Milan Jasem

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ISOMETRIES IN RIESZ GROUPS

MILAN JASEM, Košice

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Isometries in the lattice ordered groups have been studied by K. L. Swamy [8], [9] and W. B. Powell [6] for the abelian case and by J. Jakubík in [3], [4] for the general case. Isometries in the 2-isolated abelian Riesz groups have been investigated by J. Rachůnek [7].

In this paper isometries in abelian Riesz groups are studied and some of Rachůnek's results on isometries from [7] are generalised. It is also shown that the results on the relations between isometries and direct decompositions of lattice ordered groups [3], which J. Jakubík and M. Kolibiar extended to abelian distributive multilattice groups [5], can be also extended to abelian Riesz groups. Note that a Riesz group need not be a multilattice group and conversely, a multilattice group need not be a Riesz group.

First we recall some notions and notations used in the paper.

Let G be a partially ordered group. The group operation will be written additively. We denote $G^+ = \{x \in G;\ x \geqq 0\}$, $G^- = \{x \in G;\ x \leqq 0\}$. If $a_1, \ldots, a_n$ are elements of G, then we denote by $U(a_1, \ldots, a_n)$ and $L(a_1, \ldots, a_n)$ the set of all upper bounds and the set of all lower bounds of the set $\{a_1, \ldots, a_n\}$, respectively. For each $a \in G$, $|a| = U(a, -a)$.

The following notion of isometry in partially ordered groups was introduced by J. Rachůnek [7].

If G is a partially ordered group, then a bijection $f\colon G \to G$ is called an *isometry in* G if $|a - b| = |f(a) - f(b)|$ for each $a, b \in G$. An isometry f in an ordered group G is called a *0-isometry* if $f(0) = 0$.

A Riesz group is any partially ordered group which is directed and satisfies the Riesz interpolation property, i.e., for each $a_i, b_j \in G$ $(i, j = 1, 2)$ such that $a_i \leqq b_j$ $(i, j = 1, 2)$ there exists $c \in G$ such that $a_i \leqq c \leqq b_j$ $(i, j = 1, 2)$. See [1].

Throughout the paper we assume that G is an abelian Riesz group and f is a 0-isometry in G.

1. Lemma. a) *If* $x \in G^+$, *then there exist* $x_1, x_2 \in G^+$ *such that* $x = x_1 + x_2$, $f(x_1) \geqq 0$, $f(x_2) \leqq 0$, $f(x) \leqq x_1 \leqq x + f(x)$.

b) *If* $x \in G^+$, $t \in G$, $t \in [0; x] \cap [f(x); x + f(x)]$, *then* $x + f(x) = 2t$.

Proof. If $x \in G^+$, $x' = f(x)$, then $U(x) = |x| = |x'|$. Thus $x \geqq x'$, $x \geqq -x'$, hence $x + x' \geqq 0$. Because of $x \geqq 0$, $x + x' \geqq x'$. Since G is a Riesz group, there exists b' in G such that

$$0 \leqq b' \leqq x\,, \quad x' \leqq b' \leqq x + x'\,.$$

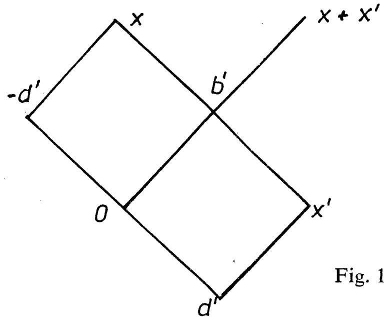

Fig. 1

(Cf. Fig. 1.) Let $b = f^{-1}(b')$. From $b' \geqq 0$, $x \geqq b'$ we get $x \in U(b') = |b'| = |b|$. Thus $x \geqq b$. Because of $x - b \geqq 0$, $x' - b' \leqq 0$, from $|x - b| = |x' - b'|$ it follows that $x - b = b' - x'$. Let $d' = x' - b'$, then $d' \leqq 0$, $d' \leqq x'$, $-d' = x - b$. Denote $d = f^{-1}(d')$. Then we obtain $x \geqq x - d$, since

$$x \in |b'| = |x' - d'| = |x - d|\,.$$

Hence $d \geqq 0$. From $|d'| = |d|$ we get $d = -d' = x - b$. Thus $x = b + d$. Because of $x + x' \geqq b'$, $b' \geqq 0$ we get $x + x' \in U(b') = |b'| = |x' - d'|$. Thus $x \geqq -d' = x - b$, hence $b \geqq 0$.

From the relations $b \geqq 0, f(b) \geqq 0$ and $|b| = |f(b)|$ we obtain $f(b) = b$. If we put $x_1 = b$ and $x_2 = d$ we obtain the required elements. We have proved that $f(x_1) = x_1$ and also $f(x_2) = -x_2$. Thus $x' = b' + d' = b - d = x_1 - x_2$ and clearly $x + x' = 2x_1$, $x - x' = 2x_2$.

It is clear that for each $t \in G$ such that $t \in [0, x] \cap [f(x), x + f(x)]$ the relation $x + f(x) = 2t$ is valid.

Hence the following assertion is valid.

2. Lemma. *Let x, x_1, x_2 be as in Lemma* 1a) *and let $x' = f(x)$. Then $f(x_1) = x_1$, $f(x_2) = -x_2$, $x' = x_1 - x_2$, $x + x' = 2x_1$, $x - x' = 2x_2$, $x \geqq x'$.*

The following assertion can be verified analogously:

3. Lemma. *If $x \in G^-$, then there exist elements $x_1, x_2 \in G^-$ such that $x = x_1 + x_2$, $f(x_1) = x_1$, $f(x_2) = -x_2$.*

4. Lemma. *Let x, x_1, x_2 be as in* 1a) *and $x' = f(x)$. If $0 \leqq y \leqq x$, $x' \leqq y \leqq x + x'$ holds for some $y \in G$, then $y = x_1$.*

Proof. Let $y \in G$ such that $0 \leqq y \leqq x$, $x' \leqq y \leqq x + x'$. Since $x_1 \leqq x$, $x_1 \leqq$

$\leqq x + x'$, there exists $y_1 \in G$ such that

$$y \leqq y_1 \leqq x, \quad x_1 \leqq y_1 \leqq x + x'.$$

From Lemma 1b) and Lemma 2 we obtain $x + x' = 2y$, $x + x' = 2y_1$, $x + x' = 2x_1$. Thus we get $2(y_1 - y) = 0$; $2(y_1 - x_1) = 0$. Since $y_1 - y \geqq 0$, $y_1 - x_1 \geqq 0$, we have $y = y_1 = x_1$.

4′. Lemma. *Let* x, x_1, x_2 *be as in* 1a) *and let* $x' = f(x)$. *If* $0 \leqq y \leqq x$, $-x' \leqq y \leqq x - x'$ *hold for some* $y \in G$, *then* $y = x_2$.

Proof. From the assumptions we have $x' \leqq y + x' \leqq x + x'$, $0 \leqq y + x' \leqq x$. In view of 4 we obtain $y + x' = x_1$. Then 2 implies that $y = x_2$.

5. Lemma. *Let* $x \in G^+$, $x = u + v$, $u, v \in G^+$, $f(u) \geqq 0$, $f(v) \leqq 0$ *and let* x_1, x_2 *be as in* 1a). *Then* $u = x_1$, $v = x_2$.

Proof. Clearly $f(u) = u$, $f(v) = -v$. Let $x' = f(x)$. Because of $x - u \geqq 0$, from $|x - u| = |f(x) - f(u)| = |x' - u|$ we infer that $x - u \geqq -x' + u$. Since $2u \geqq u$ we obtain $x + x' \geqq u$. Thus $u \leqq x$, $u \leqq x + x'$, $x_1 \leqq x$, $x_1 \leqq x + x'$. Then there exists an element $t \in G$ such that $u \leqq t \leqq x$, $x_1 \leqq t \leqq x + x'$. In view of 4 we have $t = x_1$. Thus $u \leqq x_1$. Since $x = x_1 + x_2 = u + v$, then $v - x_2 = x_1 - u \geqq 0$. Because of $x - v \geqq 0$, $f(v) = -v$ we obtain $x - v \in |x - v| = |x' - f(v)| = |x' + v|$.

Thus $x - v \geqq x' + v$. In view of 2 we infer that $2(x_2 - v) \geqq 0$. In view of $x_2 - v \leqq 0$ we have $x_2 = v$. Then clearly $x_1 = u$.

6. Lemma. *Let* $x, y \in G^+$ *such that* $x = x_1 + x_2$, $y = y_1 + y_2$, $f(x_1) \geqq 0$, $f(x_2) \leqq 0$, $f(y_1) \geqq 0$, $f(y_2) \leqq 0$ where $x_1, x_2, y_1, y_2 \in G^+$.

Then the following conditions are equivalent:

(i) $y \leqq x$;

(ii) $x_1 \geqq y_1$ *and* $x_2 \geqq y_2$.

Proof. The implication (ii) $\Rightarrow$ (i) is obvious. Let $y \leqq x$ be valid, and let $x' = f(x)$, $y' = f(y)$.

Because of $x - y = x_1 + x_2 - y_1 - y_2 \geqq 0$, from $|x - y| = |x' - y'|$ we obtain

$$x - y \geqq x' - y', \quad x - y \geqq y' - x'.$$

Thus $x - x' \geqq y - y'$, $x + x' \geqq y + y'$. In view of 2 and 5 we have $x + x' \geqq 2y_1 \geqq y_1$, $x - x' \geqq 2y_2 \geqq y_2$.

Clearly $y_1 \leqq x$, $y_2 \leqq x$. Since G is a Riesz group, there exist $u, v \in G$ such that $y_1 \leqq u \leqq x$, $x' \leqq u \leqq x + x'$, $-x' \leqq v \leqq x - x'$, $y_2 \leqq v \leqq x$. From 4,4′ it follows that $x_1 = u$, $x_2 = v$. Thus $y_1 \leqq x_1$, $y_2 \leqq x_2$.

We denote $A_1 = \{x \in G^+; f(x) \geqq 0\}$, $B_1 = \{x \in G^+; f(x) \leqq 0\}$.

7. Lemma. *The set* A_1 *is closed with respect to the operation* $+$.

Proof. Let $x, y \in A_1$, $x = x_1 + x_2$, $y = y_1 + y_2$, where $x_1, x_2, y_1, y_2 \in G^+$,

$f(x_1) \geqq 0, f(x_2) \leqq 0, f(y_1) \geqq 0, f(y_2) \leqq 0$. Then from 5 we obtain $x_1 = x$, $y_1 = y$, $x_2 = 0$, $y_2 = 0$. Using analogous notation for $x + y$ we infer from 6 that $x_1 \leqq \leqq (x + y)_1$; $y_1 \leqq (x + y)_1$ is valid.

From the above inequalities and 2 we infer that $x_1 + y_1 \leqq x + y + f(x + y)$. Since $x + y = x_1 + y_1$, we obtain $f(x + y) \geqq 0$.

Analogously we can verify

8. Lemma. *The set B_1 is closed with respect to the operation $+$.*

9. Lemma. *Let $x, y \in G^+$ and let the elements $x_1, x_2, y_1, y_2, (x + y)_1, (x + y)_2$ be determined according to* 1a). *Then $(x + y)_1 = x_1 + y_1$, $(x + y)_2 = x_2 + y_2$.*

Proof. This is a consequence of 5, 7, 8.

Summarizing, we have

10. Lemma. *The partially ordered semigroup G^+ is a direct product of partially ordered semigroups A_1 and B_1.*

Put $A = A_1 - A_1$, $B = B_1 - B_1$. Then from 10 and Thm. 2.3 [2] we infer

11. Lemma. *The partially ordered group G is a direct product of partially ordered groups A and B.*

Remark. For $g \in G$ we denote by g_A and g_B the components of g in the direct factor A and B, respectively. If $x \in G^+$ and elements x_1, x_2 are as in 1a), then according to the definition of A_1 and B_1 we have $x_1 = x_A$, $x_2 = x_B$.

The following two lemmas generalize Theorems 2.3 and 2.4 of Rachůnek [7] (in [7] it was assumed that G is a 2-isolated abelian Riesz group).

12. Lemma. *If g is an isometry in a partially ordered group H, $a, c \in H$, $a \leqq c$, $g(a) \leqq g(c)$, then $g([a, c]) = [g(a); g(c)]$.*

Proof. Because of $c - a \geqq 0$, $g(c) - g(a) \geqq 0$; then from $|c - a| = |g(c) - g(a)|$ we obtain $c - a = g(c) - g(a)$, hence $-g(c) + c = -g(a) + a$. Let $b \in [a, c]$. Since $b - a \geqq 0$, from $|b - a| = |g(b) - g(a)|$ we get $-g(b) + b \geqq -g(a) + a$. Thus $g(c) - g(b) \geqq c - b \geqq 0$, hence $g(c) \geqq g(b)$. Because of $c - b \geqq 0$, from $|c - b| = |g(c) - g(b)|$ we obtain $-g(c) + c \geqq -g(b) + b$. Thus $g(b) - g(a) \geqq \geqq b - a \geqq 0$, hence $g(b) \geqq g(a)$. We obtain $g([a, c]) \subseteq [g(a); g(c)]$. If we consider the isometry g^{-1} instead of g we get $g^{-1}[g(a), g(c)] \subseteq [a, c]$. Thus $[g(a), g(c)] \subseteq \subseteq g([a, c])$.

Analogously we can verify

13. Lemma. *If g is an isometry in a partially ordered group H, $a, c \in H$, $a \leqq c$, $g(a) \geqq g(c)$, then $g([a, c]) = [g(c), g(a)]$.*

If H is a partially ordered group, then a quadruple $\{a, b, u, v\}$ of elements of H is said to be *elementary* if $u \in L(a, b)$, $v \in U(a, b)$ and $v - a = b - u$.

14. Lemma. *Let $\{a, b, u, v\}$ be an elementary quadruple in an abelian partially ordered group H and let g be an isometry in H. Assume that $g(a) \leqq g(u)$, $g(a) \leqq \leqq g(v)$. Then $\{g(u), g(v), g(a), g(b)\}$ is an elementary quadruple.*

Proof. Let $v_1' = g(v) - g(a) + g(u)$. Then the quadruple $\{g(u), g(v), g(a), v_1'\}$ must be elementary. Let $v_1 = g^{-1}(v_1')$. Because of $u - a = b - v$ we get

$$|v_1 - v| = |g(v_1) - g(v)| = |g(u) - g(a)| = |u - a| = |b - v| = |v - b| .$$

Since $v - b \geqq 0$, we obtain $v - b \geqq v - v_1$. Thus $v_1 \geqq b$. Analogously we have

$$|v_1 - u| = |g(v_1) - g(u)| = |g(v) - g(a)| = |v - a| = |b - u| .$$

Then $b - u \geqq 0$ implies $b - u \geqq v_1 - u$. Thus $v_1 \leqq b$, hence $b = v_1$.

The following assertion can be verified similarly.

15. Lemma. *Let $\{a, b, u, v\}$ be an elementary quadruple in an abelian partially ordered group H and let g be an isometry in H. Assume that $g(b) \geqq g(u)$, $g(b) \geqq \geqq g(v)$. Then $\{g(u), g(v), g(a), g(b)\}$ is an elementary quadruple.*

16. Lemma. *For each $x \in G$ we have $f(x) = x_A - x_B$.*

Proof. Let $x \in G$. Then there exists $v \in U(0, x)$. If we put $u = x - v$, then $\{0, x, u, v\}$ is an elementary quadruple. Because of $v \geqq 0$, in view of 1a), 2 there exist elements $v_1, v_2 \in G^+$ such that $v = v_1 + v_2$, $f(v_1) = v_1$, $f(v_2) = -v_2$, $f(v) = = v_1 - v_2$. Since $u \leqq 0$, it follows from 3 that there exist elements $u_1, u_2 \in G^-$ such that $u = u_1 + u_2$, $f(u_1) = u_1$, $f(u_2) = -u_2$, $f(u) = u_1 - u_2$.

Let $z' = v_1 - u_2$. Because of $z' \geqq 0$, we obtain from 2 and 10 (by considering the isometry f^{-1}) that $f^{-1}(z') = v_1 + u_2$. If we put $z = f^{-1}(z')$, $t = v + u_2$ then $\{0, z, u_2, v_1\}$, $\{v_1, t, z, v\}$ are elementary quadruples. Since $z' = v_1 - u_2$, $f(v) = = v_1 - v_2$ we have $z' \geqq f(v)$. Because of $z \leqq t \leqq v$, 13 implies that

$$f(v) \leqq f(t) \leqq f(z) = z' .$$

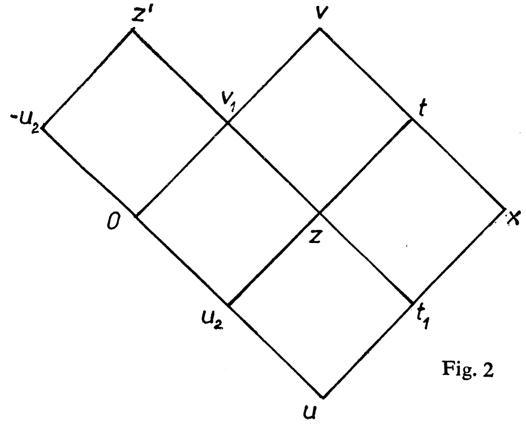

Fig. 2

Next we put $t_1 = u + v_1$. (Cf. Fig. 2.) Then we obtain $u \leqq t_1 \leqq z$, $t_1 \leqq x \leqq t$. Thus the quadruples $\{u_2, t_1, u, z\}$, $\{z, x, t_1, t\}$ are elementary. From $f(z) = v_1 - u_2$, $f(u) = u_1 - u_2$ it follows that $f(z) \geqq f(u)$. Because of $u \leqq t_1 \leqq z$, by using 12 we get $f(u) \leqq f(t_1) \leqq f(z)$. Then according to 15 we obtain that $\{f(t_1), f(t), f(x), f(z)\}$ is an elementary quadruple. Since in each Riesz group, $U(a) + U(b) = U(a + b)$ holds for each pair a, b of this group (cf. [1], Chap. V, Thm. 27), we infer

$$U(f(z) - f(x)) = U(f(z) - f(t) + f(t) - f(x)) =$$
$$= U(f(z) - f(t)) + U(f(t) - f(x)) = |f(z) - f(t)| + |f(t) - f(x)| =$$
$$= |z - t| + |t - x| = |t - z| + |t - x| = |v - v_1| + |u_2 - u| =$$
$$= U(v - v_1) + U(u_2 - u) = U(v - v_1 + u_2 - u).$$

Thus $f(z) - f(x) = v - v_1 + u_2 - u$, hence $f(x) = v_1 - v_2 + u_1 - u_2$. Clearly $u_1 = u_A$, $u_2 = u_B$, $v_1 = v_A$, $v_2 = v_B$. Thus $f(x) = (v_A + u_A) - (v_B + u_B)$. From the relation $x = u + v = (v_A + u_A) + (v_B + u_B)$ we get $x_A = v_A + u_A$, $x_B = v_B + u_B$. Hence $f(x) = x_A - x_B$.

17. Lemma. *Let H be an abelian partially ordered group and let $H = P \times Q$ be any direct decomposition of H. For each $x \in H$ define $g(x) = x_P - x_Q$. Then g is an isometry of H and $g(0) = 0$.*

Proof. It is easy to verify that $|z| = |z_P| + |z_Q|$. Let $x, y \in H$. From the relations $x - y = (x_P - y_P) + (x_Q - y_Q)$, $x - y = (x - y)_P - (x - y)_Q$ we obtain $(x - y)_P = x_P - y_P$, $(x - y)_Q = x_Q - y_Q$. Then we infer $g(x - y) = (x - y)_P - (x - y)_Q = x_P - y_P - (x_Q - y_Q) = (x_P - x_Q) - (y_P - y_Q) = g(x) - g(y)$. Thus

$$|g(x) - g(y)| = |g(x - y)| = |(g(x - y))_P| + |(g(x - y))_Q| =$$
$$= |(x - y)_P| + |-(x - y)_Q| = |(x - y)_P| + |(x - y)_Q| = |x - y|.$$

Clearly $g(0) = 0$.

Summarizing, we have

18. Theorem. *Let G be an abelian Riesz group. For each 0-isometry f in G there exists a direct decomposition $G = A \times B$ such that $f(x) = x_A - x_B$ is valid for each $x \in G$. Conversely, if $G = P \times Q$ is a direct decomposition of G and if we put $g(x) = x_P - x_Q$ for each $x \in G$, then g is a 0-isometry in G.*

The notation from Thm. 18 will be adopted also in the whole remaining part of the paper.

19. Lemma. *Let $x, y, a \in G$, $y \leqq a \leqq x$. Then the element $c' = x_A - y_B$ is the smallest element of the set $U(f(x), f(y))$ and $f(a) \in L(U(f(x), f(y)))$, $f^{-1}(c') \in [y, x]$.*

Proof. In view of 18 we have $x = x_A + x_B$, $y = y_A + y_B$, $a = a_A + a_B$, $x_A \geqq a_A \geqq y_A$, $-y_B \geqq -a_B \geqq -x_B$, $f(x) = x_A - x_B$, $f(y) = y_A - y_B$, $f(a) = a_A - a_B$. If we put $c' = x_A - y_B$, then we obtain $c' \geqq f(x)$, $c' \geqq f(y)$, $c' \geqq f(a)$.

Let $d' \in G$, $d' \in U(f(x), f(y))$. Then we have $d'_A \geqq x_A$, $d'_B \geqq -x_B$, $d'_A \geqq y_A$, $d'_B \geqq -y_B$. Thus $d' \geqq c'$. From the relation $f(a) \leqq c'$ we get $f(a) \in L(U(f(x), f(y)))$. Since $f^{-1}(c') = x_A + y_B$, the relation $f^{-1}(c') \in [y, x]$ is valid.

Analogously we can prove

20. Lemma. *Let $a, x, y \in G$, $y \leqq a \leqq x$. Then $d' = y_A - x_B$ is the greatest element of the set $L(f(x), f(y))$ and $f(a) \in U(L(f(x), f(y)))$, $f^{-1}(d') \in [y, x]$.*

21. Lemma. *Let $x, y \in G$, $y \leqq x$. Then $f([y, x]) = [y_A - x_B, x_A - y_B]$.*

Proof. It follows from 19 and 20 that $f([y, x]) \subseteq [y_A - x_B, x_A - y_B]$. Let $p' \in G$ such that $y_A - x_B \leqq p' \leqq x_A - y_B$. Then we get $y_A \leqq p'_A \leqq x_A$, $-x_B \leqq p'_B \leqq -y_B$. If we put $p = f^{-1}(p')$, then we have $p = p'_A - p'_B$. Thus $y \leqq p \leqq x$, hence $[y_A - x_B, x_A - y_B] \subseteq f([y, x])$.

The following result generalizes Theorem 2.2 of Rachůnek [7] (in [7] it was assumed that G is a 2-isolated abelian Riesz group).

22. Theorem. *If g is an isometry in G and $x, y \in G$, then*

$$g(U(L(x, y)) \cap L(U(x, y))) = U(L(g(x), g(y))) \cap L(U(g(x), g(y))) .$$

Proof. If g is a translation, the assertion obviously holds. Since each isometry is a superposition of a translation and a 0-isometry, it suffices to consider the case when g is a 0-isometry. Let $a \in U(L(x, y)) \cap L(U(x, y))$. Then there exist elements $v \in U(x, y)$, $u \in L(x, y)$ such that $u \leqq a \leqq v$. In view of 18 and 21 we have $g(a)$, $g(x), g(y) \in [u_A - v_B, v_A - u_B]$. Let $z'_1 \in U(g(x), g(y))$, $t'_1 \in L(g(x), g(y))$. Then there

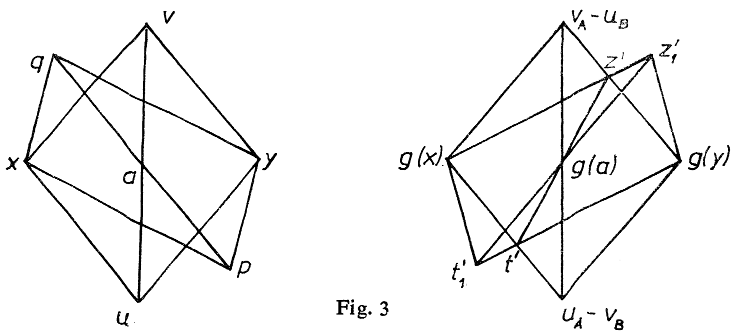

Fig. 3

exist elements z', t' such that $g(x) \leqq z' \leqq v_A - u_B$, $g(y) \leqq z' \leqq z'_1$, $g(x) \geqq t' \geqq u_A - v_B$, $g(y) \geqq t' \geqq t'_1$. Then we infer $x_A \leqq z'_A$, $-x_B \leqq z'_B$, $x_A \geqq t'_A$, $-x_B \geqq t'_B$, $y_A \leqq z'_A$, $-y_B \leqq z'_B$, $y_A \geqq t'_A$, $-y_B \geqq t'_B$. If we put $q = z'_A - t'_B$, $p = t'_A - z'_B$ then we obtain $q \in U(x, y)$, $p \in L(x, y)$, because of $q_A = z'_A$, $q_B = -t'_B$, $p_A = t'_A$, $p_B = -z'_B$. Thus $p \leqq a \leqq q$. In view of 21 we have $g(a) \in [t'_A + t'_B, z'_A + z'_B]$. (Cf. Fig. 3.)

Since $z'_A + z'_B = z' \leqq z'_1$, $t'_A + t'_B = t' \geqq t'_1$, we have $t'_1 \leqq g(a) \leqq z'_1$. Hence $g(U(L(x, y)) \cap L(U(x, y))) \subseteq U(L(g(x), g(y))) \cap L(U(g(x), g(y)))$.

Let $x' = g(x)$ and $y' = g(y)$. If we consider the 0-isometry g^{-1} instead of g then we get, for x', y', $g^{-1}(U(L(x', y')) \cap L(U(x', y'))) \subseteq U(L(g^{-1}(x'), g^{-1}(y'))) \cap \cap L(U(g^{-1}(x'), g^{-1}(y')))$. Hence $g^{-1}(U(L(g(x), g(y))) \cap L(U(g(x), g(y)))) \subseteq \subseteq U(L(x, y)) \cap L(U(x, y))$. Then we obtain $U(L(g(x), g(y))) \cap L(U(g(x), g(y))) \subseteq \subseteq g(U(L(x, y)) \cap L(U(x, y)))$.

23. Lemma. *Let $x, y, a \in G$ such that $f(y) \leqq f(a) \leqq f(x)$. Then the element $x_A + y_B$ is the smallest element of the set $U(x, y)$ and $a \leqq x_A + y_B$.*

Proof. In view of 18 we have $y_A \leqq a_A \leqq x_A$, $-y_B \leqq -a_B \leqq -x_B$. Thus $a = = a_A + a_B \leqq x_A + y_B$, $x_A + y_B \geqq x$, $x_A + y_B \geqq y$. Hence $x_A + y_B \in U(x, y)$. Let $v \in G$, $v \in U(x, y)$. Then 18 implies that $v_A \geqq x_A$, $v_B \geqq x_B$, $v_A \geqq y_A$, $v_B \geqq y_B$. Thus $v = v_A + v_B \geqq x_A + y_B$.

Analogously we can verify

24. Lemma. *Let $a, x, y \in G$ such that $f(y) \leqq f(a) \leqq f(x)$. Then $y_A + x_B$ is the greatest element of the set $L(x, y)$ and $a \geqq y_A + x_B$.*

25. Lemma. *Let $x, y \in G$ such that $f(y) \leqq f(x)$. Then $[f(y), f(x)] = f([y_A + x_B, x_A + y_B])$.*

Proof. In view of 23 and 24 we obtain $[f(y), f(x)] \subseteq f([y_A + x_B, x_A + y_B])$. Let $a \in G$, $a \in [y_A + x_B, x_A + y_B]$, then from 21 we get $f(a) \in [f(y), f(x)]$. Thus $f([y_A + x_B, x_A + y_B]) \subseteq [f(y), f(x)]$.

26. Lemma. *H is a directed convex subset of G if and only if $f(H)$ is a directed convex subset of G.*

Proof. Let H be a directed convex subset of G. a) Let $z' \in G$ such that $f(y) \leqq \leqq z' \leqq f(x)$ for some $x, y \in H$. If we put $z = f^{-1}(z')$, then in view of 25 we obtain $y_A + x_B \leqq z \leqq x_A + y_B$. Since H is a convex directed subset of G, from 23 and 24 we obtain $y_A + x_B$, $x_A + y_B \in H$. Then by the convexity of H, $z \in H$. Thus $z' \in f(H)$, hence $f(H)$ is a convex subset of G.
b) Let $x', y' \in f(H)$, $x = f^{-1}(x')$, $y = f^{-1}(y')$. Then there exist elements $u, v \in H$ such that $u \in L(x, y)$, $v \in U(x, y)$. Since $u \leqq v_A + u_B \leqq v$, $u \leqq u_A + v_B \leqq v$, by the convexity of H we get $v_A + u_B$, $u_A + v_B \in H$. It follows from 21 that $f([u, v]) = = [f(u_A + v_B), f(v_A + u_B)]$. Since $x, y \in [u, v]$, we obtain $f(v_A + u_B) \in U(f(x), f(y))$, $f(u_A + v_B) \in L(f(x), f(y))$. Thus $f(H)$ is a directed subset of G.

If we consider the 0-isometry f^{-1} we can prove the sufficiency of the condition.

27. Proposition. *H is a directed convex subgroup of G if and only if $f(H)$ is a directed convex subgroup of G.*

Proof. Let H be a directed convex subgroup of G. In view of 26 it suffices to prove that $f(H)$ is a subgroup of G. Let $x', y' \in f(H)$, $x = f^{-1}(x')$, $y = f^{-1}(y')$. Then 18 implies that $x' = x_A - x_B$, $y' = y_A - y_B$. Hence we have

$$x' - y' = (x_A - x_B) - (y_A - y_B) = (x_A - y_A) - (x_B - y_B) =$$
$$= (x - y)_A - (x - y)_B = f(x - y).$$

Thus $x' - y' \in f(H)$.

If we consider the 0-isometry f^{-1} we can similarly prove the sufficiency of the condition.

The following example shows that the image of a convex subgroup of G under a 0-isometry need not be a convex subgroup of G and also, that the image of a directed subgroup of G under a 0-isometry need not be a directed subgroup.

Example. Let R be the additive group of all real numbers with the natural order and $H = R \times R$. Then the mapping $f: f((x_1, x_2)) = (x_1, -x_2)$ is a 0-isometry in H.

The subgroup $H_1 = \{(x, x); x \in R\}$ of H is directed, but $f(H_1)$ is trivially ordered.

The subgroup $H_2 = \{(x, -x), x \in R\}$ of H is convex, but $f(H_2)$ is not a convex subgroup of H.

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Author's address: 040 01 Košice, Švermova 9, Czechoslovakia (Katedra matematiky VŠT).