group, \( U(a) + U(b) = U(a + b) \) holds for each pair \( a, b \) of this group (cf. [1], Chap. V, Thm. 27), we infer

\[
U(f(z) - f(x)) = U(f(z) - f(t) + f(t) - f(x)) = \\
= U(f(z) - f(t)) + U(f(t) - f(x)) = |f(z) - f(t)| + |f(t) - f(x)| = \\
= |z - t| + |t - x| = |t - z| + |t - x| = |v - v_1| + |u_2 - u| = \\
= U(v - v_1) + U(u_2 - u) = U(v - v_1 + u_2 - u).
\]

Thus \( f(z) - f(x) = v - v_1 + u_2 - u \), hence \( f(x) = v_1 - v_2 + u_1 - u_2 \). Clearly \( u_1 = u_A \), \( u_2 = u_B \), \( v_1 = v_A \), \( v_2 = v_B \). Thus \( f(x) = (v_A + u_A) - (v_B + u_B) \). From the relation \( x = u + v = (v_A + u_A) + (v_B + u_B) \) we get \( x_A = v_A + u_A \), \( x_B = v_B + u_B \). Hence \( f(x) = x_A - x_B \).

17. Lemma. Let \( H \) be an abelian partially ordered group and let \( H = P \times Q \) be any direct decomposition of \( H \). For each \( x \in H \) define \( g(x) = x_P - x_Q \). Then \( g \) is an isometry of \( H \) and \( g(0) = 0 \).

Proof. It is easy to verify that \( |z| = |z_P| + |z_Q| \). Let \( x, y \in H \). From the relations \( x - y = (x_P - y_P) + (x_Q - y_Q) \), \( x - y = (x - y)_P - (x - y)_Q \) we obtain \( (x - y)_P = x_P - y_P \), \( (x - y)_Q = x_Q - y_Q \). Then we infer \( g(x - y)_P - (x - y)_Q = x_P - x_Q - (y_P - y_Q) = g(x) - g(y) \). Thus

\[
|g(x) - g(y)| = |g(x - y)| = |(g(x - y))_P| + |(g(x - y))_Q| = \\
= |(x - y)_P| + |(x - y)_Q| = |x - y|.
\]

Clearly \( g(0) = 0 \).

Summarizing, we have

18. Theorem. Let \( G \) be an abelian Riesz group. For each 0-isometry \( f \) in \( G \) there exists a direct decomposition \( G = A \times B \) such that \( f(x) = x_A - x_B \) is valid for each \( x \in G \). Conversely, if \( G = P \times Q \) is a direct decomposition of \( G \) and if we put \( g(x) = x_P - x_Q \) for each \( x \in G \), then \( g \) is a 0-isometry in \( G \).

The notation from Thm. 18 will be adopted also in the whole remaining part of the paper.

19. Lemma. Let \( x, y, a \in G \), \( y \leq a \leq x \). Then the element \( c' = x_A - y_B \) is the smallest element of the set \( U(f(x), f(y)) \) and \( f(a) \in L(U(f(x), f(y)), f^{-1}(c') \in [y, x] \).

Proof. In view of 18 we have \( x = x_A + x_B \), \( y = y_A + y_B \), \( a = a_A + a_B \), \( x_A \geq a_A \geq y_A \), \( -y_A \geq -a_A \geq -x_A \), \( f(x) = x_A - x_B \), \( f(y) = y_A - y_B \), \( f(a) = a_A - a_B \). If we put \( c' = x_A - y_B \), then we obtain \( c' \geq f(x) \), \( c' \geq f(y) \), \( c' \geq f(a) \).
Let $d' \in G$, $d' \in U(f(x), f(y))$. Then we have $d' \geq x_A$, $d' \geq -x_B$, $d' \geq y_A$, $d' \geq -y_B$. Thus $d' \geq c'$. From the relation $f(a) \leq c'$ we get $f(a) \in L(U(f(x), f(y)))$. Since $f^{-1}(c') = x_A + y_B$, the relation $f^{-1}(c') \in [y, x]$ is valid.

Analogously we can prove

**20. Lemma.** Let $a, x, y \in G$, $y \leq a \leq x$. Then $d' = y_A - x_B$ is the greatest element of the set $L(f(x), f(y))$ and $f(a) \in U(L(f(x), f(y)))$, $f^{-1}(d') \in [y, x]$.

**21. Lemma.** Let $x, y \in G$, $y \leq x$. Then $f([y, x]) = [y_A - x_B, x_A - y_B]$.

**Proof.** It follows from 19 and 20 that $f([y, x]) \subseteq [y_A - x_B, x_A - y_B]$. Let $p' \in G$ such that $y_A - x_B \leq p' \leq x_A - y_B$. Then we get $y_A \leq p' \leq x_A$, $-x_B \leq \leq p'_B \leq -y_B$. If we put $p = f^{-1}(p')$, then we have $p = p'_A - p'_B$. Thus $y \leq p \leq x$, hence $[y_A - x_B, x_A - y_B] \subseteq f([y, x])$.

The following result generalizes Theorem 2.2 of Rachůnek [7] (in [7] it was assumed that $G$ is a 2-isolated abelian Riesz group).

**22. Theorem.** If $g$ is an isometry in $G$ and $x, y \in G$, then

$$g(U(L(x, y)) \cap L(U(x, y))) = U(L(g(x), g(y))) \cap L(U(g(x), g(y))).$$

**Proof.** If $g$ is a translation, the assertion obviously holds. Since each isometry is a superposition of a translation and a 0-isometry, it suffices to consider the case when $g$ is a 0-isometry. Let $a \in U(L(x, y)) \cap L(U(x, y))$. Then there exist elements $v \in U(x, y), u \in L(x, y)$ such that $u \leq a \leq v$. In view of 18 and 21 we have $g(a), g(x), g(y) \in [u_A - v_B, v_A - u_B]$. Let $z_i \in U(g(x), g(y)), t'_i \in L(g(x), g(y))$. Then there exist elements $z', t'$ such that $g(x) \leq z' \leq v_A - u_B$, $g(y) \leq z' \leq z'_i$, $g(x) \geq t' \geq u_A - v_B$, $g(y) \geq t' \geq t'_i$. Then we infer $x_A \leq z'_A$, $-x_B \leq z'_B$, $x_A \geq t'_A$, $-x_B \geq t'_B$, $y_A \leq z'_A$, $-y_B \leq z'_B$, $y_A \geq t'_A$, $-y_B \geq t'_B$. If we put $q = z'_A - t'_B, p = t'_A - z'_B$, then we obtain $q \in U(x, y), p \in L(x, y)$, because of $q_A = z'_A, q_B = t'_B, p_A = t'_A, p_B = -z'_B$. Thus $p \leq a \leq q$. In view of 21 we have $g(a) \in [t'_A + t'_B, z'_A + z'_B]$. (Cf. Fig. 3.)
Since \( z'_A + z'_B = z' \leq z'_1 \), \( t'_A + t'_B = t' \geq t'_1 \), we have \( t'_1 \leq g(a) \leq z'_1 \). Hence 
\( g(U(L(x, y)) \cap L(U(x, y))) \subseteq U(L(g(x), g(y))) \cap L(U(g(x), g(y))). \)

Let \( x' = g(x) \) and \( y' = g(y) \). If we consider the 0-isometry \( g^{-1} \) instead of \( g \) then we get, for \( x', y' \), \( g^{-1}(U(L(x', y'))) \cap L(U(x', y')) \subseteq U(L(g^{-1}(x'), g^{-1}(y'))) \cap L(U(g^{-1}(x'), g^{-1}(y'))). \) Hence \( g^{-1}(U(L(g(x), g(y))) \cap L(U(g(x), g(y)))) \subseteq U(L(x, y)) \cap L(U(x, y)). \) Then we obtain \( U(L(g(x), g(y))) \cap L(U(g(x), g(y))) \subseteq g(U(L(x, y)) \cap L(U(x, y))). \)

23. Lemma. Let \( x, y, a \in G \) such that \( f(y) \leq f(a) \leq f(x) \). Then the element \( x_A + y_B \) is the smallest element of the set \( U(x, y) \) and \( a \leq x_A + y_B. \)

Proof. In view of 18 we have \( y_A \leq a_A \leq x_A, y_B \leq -a_B \leq -x_B \). Thus \( a = a_A + a_B \leq x_A + y_B, x_A + y_B \leq x, x_A + y_B \geq y \). Hence \( x_A + y_B \in U(x, y) \). Let \( v \in G, v \in U(x, y) \). Then 18 implies that \( v_A \geq x_A, v_B \geq x_B, v_A \geq y_A, v_B \geq y_B \). Thus \( v = v_A + v_B \geq x_A + y_B. \)

Analogously we can verify

24. Lemma. Let \( a, x, y \in G \) such that \( f(y) \leq f(a) \leq f(x) \). Then \( y_A + x_B \) is the greatest element of the set \( L(x, y) \) and \( a \geq y_A + x_B. \)

25. Lemma. Let \( x, y \in G \) such that \( f(y) \leq f(x) \). Then \( [f(y), f(x)] = f([y_A + x_B, x_A + y_B]). \)

Proof. In view of 23 and 24 we obtain \( [f(y), f(x)] \subseteq f([y_A + x_B, x_A + y_B]). \) Let \( a \in G, a \in [y_A + x_B, x_A + y_B], \) then from 21 we get \( f(a) \in [f(y), f(x)]. \) Thus \( f([y_A + x_B, x_A + y_B]) \subseteq [f(y), f(x)]. \)

26. Lemma. \( H \) is a directed convex subset of \( G \) if and only if \( f(H) \) is a directed convex subset of \( G. \)

Proof. Let \( H \) be a directed convex subset of \( G. \) a) Let \( z' \in G \) such that \( f(y) \leq z' \leq f(x) \) for some \( x, y \in H. \) If we put \( z = f^{-1}(z') \), then in view of 25 we obtain \( y_A + x_B \leq z \leq x_A + y_B. \) Since \( H \) is a convex directed subset of \( G, \) from 23 and 24 we obtain \( y_A + x_B, x_A + y_B \in H. \) Then by the convexity of \( H, z \in H. \) Thus \( z' \in f(H), \) hence \( f(H) \) is a convex subset of \( G. \)

b) Let \( x', y' \in f(H), x = f^{-1}(x'), y = f^{-1}(y'). \) Then there exist elements \( u, v \in H \) such that \( u \in L(x, y), v \in U(x, y). \) Since \( u \leq v_A + u_B \leq v, u \leq u_A + v_B \leq v, \) by the convexity of \( H \) we get \( v_A + u_B, u_A + v_B \in H. \) It follows from 21 that \( f([u, v]) = [f(u_A + v_B), f(v_A + u_B)]. \) Since \( x, y \in [u, v], \) we obtain \( f(u_A + v_B) \in U(f(x), f(y)), f(u_A + v_B) \in L(f(x), f(y)). \) Thus \( f(H) \) is a directed subset of \( G. \)

If we consider the 0-isometry \( f^{-1} \) we can prove the sufficiency of the condition.

27. Proposition. \( H \) is a directed convex subgroup of \( G \) if and only if \( f(H) \) is a directed convex subgroup of \( G. \)

Proof. Let \( H \) be a directed convex subgroup of \( G. \) In view of 26 it suffices to prove that \( f(H) \) is a subgroup of \( G. \) Let \( x', y' \in f(H), x = f^{-1}(x'), y = f^{-1}(y'). \) Then 18
implies that $x' = x_A - x_B$, $y' = y_A - y_B$. Hence we have
\[
x' - y' = (x_A - x_B) - (y_A - y_B) = (x_A - y_A) - (x_B - y_B) = (x - y)_A - (x - y)_B = f(x - y).
\]
Thus $x' - y' \in f(H)$.

If we consider the 0-isometry $f^{-1}$ we can similarly prove the sufficiency of the condition.

The following example shows that the image of a convex subgroup of $G$ under a 0-isometry need not be a convex subgroup of $G$ and also, that the image of a directed subgroup of $G$ under a 0-isometry need not be a directed subgroup.

Example. Let $R$ be the additive group of all real numbers with the natural order and $H = R \times R$. Then the mapping $f: f((x_1, x_2)) = (x_1, -x_2)$ is a 0-isometry in $H$.

The subgroup $H_1 = \{(x, x); x \in R\}$ of $H$ is directed, but $f(H_1)$ is trivially ordered.

The subgroup $H_2 = \{(x, -x), x \in R\}$ of $H$ is convex, but $f(H_2)$ is not a convex subgroup of $H$.

References


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