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THE UNIFORM CLASSIFICATION OF BOUNDEDLY COMPACT LOCALLY CONVEX SPACES

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Recently, Mankiewicz and Vilímovský [10] accomplished the uniform classification of Montel-Fréchet spaces by showing that a locally convex space is uniformly homeomorphic to a Montel-Fréchet space iff it is isomorphic to it. This together with Enflo's result [6] that a space, uniformly homeomorphic to a Hilbert space, is isomorphic to it, constitutes the only cases known till now where uniform equivalence of a given space implies linear-topological equivalence. That this implication does not hold for all locally convex spaces is demonstrated by the famous example due to Aharoni and Lindenstrauss [1] of two uniformly (even Lipschitz) homeomorphic Banach spaces which are not isomorphic. For further information on the Banach space case we refer to [7] and the bibliography therein.

In this paper we enlarge the range of locally convex spaces for which the above implication holds. We extend the result of Mankiewicz and Vilímovský to the class of all boundedly compact locally convex spaces, generalizing in this way also the results from [9]. This enables us to give the classification of all Banach spaces with respect to weak-star uniform equivalence of their dual spaces: Two Banach spaces are isomorphic if and only if their dual spaces are uniformly homeomorphic in their weak-star topologies.

We consider only vector spaces over the field of reals. A topological vector space E is called *boundedly compact* if each bounded subset of E is relatiely compact [5]. Given a locally convex space E, we denote by $\mathfrak{U}(E)$ the set of absolutely convex closed neighbourhoods of zero, by $\mathfrak{P}(E)$ the set of all continuous seminorms on E, and by $\mathfrak{B}(E)$ the set of all absolutely convex closed bounded sets. Isomorphic means always linearly isomorphic. A mapping $f\colon E \to F$ is called *Lipschitz* if for each $q \in \mathfrak{P}(F)$ there is a $p \in \mathfrak{P}(E)$ and a constant K such that

$$q(f(x_1) - f(x_2)) \leqq K p(x_1 - x_2) \quad (x_1, x_2 \in E) . \tag{1}$$

The spaces E and F are called *uniformly homeomorphic* (*Lipschitz homeomorphic*) if there is a one-to-one mapping f of E onto F such that both f and f^{-1} are uniformly continuous (Lipschitz, respectively).

It is convenient for our purposes to introduce the set of Lipschitz constants of f

in the following way. The Lipschitz constant $\mathscr{K}(f)$ denotes the function $K_f\colon \mathfrak{P}(E) \times \times \mathfrak{P}(F) \to R^+ \cup \{+\infty\}$, where $K_f(p, q)$ is the smallest constant K such that (1) holds, if there is such K at all. Otherwise we set $K_f(p, q) = +\infty$. The Lipschitz homeomorphism constant $\mathscr{H}(f)$ stands for the pair $(\mathscr{K}(f), \mathscr{K}(f^{-1}))$. Finally, we write $\mathscr{K}(f) \leqq \mathscr{K}(g)$ if $K_f(p, q) \leqq K_g(p, q)$ for all p and q, and $\mathscr{H}(f) \leqq \mathscr{H}(g)$ if $\mathscr{K}(f) \leqq \mathscr{K}(g)$ and $\mathscr{K}(f^{-1}) \leqq \mathscr{K}(g^{-1})$.

The following lemma was shown in [10] (compare also the non-compact version in [7]):

Lemma 1. *Let E and F be boundedly compact locally convex spaces. If E and F are uniformly homeomorphic, then they are Lipschitz homeomorphic.*

The following result constitutes the basis of our compactness arguments:

Lemma 2. *Suppose E and F are boundedly compact locally convex spaces. Let f and f_α $(\alpha \in A)$ be Lipschitz homeomorphisms from E to F with $f_\alpha(0) = f(0) = 0$ and $\mathscr{H}(f_\alpha) \leqq \mathscr{H}(f)$. Let $\mathscr{D}$ be an ultrafilter on A, and set $g(x) = \lim_{\mathscr{D}} f_\alpha(x)$ $(x \in E)$. Then g is a well-defined Lipschitz homeomorphism from E to F with $\mathscr{H}(g) \leqq \mathscr{H}(f)$.*

Proof. By assumption, we have $q(f_\alpha(x)) \leqq K_f(p, q)\, p(x)$, where for each $q \in \mathfrak{P}(F)$ there is a $p \in \mathfrak{P}(E)$ such that $K_f(p, q) < \infty$. This shows that for a fixed $x \in E$, $\{f_\alpha(x)\colon \alpha \in A\}$ is a bounded subset of F. Therefore g is correctly defined. As a limit of mappings with uniformly bounded Lipschitz constants, g is clearly Lipschitz and $\mathscr{K}(g) \leqq \mathscr{K}(f)$. Now define

$$h(y) = \lim_{\mathscr{D}} f_\alpha^{-1}(y) \quad \text{for} \quad y \in F .$$

Then, as above, $\mathscr{K}(h) \leqq \mathscr{K}(f^{-1})$, and we will show that $h \circ g = \mathrm{id}_E$. So, given $p \in \mathfrak{P}(E)$ and $x \in E$, we have

$$p(h(g(x)) - x) \leqq p(h(g(x)) - f_\alpha^{-1}(g(x))) + p(f_\alpha^{-1}(g(x)) - f_\alpha^{-1}(f_\alpha(x))) .$$

The first term on the right-hand side converges to zero as α passes along $\mathscr{D}$, by the definition of h. The second term can be estimated as follows:

$$p(f_\alpha^{-1}(g(x)) - f_\alpha^{-1}(f_\alpha(x))) \leqq K_{f^{-1}}(q, p)\, q(g(x) - f_\alpha(x)) .$$

The right-hand side of this inequality clearly also tends to zero, showing that $h \circ g = = \mathrm{id}_E$. Similarly, $g \circ h = \mathrm{id}_F$, so g is a Lipschitz homeomorphism with the constants required above.

Given a locally convex space E and a subspace X, we denote by J_X the embedding of X into E. For $U \in \mathfrak{U}(E)$, E_U denotes the Banach space canonically associated with U, namely the completion of E/N_U under the gauge norm of U, where $N_U = = \{x \in E\colon x \in \lambda U \text{ for all } \lambda > 0\}$. $Q_U\colon E \to E_U$ stands for the corresponding quotient map.

The following lemma is crucial for the proof of Theorem. It shows that we can linearize f "locally" without changing its homeomorphism properties.

Lemma 3. *Assume that E and F are boundedly compact locally convex spaces and $f: E \to F$ is a Lipschitz homeomorphism. Let $X \subset E$ be a finite dimensional subspace and let $V \in \mathfrak{U}(F)$. Then there exists a Lipschitz homeomorphism g from E to F with $\mathscr{H}(g) \leqq \mathscr{H}(f)$ and such that the mapping $Q_V \circ g \circ J_X$ is linear.*

Proof. Applying a translation if necessary we ensure $f(0) = 0$. Choose a seminorm p on E which is a norm on X. Then the restriction of f to X, denoted by f_X, is Lipschitz, considered as a mapping from the normed space X into F. Consequently, for each $q \in \mathfrak{P}(F)$

(2) $$q(f_X(x_1) - f_X(x_2)) \leqq K_{f_X}(p, q)\, p(x_1 - x_2) \quad (x_1, x_2 \in X)$$

where $K_{f_X}(p, q) < \infty$ for all q. Define

$$B = \{y \in F: q(y) \leqq K_{f_X}(p, q) \text{ for all } q \in \mathfrak{P}(F)\}.$$

B is a closed, bounded, absolutely convex subset of F, so it generates a Banach space $F_B = \bigcap\{\lambda B: \lambda > 0\}$, equipped with the gauge norm of B. Then (2) shows that f_X, in fact, acts as a Lipschitz mapping from X into F_B. Call this mapping, for distinction, $f_{X,B}$.

Next we consider the canonical embedding $I_{B,V}$ of F_B into F_V, which is compact by our assumption. Therefore, by [4], $I_{B,V}$ factors through a reflexive Banach space, say Z, i.e. there are bounded linear operators $S: F_B \to Z$ and $T: Z \to F_V$ such that $I_{B,V} = TS$. Now $S \circ f_{X,B}$ is a Lipschitz mapping with values in a reflexive space, therefore, according to the vector-valued Rademacher's Theorem (cf. [3], [8], and [2]), it is almost everywhere in X Gâteaux differentiable (with a linear derivative). Let $x_0 \in X$ be such a point. Then clearly $T \circ S \circ f_{X,B} = Q_V \circ f \circ J_X$ is also Gâteaux differentiable in x_0.

Now we are ready to define g: Introduce mappings $f_n: E \to F$ ($n \in \mathbb{N}$) by setting for $x \in E$,

$$f_n(x) = n(f(x_0 + n^{-1}x) - f(x_0)).$$

It is readily checked that f_n is Lipschitz with $\mathscr{H}(f_n) \leqq \mathscr{H}(f)$. By virtue of $f_n^{-1}(y) = n(f^{-1}(f(x_0) + n^{-1}y) - x_0)$ it follows in the same way that f_n is a Lipschitz homeomorphism and $\mathscr{H}(f_n) \leqq \mathscr{H}(f)$. This puts us in a position to apply Lemma 2. With a non-trivial ultrafilter $\mathscr{D}$ on $\mathbb{N}$ were set

$$g(x) = \lim_{\mathscr{D}} f_n(x).$$

This is a Lipschitz homeomorphism, $\mathscr{H}(g) \leqq \mathscr{H}(f)$, and we get for $x \in X$

$$(Q_V \circ g \circ J_X)(x) = \lim_{\mathscr{D}} Q_V f_n(x) = \lim_{n \to \infty} (Q_V f(x_0 + n^{-1}x) - Q_V f(x_0))/n^{-1}$$

which is the derivative of $Q_V \circ f \circ J_X$, hence is linear.

The following is the main result of this paper. It generalizes the result of [10] on Montel-Fréchet spaces as well as Theorem 4.5 and Corollaries 3.11 and 3.12 of [9], where partial results for LF-spaces we obtained.

Theorem. *Let E be a locally convex space which is uniformly homeomorphic to a boundedly compact locally convex space F. Then E is isomorphic to F.*

Proof. As is easily checked, E is also boundedly compact. By Lemma 1, E and F are Lipschitz homeomorphic. Let $f: E \to F$ be a Lipschitz homeomorphism. Denote by I the set of all pairs (X, V) with $X \subset E$ a finite dimensional subspace and $V \in \mathfrak{U}(F)$. Let the order on I be defined by $(X_1, V_1) \leqq (X_2, V_2)$ iff $X_1 \subseteq X_2$ and $V_1 \supseteq V_2$. Take an ultrafilter $\mathscr{D}$ on I which dominates the natural order-filter. Find for each $(X, V) \in I$ a mapping $g_{(X,V)}: E \to F$ satisfying the conclusions of Lemma 3. In particular, $\mathscr{H}(g_{(X,V)}) \leqq \mathscr{H}(f)$, so we can pass to

$$g(x) = \lim_{\mathscr{D}} g_{(X,V)}(x) \quad (x \in E).$$

By Lemma 2, g is a Lipschitz homeomorphism from E to F. Given $(X_0, V_0) \in I$, $Q_{V_0} g_{(X,V)} J_{X_0}$ is linear for all $(X, V) \geqq (X_0, V_0)$, hence $Q_{V_0} g J_{X_0}$, and thus g itself is linear.

The following corollary gives the classification of all Banach spaces with respect to the uniform structure of their dual spaces, equipped with the weak-star topology. For the class of Banach spaces with separable duals, this was already shown in [10].

Corollary. *Let X and Y be Banach spaces and assume that their duals X^* and Y^* are uniformly homeomorphic with respect to the weak-star topologies. Then X and Y are isomorphic.*

Proof. It immediately follows from Theorem that X^* and Y^* are isomorphic in their weak-star topologies, which implies that X and Y are isomorphic with respect to their norm-topologies.

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