Sven Heinrich
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THE UNIFORM CLASSIFICATION OF BOUNDEDLY COMPACT
LOCALLY CONVEX SPACES

S. HEINRICH, Berlin

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Recently, Mankiewicz and Vilimovský [10] accomplished the uniform classifica­
tion of Montel-Fréchet spaces by showing that a locally convex space is uniformly
homeomorphic to a Montel-Fréchet space iff it is isomorphic to it. This together with
Enflo’s result [6] that a space, uniformly homeomorphic to a Hilbert space, is iso­
morphic to it, constitutes the only cases known till now where uniform equivalence of
a given space implies linear-topological equivalence. That this implication does not
hold for all locally convex spaces is demonstrated by the famous example due to
Aharoni and Lindenstrauss [1] of two uniformly (even Lipschitz) homeomorphic
Banach spaces which are not isomorphic. For further information on the Banach
space case we refer to [7] and the bibliography therein.

In this paper we enlarge the range of locally convex spaces for which the above
implication holds. We extend the result of Mankiewicz and Vilimovský to the class
of all boundedly compact locally convex spaces, generalizing in this way also the
results from [9]. This enables us to give the classification of all Banach spaces with
respect to weak-star uniform equivalence of their dual spaces: Two Banach spaces
are isomorphic if and only if their dual spaces are uniformly homeomorphic in their
weak-star topologies.

We consider only vector spaces over the field of reals. A topological vector space $E$
is called boundedly compact if each bounded subset of $E$ is relatively compact [5].
Given a locally convex space $E$, we denote by $U(E)$ the set of absolutely convex
closed neighbourhoods of zero, by $\Psi(E)$ the set of all continuous seminorms on $E$,
and by $\mathcal{B}(E)$ the set of all absolutely convex closed bounded sets. Isomorphic means
always linearly isomorphic. A mapping $f: E \to F$ is called Lipschitz if for each
$q \in \Psi(F)$ there is a $p \in \Psi(E)$ and a constant $K$ such that

$$q(f(x_1) - f(x_2)) \leq K p(x_1 - x_2)$$

for all $x_1, x_2 \in E$.

The spaces $E$ and $F$ are called uniformly homeomorphic (Lipschitz homeomorphic)
if there is a one-to-one mapping $f$ of $E$ onto $F$ such that both $f$ and $f^{-1}$ are uniformly
continuous (Lipschitz, respectively).

It is convenient for our purposes to introduce the set of Lipschitz constants of $f$

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in the following way. The Lipschitz constant \( \mathcal{H}(f) \) denotes the function \( K_f: \Psi(E) \times \Psi(F) \to R^+ \cup \{ +\infty \} \), where \( K_f(p, q) \) is the smallest constant \( K \) such that (1) holds, if there is such \( K \) at all. Otherwise we set \( K_f(p, q) = +\infty \). The Lipschitz homeomorphism constant \( \mathcal{H}(f) \) stands for the pair \( (\mathcal{H}(f), \mathcal{H}(f^{-1})) \). Finally, we write \( \mathcal{H}(f) \leq \mathcal{H}(g) \) if \( K_f(p, q) \leq K_g(p, q) \) for all \( p \) and \( q \), and \( \mathcal{H}(f) \leq \mathcal{H}(g) \) if \( \mathcal{H}(f) \leq \mathcal{H}(g) \) and \( \mathcal{H}(f^{-1}) \leq \mathcal{H}(g^{-1}) \).

The following lemma was shown in [10] (compare also the non-compact version in [7]):

**Lemma 1.** Let \( E \) and \( F \) be boundedly compact locally convex spaces. If \( E \) and \( F \) are uniformly homeomorphic, then they are Lipschitz homeomorphic.

The following result constitutes the basis of our compactness arguments:

**Lemma 2.** Suppose \( E \) and \( F \) are boundedly compact locally convex spaces. Let \( f \) and \( f_a (a \in A) \) be Lipschitz homeomorphisms from \( E \) to \( F \) with \( f(0) = f_a(0) = 0 \) and \( \mathcal{H}(f_a) \leq \mathcal{H}(f) \). Let \( \mathcal{O} \) be an ultrafilter on \( A \), and set \( g(x) = \lim_{a \to \mathcal{O}} f_a(x) \) \( (x \in E) \).

Then \( g \) is a well-defined Lipschitz homeomorphism from \( E \) to \( F \) with \( \mathcal{H}(g) \leq \mathcal{H}(f) \).

**Proof.** By assumption, we have \( q(f_a(x)) \leq K_f(p, q) p(x) \), where for each \( q \in \Psi(F) \) there is a \( p \in \Psi(E) \) such that \( K_f(p, q) < \infty \). This shows that for a fixed \( x \in E \), \( \{ f_a(x): a \in A \} \) is a bounded subset of \( F \). Therefore \( g \) is correctly defined. As a limit of mappings with uniformly bounded Lipschitz constants, \( g \) is clearly Lipschitz and \( \mathcal{H}(g) \leq \mathcal{H}(f) \). Now define

\[
    h(y) = \lim_{a \to \mathcal{O}} f_a^{-1}(y) \quad \text{for} \quad y \in F.
\]

Then, as above, \( \mathcal{H}(h) \leq \mathcal{H}(f^{-1}) \), and we will show that \( h \circ g = \text{id}_E \). So, given \( p \in \Psi(E) \) and \( x \in E \), we have

\[
    p(h(g(x)) - x) \leq p(h(g(x)) - f_a^{-1}(g(x))) + p(f_a^{-1}(g(x)) - f_a^{-1}(f_a(x))).
\]

The first term on the right-hand side converges to zero as \( a \) passes along \( \mathcal{O} \), by the definition of \( h \). The second term can be estimated as follows:

\[
    p(f_a^{-1}(g(x)) - f_a^{-1}(f_a(x))) \leq K_{f_a^{-1}}(q, p) q(g(x) - f_a(x)).
\]

The right-hand side of this inequality clearly also tends to zero, showing that \( h \circ g = \text{id}_E \). Similarly, \( g \circ h = \text{id}_F \), so \( g \) is a Lipschitz homeomorphism with the constants required above.

Given a locally convex space \( E \) and a subspace \( X \), we denote by \( J_X \) the embedding of \( X \) into \( E \). For \( U \in \U(E) \), \( E_U \) denotes the Banach space canonically associated with \( U \), namely the completion of \( E/N_U \) under the gauge norm of \( U \), where \( N_U = \{ x \in E: x \in \lambda U \text{ for all } \lambda > 0 \} \). \( Q_U: E \to E_U \) stands for the corresponding quotient map.

The following lemma is crucial for the proof of Theorem. It shows that we can linearize \( f \) "locally" without changing its homeomorphism properties.
Lemma 3. Assume that $E$ and $F$ are boundedly compact locally convex spaces and $f: E \to F$ is a Lipschitz homeomorphism. Let $X \subset E$ be a finite dimensional subspace and let $V \in \mathfrak{U}(F)$. Then there exists a Lipschitz homeomorphism $g$ from $E$ to $F$ with $\mathcal{H}(g) \leq \mathcal{H}(f)$ and such that the mapping $Q_V \circ g \circ J_X$ is linear.

Proof. Applying a translation if necessary we ensure $f(0) = 0$. Choose a seminorm $p$ on $E$ which is a norm on $X$. Then the restriction of $f$ to $X$, denoted by $f_X$, is Lipschitz, considered as a mapping from the normed space $X$ into $F$. Consequently, for each $q \in \Psi(F)$

$$q(f_X(x_1) - f_X(x_2)) \leq K_{f_X}(p, q) \ p(x_1 - x_2) \quad (x_1, x_2 \in X)$$

where $K_{f_X}(p, q) < \infty$ for all $q$. Define

$$B = \{ y \in F : q(y) \leq K_{f_X}(p, q) \text{ for all } q \in \Psi(F) \}.$$ 

$B$ is a closed, bounded, absolutely convex subset of $F$, so it generates a Banach space $F_B = \bigcap \{ \lambda B : \lambda > 0 \}$, equipped with the gauge norm of $B$. Then (2) shows that $f_X$, in fact, acts as a Lipschitz mapping from $X$ into $F_B$. Call this mapping, for distinction, $f_{X,B}$.

Next we consider the canonical embedding $I_{B,Y}$ of $F_B$ into $F_Y$, which is compact by our assumption. Therefore, by [4], $I_{B,Y}$ factors through a reflexive Banach space, say $Z$, i.e. there are bounded linear operators $S: F_B \to Z$ and $T: Z \to F_Y$ such that $I_{B,Y} = TS$. Now $S \circ f_{X,B}$ is a Lipschitz mapping with values in a reflexive space, therefore, according to the vector-valued Rademacher’s Theorem (cf. [3], [8], and [2]), it is almost everywhere in $X$ Gâteaux differentiable (with a linear derivative). Let $x_0 \in X$ be such a point. Then clearly $T \circ S \circ f_{X,B} = Q_Y \circ f \circ J_X$ is also Gâteaux differentiable in $x_0$.

Now we are ready to define $g$: Introduce mappings $f_n: E \to F$ ($n \in \mathbb{N}$) by setting for $x \in E$,

$$f_n(x) = n(f(x_0 + n^{-1}x) - f(x_0)).$$

It is readily checked that $f_n$ is Lipschitz with $\mathcal{H}(f_n) \leq \mathcal{H}(f)$. By virtue of $f_n^{-1}(y) = n(f^{-1}(f(x_0) + n^{-1}y) - x_0)$ it follows in the same way that $f_n$ is a Lipschitz homeomorphism and $\mathcal{H}(f_n) \leq \mathcal{H}(f)$. This puts us in a position to apply Lemma 2.

With a non-trivial ultrafilter $\mathcal{D}$ on $\mathbb{N}$ were set

$$g(x) = \lim_{\mathcal{D}} f_n(x).$$

This is a Lipschitz homeomorphism, $\mathcal{H}(g) \leq \mathcal{H}(f)$, and we get for $x \in X$

$$(Q_Y \circ g \circ J_X)(x) = \lim_{\mathcal{D}} Q_Y f_n(x) = \lim_{n \to \infty} Q_Y (f(x_0 + n^{-1}x) - f(x_0))/n^{-1}$$

which is the derivative of $Q_Y \circ f \circ J_X$, hence is linear.

The following is the main result of this paper. It generalizes the result of [10] on Montel-Fréchet spaces as well as Theorem 4.5 and Corollaries 3.11 and 3.12 of [9], where partial results for LF-spaces we obtained.

Theorem. Let $E$ be a locally convex space which is uniformly homeomorphic to a boundedly compact locally convex space $F$. Then $E$ is isomorphic to $F$. 

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Proof. As is easily checked, $E$ is also boundedly compact. By Lemma 1, $E$ and $F$ are Lipschitz homeomorphic. Let $f: E \to F$ be a Lipschitz homeomorphism. Denote by $I$ the set of all pairs $(X, V)$ with $X \subseteq E$ a finite dimensional subspace and $V \in \mathcal{U}(F)$. Let the order on $I$ be defined by $(X_1, V_1) \preceq (X_2, V_2)$ iff $X_1 \subseteq X_2$ and $V_1 \supseteq V_2$. Take an ultrafilter $\mathcal{D}$ on $I$ which dominates the natural order-filter. Find for each $(X, V) \in I$ a mapping $g_{(X,V)}: E \to F$ satisfying the conclusions of Lemma 3. In particular, $\mathcal{H}(g_{(X,V)}) \leq \mathcal{H}(f)$, so we can pass to

$$g(x) = \lim_{\mathcal{D}} g_{(X,V)}(x) \quad (x \in E).$$

By Lemma 2, $g$ is a Lipschitz homeomorphism from $E$ to $F$. Given $(X_0, V_0) \in I$, $Q_{V_0}g_{(X,V)} J_{X_0}$ is linear for all $(X, V) \supseteq (X_0, V_0)$, hence $Q_{V_0}g_{J_{X_0}}$, and thus $g$ itself is linear.

The following corollary gives the classification of all Banach spaces with respect to the uniform structure of their dual spaces, equipped with the weak-star topology. For the class of Banach spaces with separable duals, this was already shown in [10].

**Corollary.** Let $X$ and $Y$ be Banach spaces and assume that their duals $X^*$ and $Y^*$ are uniformly homeomorphic with respect to the weak-star topologies. Then $X$ and $Y$ are isomorphic.

**Proof.** It immediately follows from Theorem that $X^*$ and $Y^*$ are isomorphic in their weak-star topologies, which implies that $X$ and $Y$ are isomorphic with respect to their norm-topologies.

References


Author’s address: Akademie der Wissenschaften der DDR, Institut für Mathematik, DDR - 1086 Berlin, Mohrenstr. 39, Postfach 1304.