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NONOSCILLATORY SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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In the paper a result of J. Ohriska in [3] concerning oscillation of the second order linear differential equation with delay is extended to an $n$-th order differential equation with deviating argument. The main tool in establishing the results are Kiguradze lemmas.

We consider the differential equation

$$L_n y(t) + f(t, y[g(t)]) = 0$$

where $n > 1$,

$$L_n y(t) = p_n(t) \left[ p_{n-1}(t) \left( \ldots \left[ p_1(t) \left( p_0(t) y'(t) \right)' \right] \ldots \right) \right]'$$

$p_i, i = 0, 1, \ldots, n$ are positive and continuous functions on $(t_0, \infty)$, $f$ is real valued and continuous on $D = (t_0, \infty) \times \mathbb{R}$, $g: (t_0, \infty) \to (t_0, \infty)$ is continuous and $t_0 \in \mathbb{R}$.

The expressions

$$L_i y(t) = p_i(t) \left[ L_{i-1} y(t) \right]' , \quad i = 1, 2, \ldots, n,$$

are called the quasi-derivatives of $y$ at the point $t \in (t_0, \infty)$. We restrict our considerations to those solutions of (1) which exist on some ray $(t_0, \infty)$ and satisfy the condition

$$\sup \{|y(t)|; t_1 \leq t < \infty\} > 0 \quad \text{for any} \quad t_1 \in (t_0, \infty).$$

Such a solution is called oscillatory if it has arbitrarily large zeros. Otherwise a solution is called nonoscillatory. The equation (1) is called oscillatory if all solutions of (1) are oscillatory.

Sometimes we will require the following conditions to be satisfied:

$$\int_0^\infty p_i^{-1}(t) \, dt = \infty, \quad i = 1, 2, \ldots, n - 1;$$

$$y f(t, y) \geq 0 \quad \text{for all} \quad (t, y) \in D, \quad \text{and for any interval of the form} \quad (t_1, \infty) \quad \text{with} \quad t_1 \geq t_0 \quad \text{and any function} \quad h \in C((t_1, \infty)), f[t, h(t)] \equiv 0 \quad \text{implies} \quad h(t) \equiv 0;$$

$$y f(t, y) \leq 0 \quad \text{for all} \quad (t, y) \in D, \quad \text{and for any interval of the form} \quad (t_1, \infty) \quad \text{with} \quad t_1 \geq t_0 \quad \text{and any function} \quad h \in C((t_1, \infty)), f[t, h(t)] \equiv 0 \quad \text{implies} \quad h(t) \equiv 0;$$

$$\lim_{t \to \infty} g(t) = \infty.$$
To obtain the main results we need three lemmas. The first is adapted from the papers [1], [4], [5], [7] and contains a generalization of the well-known first two Kiguradze lemmas.

**Lemma 1.** Let the condition (4) be satisfied and let \( y \) be a positive function on the interval \( (t_1, \infty) \), \( t_1 \geq t_0 \), such that \( L_n y \) exists on \( (t_1, \infty) \), is of constant sign and is not identically zero on any interval of the form \( (t_2, \infty) \), \( t_2 \geq t_1 \).

Then there exists an integer \( l, 0 \leq l \leq n \), with \( n + 1 \) odd for \( L_n y \leq 0 \) or \( n + 1 \) even for \( L_n y \geq 0 \), such that

\[
I \leq n - 1 \quad \text{implies} \quad (-1)^{l+j} L_j y(t) > 0 \quad \text{for every} \quad t \geq t_1
\]

\[
I > 1 \quad \text{implies} \quad L_i y(t) > 0 \quad \text{for all large} \quad t \quad (i = 1, 2, \ldots, l - 1)
\]

Further, for every \( i = 0, 1, \ldots, n - 1 \), \( \lim_{t \to \infty} L_i y(t) \) exists in the extended real line \( \mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\} \) whereby

\[
\lim_{t \to \infty} L_i y(t) = c_i \geq 0
\]

for \( I \leq n - 1 \), \( \lim_{t \to \infty} L_i y(t) = 0 \quad (j = l + 1, \ldots, n - 1) \),

\[
\lim_{t \to \infty} L_i y(t) = \infty \quad (i = 0, \ldots, l - 2)
\]

Remark. If \( 1 \leq l \leq n - 1 \), the lemma gives no exact result about \( c_i \) and \( c_{l-1} = = \lim_{t \to \infty} L_{l-1} y(t) \). If \( c_l > 0 \), then \( c_{l-1} = \infty \). For \( c_l = 0 \), \( c_{l-1} > 0 \) may be finite or infinite as the example of functions

\[
y_1(t) = \arctan t, \quad y'_1(t) = 1/(1 + t^2), \quad y''_1(t) = -2t/(1 + t^2)^2,
\]

\[
y_2(t) = \ln t, \quad y'_2(t) = 1/t, \quad y''_2(t) = -1/t^2
\]

with \( l = 1, n = 2, p_0 \equiv p_1 \equiv p_2 \equiv 1 \) shows. Similarly, if \( l = n \), then \( \lim_{t \to \infty} L_{n-1} y(t) = = c_{n-1} > 0 \) may be finite or infinite.

Define functions

\[
I_0 \equiv 1, \quad I_k(t, a; p_i, p_{i+1}, \ldots, p_{k-1}) =
\]

\[
= \int_a^t \int_a^{t_{i_1}} \int_a^{t_{i_2}} \cdots \int_a^{t_{i_{k-1}}} \int_a^{t_{i_k}} p_i^{-1}(t) \ dt_i \ dt_{i+1} \ dt_{i-1} \ dt_t,
\]

\[
1 \leq k \leq n - 1, \quad t_0 \leq a \leq t < \infty.
\]

Then the functions

\[
x_j(t, a) = p_0^{-1}(t) I_j(t, a; p_1, p_2, \ldots, p_{j-1}), \quad j = 1, 2, \ldots, n,
\]
form a fundamental system of solutions of the equation $L_n x(t) = 0$ in $(a, \infty)$ and

$$(10) \quad L_{j-1} x_j(t, a) \equiv 1, \quad L_i x_j(t, a) \equiv 0 \text{ for } i \geq j,$$

$\quad L_i x_j(t, a) > 0 \text{ in } (a, \infty) \text{ for } i < j - 1.$

For the sake of brevity, denote

$$(11) \quad P_0(t, a) \equiv 1, \quad P_j(t, a) = I_j(t, a; p_1, \ldots, p_j),$$

$\quad j = 1, 2, \ldots, n - 1, \quad t_0 \leq a \leq b < \infty$

and

$$(12) \quad Q_a(t, a) \equiv 1, \quad Q_j(t, a) = I_{n-j}(t, a; p_{n-1}, p_{n-2}, \ldots, p_j),$$

$\quad j = 1, 2, \ldots, n - 1, \quad t_0 \leq a \leq t < \infty.$

In the case all $p_i \equiv 1$

$$P_j(t, a) = \frac{(t - a)^j}{j!}, \quad j = 0, 1, \ldots, n - 1$$

and

$$Q_j(t, a) = \frac{(t - a)^{n-j}}{(n-j)!}, \quad j = 1, \ldots, n - 1, n.$$

Remark. If $l$ from Lemma 1 satisfies $0 \leq l \leq n - 1$, then by the variation of constants formula ([1], p. 96, (9.01)) with $a$ sufficiently great and with respect to (11) and

$$L_0 y(t) = \sum_{j=0}^{l} L_j y(a) P_j(t, a) + \int_{a}^{t} p_{l+1}(s) L_{l+1} y(s) P_l(t, s) \, ds, \quad t \geq a$$

where $L_j y(a) > 0$, $j = 0, 1, \ldots, l$, and $L_{l+1} y(t) \leq 0$, $t \geq a$, we get that

$$L_0 y(t) \leq \sum_{j=0}^{l} L_j y(a) P_j(t, a).$$

In the general case, when $y$ is either positive in a neighbourhood of infinity or negative, we come to the inequality

$$(13) \quad |L_0 y(t)| \leq \sum_{j=0}^{l} |L_j y(a)| P_j(t, a).$$

Further, (4) implies that

$$\lim_{t \to \infty} \frac{P_j(t, a)}{P_{l+1}(t, a)} = 0, \quad j = 0, 1, \ldots, l$$

and hence, by (13),

$$(14) \quad \lim_{t \to \infty} \frac{L_0 y(t)}{P_k(t, a)} = 0, \quad k = l + 1, \ldots, n.$$
In the case $l = 0$, by Lemma 1, $|L_0y|$ is a nonincreasing function and hence there exists

$$\lim_{t \to \infty} \frac{L_0 y(t)}{P_0(t, a)} = c_0.$$  

In the case $1 \leq l \leq n - 1$, by Lemma 2.1 ([6], p. 298),

$$\lim_{t \to \infty} \frac{L_0 y(t)}{P_l(t, a)} = \lim_{t \to \infty} L_j y(t) = c_j, \quad j = 0, \ldots, l$$

whereby $|c_j| = \infty$ for $j = 0, \ldots, l - 2$, and $0 < |c_{l-1}| \leq \infty$. Thus the following statement is true:

If the conditions of Lemma 1 are satisfied and there is an integer $k$, $0 \leq k \leq n - 2$, such that

$$\lim_{t \to \infty} \frac{L_0 y(t)}{P_k(t, a)} > 0, \quad \lim_{t \to \infty} \frac{L_0 y(t)}{P_{k+1}(t, a)} = 0,$$

then by the former relation $k \leq l$ and by the latter $k \geq l - 1$, hence $k$ is either $l - 1$ or $l$.

U. Elias in [2] has generalized the third Kiguradze lemma. From his results (Theorem 3, case $j = k + 1$) the following lemma is important for our considerations.

**Lemma 2.** Let $l$ be an integer, $1 \leq l \leq n - 1$, $a \in (t_0, \infty)$. If the function $y$ satisfies

$L_0 y(a), \ldots, L_{l-1} y(a) \geq 0$, \quad $L_{l+1} y(t) \leq 0$ for \quad $a \leq t < \infty$,

then

(a) \quad \left( \frac{L_i y(t)}{L_i x_{i+1}(t, a)} \right)' \leq 0, \quad i = 0, 1, \ldots, l, \quad a < t < \infty \quad \text{and}

(b) \quad L_i y(t) \geq L_{i+1} y(t) \frac{L_0 x_{i+1}(t, a)}{L_i x_{i+1}(t, a)}, \quad i = 0, 1, \ldots, l, \quad a < t < \infty.

Hence by (a), (b), (2), (9), (11),

(c) \quad \frac{L_0 y(t)}{P_l(t, a)} \text{ is a nonincreasing function in } (a, \infty),

(d) \quad L_0 y(t) \geq L_i y(t) \frac{P_l(t, a)}{L_i x_{i+1}(t, a)}, \quad i = 1, \ldots, l, \quad a < t < \infty

and with respect to (10),

(e) \quad L_0 y(t) \geq L_i y(t) P_l(t, a), \quad a \leq t < \infty.

**Lemma 3.** Let the conditions (4), (5), (7) (the conditions (4), (6), (7)) be satisfied and let $u$ be a nonoscillatory solution of the equation (1). Denote by $\delta$ the sign of
If \( u(t) \) is nonoscillatory and positive in a neighborhood of infinity. Then there exists a number \( t_1, t_1 \geq t_0 \), and an integer \( l \), \( 0 \leq l \leq n \), with \( n + l \) odd \((n + l) \text{ even})\), such that

(a) for \( l \leq n - 1 \), \((-1)^{l+1} \delta L_j u(t) > 0 \) for every \( t \geq t_1 \), \( j = l, l + 1, \ldots, n - 1 \), and

\[
\lim_{t \to \infty} L_i u(t) = c_i \text{ is finite, whereby } \delta c_i \geq 0;
\]

(b) for \( l \leq n - 2 \),

\[
\lim_{t \to \infty} L_j u(t) = 0, \quad j = l + 1, \ldots, n - 1;
\]

(c) for \( l \geq 2 \),

\[
\delta L_i u(t) > 0 \text{ for all large } t, \quad i = 1, 2, \ldots, l - 1
\]

and

\[
\lim_{t \to \infty} \delta L_i u(t) = \infty, \quad i = 0, \ldots, l - 2;
\]

(d) for \( l \leq n - 1 \), \( u \) is a solution of the integro-differential equation

\[
L_i y(t) = c_i + (-1)^{n-l+1} \int_{t}^{\infty} p_{n-1}(s) f(s, y[g(s)]) Q_{l+1}(s, t) ds.
\]

Proof. Suppose \( u \) is nonoscillatory and positive in a neighborhood of infinity. If \( u \) is negative, the proof can be done in a similar way. With respect to (7), there exists a \( t_1, t_1 \geq t_0 \), such that \( u(t) > 0 \) and also \( u[g(t)] > 0 \) for \( t \geq t_1 \). Then on the basis of (1), (5) implies ((6) implies) that \( L_n u \leq 0 \) \((L_n u \geq 0)\) on \( (t_1, \infty) \) and \( L_n u \) is not identically zero on any interval of the form \( (t_2, \infty) \), \( t_2 \geq t_1 \). Hence Lemma 1 can be applied. By that lemma the statements (a), (b), (c) are true.

Suppose now that \( l \leq n - 1 \). If \( l = n - 1 \), then integrating we obtain

\[
L_{n-1} u(t) = c_{n-1} + \int_{t}^{\infty} p_{n-1}(s) f(s, u[g(s)]) ds
\]

and hence in the case \( l = n - 1 \), (17) is satisfied by \( u \). When \( l \leq n - 2 \), then taking into account (b), by repeated integration we get

\[
L_j u(t) = (-1)^{n-j+1} \int_{t}^{\infty} p_{n-1}(s) f(s, u[g(s)]) Q_{j+1}(s, t) ds,
\]

\( j = n - 1, \ldots, l + 1 \).

Finally, integrating (18) for \( j = l + 1 \) we come to the conclusion that \( u \) is a solution of (17).

Now we can solve the first problem which is to find a sufficient condition for \( c_i \) in Lemma 3 to be zero.

Remark. It is clear that the number \( l \) in Lemma 3 is uniquely determined. This justifies the following

Definition 1. Suppose that the conditions (4), (5), (7) (the conditions (4), (6), (7)) are satisfied. Let \( u \) be a nonoscillatory solution of the equation (1) and \( \delta \) its sign in
a sufficiently small neighborhood of $\infty$. We say that $u$ has property $P_l$ with $l \in \{0, 1, \ldots, n\}$ and $n + l$ is odd ($n + l$ is even) if it has properties (a), (b), (c) from Lemma 3.

We recall that under the conditions (4), (5), (7) (the conditions (4), (6), (7)) each nonoscillatory solution of (1) has property $P_l$, with $l \in \{0, 1, \ldots, n\}$.

**Theorem 1.** Let the conditions (4), (5), (7) (the conditions (4), (6), (7)) be satisfied and let $u$ be a nonoscillatory solution of the equation (1) with property $P_l$, where $0 \leq l \leq n - 1$.

Let there exist a function $G = G(t, y); D_1 = \langle t_0, \infty \rangle \times \langle 0, \infty \rangle \to \langle 0, \infty \rangle$, which is continuous, nondecreasing in $y$ for each fixed $t$ and such that

\[
|f(t, y)| \geq G(t, |y|) \quad ((t, y) \in D).
\]

Then the condition: For each $k > 0$ and each $a$ from a neighborhood of $\infty$ either

\[
\int_a^\infty p_n^{-1}(s) Q_{l+1}(s, t) G(s, k p_0^{-1}[g(s)] P_l[g(s), a]) \, ds = \infty
\]

for all $t \geq a$ or

\[
\lim_{t \to \infty} \int_t^\infty p_n^{-1}(s) Q_{l+1}(s, t) G(s, k p_0^{-1}[g(s)] P_l[g(s), a]) \, ds > 0
\]

implies that

\[
c_i = \lim_{t \to \infty} L_t u(t) = 0.
\]

**Remark.** Suppose that $m: \langle t_0, \infty \rangle \to \langle 0, \infty \rangle$ is a continuous function and let us investigate

\[
\int_t^\infty p_n^{-1}(s) Q_{l+1}(s, t) m(s) \, ds
\]

which represents the general form of the integrals in (20) or (20'). As the function $p_n^{-1}(s) Q_{l+1}(s, t) m(s)$ is nonincreasing in the variable $t$ for $t < s$, $s$ being fixed, and nonnegative, two cases are possible concerning (20'). Either $\int_t^\infty p_n^{-1}(s) Q_{l+1}(s, t) \cdot m(s) \, ds = \infty$ for all $t \geq a$, or there is a $t_1$, $a \leq t_1 < \infty$, such that $\int_t^{t_1} p_n^{-1}(s) \cdot Q_{l+1}(s, t) m(s) \, ds < \infty$ for all $t \geq t_1$ and this function is nonincreasing in $\langle t_1, \infty \rangle$. Hence $\lim_{t \to \infty} \int_t^{t_1} p_n^{-1}(s) Q_{l+1}(s, t) m(s) \, ds$ exists and is finite and nonnegative.

In the case $l = n - 1$, the condition (20') cannot hold, because if $\int_t^{\infty} p_n^{-1}(s) m(s) \, ds$ exists, then $\lim_{t \to \infty} \int_t^{\infty} p_n^{-1}(s) m(s) \, ds = 0$.

**Proof of Theorem 1.** Suppose the conditions (4), (5), (7) and (19) are satisfied and $c_i \neq 0$. Then, by Lemma 3, $n + l$ is odd, and hence (17) implies that the equality

\[
L_t u(t) = c_i + \int_t^\infty p_n^{-1}(s) Q_{l+1}(s, t) f(s, u[g(s)]) \, ds
\]
is true. Denote by \( \delta \) the sign of \( u \) in a sufficiently small neighbourhood of \( \infty \). Let \( t_1 \) be such that

\[
\delta L_i u(t) > 0 \quad \text{in} \quad \langle t_i, \infty \rangle, \quad i = 0, \ldots, l, \\
\delta L_{i+1} u(t) \leq 0 \quad \text{in} \quad \langle t_i, \infty \rangle.
\]

By (7), there exists an \( a \geq t_1 \) such that \( g(t) \geq t_1 \) for each \( t \geq a \). We shall distinguish two cases:

1. \( l = 0 \).

By (23), \( \delta L_0 u \) is nonincreasing in \( \langle t_1, \infty \rangle \) and as it converges to \( \delta c_0 \), we have

\[
\delta L_0 u \geq \delta c_0 > 0.
\]

This implies

\[
|u(t)| \geq \frac{|c_0|}{p_0(t)}, \quad t \in \langle t_1, \infty \rangle.
\]

(22) can be written in the form

\[
\delta L_0 u(t) = \delta c_0 + \delta \int_{t}^{\infty} p_n^{-1}(s) Q_i(s, t) f(s, u[g(s)]) \, ds.
\]

By (5), this means

\[
|L_0 u(t)| \geq |c_0| + \int_{t}^{\infty} p_n^{-1}(s) Q_i(s, t) |f(s, u[g(s)])| \, ds
\]

and hence, (19) and (24) yield

\[
|L_0 u(t)| \geq |c_0| + \int_{t}^{\infty} p_n^{-1}(s) G(s, [u[g(s)]]Q_i(s, t) \, ds \geq
\]

\[
\geq |c_0| + |c_0| \int_{t}^{\infty} p_n^{-1}(s) \frac{G(s, [u[g(s)]]}{|c_0|} Q_i(s, t) \, ds
\]

which gives

\[
1 \geq \frac{|c_0|}{|L_0 u(t)|} \left( 1 + \int_{t}^{\infty} p_n^{-1}(s) Q_i(s, t) \frac{G(s, [u[g(s)]]}{|c_0|} \right) \, ds
\]

and this contradicts (20) or (20') because

\[
\lim_{t \to \infty} \frac{|c_0|}{|L_0 u(t)|} = 1.
\]

2. \( 1 \leq l \leq n - 1.\)

By Lemma 2, \( \delta L_0 u(t) \geq \delta L_i u(t) P_i(t, a) \) for all \( t \geq a \) and hence, with respect to (22),

\[
\delta L_0 u(t) \geq \delta L_i u(t) P_i(t, a) =
\]

\[
= \delta c_i P_i(t, a) + P_i(t, a) \int_{t}^{\infty} p_n^{-1}(s) Q_{i+1}(s, t) |f(s, u[g(s)])| \, ds \geq
\]

\[
\geq |c_i| P_i(t, a) + P_i(t, a) \int_{t}^{\infty} p_n^{-1}(s) Q_{i+1}(s, t) G(s, |u[g(s)]|) \, ds, \quad t \geq a.
\]
Thus

\[ |L_0 u(t)| \geq |c_i| P_i(t, a) + P_i(t, a) \int_t^\infty p_n^{-1}(s) Q_{l+1}(s, t) \cdot G(s, |u[g(s)]|) \, ds, \quad a \leq t < \infty. \]

This implies that

\[ |L_0 u(t)| \geq |c_i| P_i(t, a) \]

and

\[ |u(t)| \geq |c_i| x_{l+1}(t, a), \quad t \geq a. \]

Therefore

\[ |L_0 u(t)| \geq |c_i| P_i(t, a) + |c_i| P_i(t, a) \int_t^\infty p_n^{-1}(s) Q_{l+1}(s, t) \frac{G(s, |c_i| x_{l+1}[g(s), a])}{|c_i|} \, ds \]

for all \( t \geq b \) such that \( g(t) \geq a \) for \( t \geq b \). Then

\[ 1 \geq \frac{|c_i| P_i(t, a)}{|L_0 u(t)|} \left( 1 + \int_t^\infty p_n^{-1}(s) Q_{l+1}(s, t) \frac{G(s, |c_i| x_{l+1}[g(s), a])}{|c_i|} \, ds \right). \]

As \( \lim_{t \to \infty} L_0 u(t) = c_i \neq 0 \), Lemma 3, [5], p. 199 yields

\[ \lim_{t \to \infty} \frac{L_0 u(t)}{P_i(t, a)} = \lim_{t \to \infty} L_1 u(t) = c_i, \]

which shows that (29) contradicts (20) or (20').

Let now the conditions (4), (6), (7), (19) be satisfied. Then, by Lemma 3, \( n + l \) is even, and instead of (22) we have

\[ L_1 u(t) = c_i - \int_t^\infty p_n^{-1}(s) Q_{l+1}(s, t) f(s, u[g(s)]) \, ds. \]

The relations (23), (24) remain valid.

When \( l = 0 \), from (22') we get

\[ \delta L_0 u(t) = \delta c_0 - \delta \int_t^\infty p_n^{-1}(s) Q_1(s, t) f(s, u[g(s)]) \, ds. \]

Again we come to (26) and (27) which implies that (21) is true.

When \( 1 \leq l \leq n - 1 \), we obtain (28) and (29). This gives that (20) or (20') implies (21).

**Corollary 1.** Suppose that all assumptions of Theorem 1 are satisfied but (19) is replaced by

\[ |f(t, y)| \geq \alpha(t) |y| \quad ((t, y) \in D), \]

where \( \alpha \in C([t_0, \infty)) \) is a nonnegative function. Let \( u \) be a nonoscillatory solution of the equation (1) with property \( P_1 \) and let \( l \) satisfy \( 0 \leq l \leq n - 1. \)
Then the condition: For each $a$ from a neighbourhood of $\infty$ either

\begin{equation}
\int_{t}^{\infty} p_{n}^{-1}(s) Q_{t+1}(s, t) \alpha(s) p_{0}^{-1}[g(s)] P_{l}[g(s), a] \, ds = \infty
\end{equation}

for all $t \geq a$, or

\begin{equation}
\lim_{t \to \infty} \int_{t}^{\infty} p_{n}^{-1}(s) Q_{t+1}(s, t) \alpha(s) p_{0}^{-1}[g(s)] P_{l}[g(s), a] \, ds > 0,
\end{equation}

is sufficient for the equality

\[ c_{l} = \lim_{t \to \infty} L_{t} u(t) = 0 \]

to hold.

Remark. In the special case $g(t) \equiv t$ the condition $31'$ is weaker than the condition $(38')$ in [7], p. 127,

\[ \int_{t}^{\infty} \frac{\alpha(s)}{p_{0}(s) p_{n}(s)} P_{l}(s, t) Q_{t+1}(s, t) \, ds = \infty \]

and hence Corollary 1 improves and generalizes the sufficient condition in Corollary 1 to Theorem 6 in that paper when $h = h(t, y)$.

Denote

\[ h(t) = \max \left[ t, \max_{s \leq t} g(s) \right] \quad \text{for all } t \geq a, \]

where $a$ has the same meaning as in Theorem 1. Clearly $h(t) \geq t$ and $h$ is non-decreasing in $(a, \infty)$.

**Theorem 2.** Let $1 \leq l \leq n - 1$ be an integer. Let the conditions $(4), (5), (7), (19), (20)$ or $(20')$ (the conditions $(4), (6), (7), (19), (20)$ or $(20')$) be satisfied. Let $n + l$ be odd ($n + l$ be even). Let the function $G$ be such that

\begin{equation}
G(t, ky) \geq k G(t, y)
\end{equation}

for each $k > 0$ and each $(t, y) \in D_{1}$.

Then the condition: For each $a$ from a neighbourhood of $\infty$ either

\begin{equation}
\int_{t}^{\infty} p_{n}^{-1}(s) Q_{t+1}(s, t) G \left( s, p_{0}^{-1}[g(s)] \frac{P_{l}[g(s), a]}{P_{l}[h(s), a]} \right) \, ds = \infty
\end{equation}

for all $t \geq a$ or

\begin{equation}
\lim_{t \to \infty} \sup_{a} P_{l}(t, a) \int_{t}^{\infty} p_{n}^{-1}(s) Q_{t+1}(s, t) G \left( s, p_{0}^{-1}[g(s)] \frac{P_{l}[g(s), a]}{P_{l}[h(s), a]} \right) \, ds > 1
\end{equation}

is a sufficient condition that there exists no nonoscillatory solution $u$ of the equation (1) with property $P_{l}$.

Proof. Let $u$ be an arbitrary nonoscillatory solution of (1) such that the integer $l$ from Lemma 3 satisfies $1 \leq l \leq n - 1$. As all assumptions of Theorem 1 are
satisfied, $c_i = 0$ in (22) or (22') and hence $u$ satisfies (28) in the form

$$|L_0 u(t)| \geq P_i(t, a) \int_t^\infty p_n^{-1}(s) Q_{i+1}(s, t) G(s, |u[g(s)]|) \, ds.$$  

As $|L_0 u|$ is an increasing function and $h(t) \geq t$,

$$|L_0 u[h(t)]| \geq |L_0 u(t)| \quad \text{for all } t \geq a.$$  

By Lemma 2, $|L_0 u(t)| / P_i(t, a)$ is a nonincreasing function in $(a, \infty)$ and hence, $g(s) \leq h(s)$ implies

$$|u[g(s)]| = \frac{|L_0 u[g(s)]|}{p_0[g(s)]} \geq \frac{1}{P_i[g(s), a]} \frac{P_i[h(s), a]}{|L_0 u[h(s)]|}.$$  

Further, $h$ is nondecreasing and therefore

$$|L_0 u[h(s)]| \geq |L_0 u[h(t)]|, \quad s \geq t.$$  

Putting (34), (35) and (36) into (28') we get

$$|L_0 u[h(t)]| \geq P_i(t, a) \int_t^\infty p_n^{-1}(s) Q_{i+1}(s, t) G(s, \frac{1}{P_0[g(s)]} \frac{P_i[h(s), a]}{|L_0 u[h(t)]|}) \, ds.$$  

Now using (32) we come to the inequality

$$1 \geq P_i(t, a) \int_t^\infty p_n^{-1}(s) Q_{i+1}(s, t) G(s, \frac{p_0^{-1}[g(s)]}{P_i[h(s), a]} \frac{P_i[g(s), a]}{|L_0 u[h(t)]|}) \, ds$$

which contradicts (33) or (33').

**Corollary 2.** If the assumptions of Theorem 2 are satisfied but (19) is replaced by (30), (20) or (20') by (31) or (31') and (32) is omitted, then the condition: For each $a$ from a neighbourhood of $\infty$ either

$$\int_t^\infty p_n^{-1}(s) Q_{i+1}(s, t) \alpha(s) p_0^{-1}[g(s)] \frac{P_i[g(s), a]}{P_i[h(s), a]} \, ds = \infty$$

for all $t \geq a$ or

$$\lim_{t \to -\infty} \sup P_i(t, a) \int_t^\infty p_n^{-1}(s) Q_{i+1}(s, t) \alpha(s) p_0^{-1}[g(s)] \frac{P_i[g(s), a]}{P_i[h(s), a]} \, ds > 1$$

where $a$ is a sufficiently great number is a sufficient condition that there exists no nonoscillatory solution $u$ of the equation (1) with property $P_i$.

The next theorem concerns all solutions of the equation (1). Similarly as in [1] we introduce the definitions.

**Definition 2.** The equation (1) is said to have property $A$ if for $n$ even each solution $u$ of that equation is oscillatory and for $n$ odd each solution is either oscillatory or satisfies the conditions:

(a) There exists a $t_1, t_1 \geq t_0$, such that $(-1)^j \delta L_j u(t) > 0$ for every $t \geq t_1$, $j = 0, 1, \ldots, n - 1$
(b) $\lim_{t \to \infty} L_j u(t) = 0$, $j = 0, 1, \ldots, n - 1$.

Definition 3. The equation (1) is said to have property B if for $n$ even each solution of that equation is either oscillatory or satisfies conditions (a), (b) from Definition 2 or the conditions

(c) There exists a $t_2, t_2 \geq t_0$, such that $\delta L_i u(t) > 0$ for every $t \geq t_2$, $i = 0, 1, \ldots, n - 1$;

(d) $\lim_{t \to \infty} \delta L_i u(t) = \infty$, $i = 0, \ldots, n - 1$,

and for $n$ odd each of its solutions is either oscillatory or satisfies conditions (c) and (d).

In both definitions $\delta$ means the sign of the nonoscillatory solution $u$ in a neighbourhood of infinity.

Theorem 3. Let the conditions (4), (5), (7), (19) and (32) be satisfied. Further, let the conditions (20) or (20') and (33) or (33') be fulfilled for $l = n - 1, n - 3, \ldots, 1$ provided $n$ is even ($l = n - 1, n - 3, \ldots, 2$ provided $n$ is odd).

Then the equation (1) has property A.

Proof. Let $u$ be a nonoscillatory solution of the equation (1). Then, by Lemma 3, there exists an integer $l, 0 \leq l \leq n$, with $n + l$ odd, such that the statement of that lemma is true. Hence $l$ is one of the numbers $n - 1, n - 3, \ldots, 1$ when $n$ is even and $l$ belongs to the set consisting of the numbers $n - 1, n - 3, \ldots, 2, 0$ when $n$ is odd. By Theorem 2, for $l \neq 0$ no such solution exists. Hence, if $n$ is even, each solution $u$ of (1) is oscillatory and if $n$ is odd, $u$ is either oscillatory or possesses properties (a), (b) from Definition 2. Thus the equation (1) has property A.

Corollary 3. Let the conditions (4), (5), (7), (30) be satisfied. Further, let the conditions (31) or (31'), (37) or (37') be fulfilled for $l = n - 1, n - 3, \ldots, 1$ provided $n$ is even ($l = n - 1, n - 3, \ldots, 2$ provided $n$ is odd). Then the equation (1) has property A.

Remark. In the case $n = 2, l = 1, p_0 = p_1 = p_2 = 1$, $g(t) \leq t$, (31) or (31') and (37) or (37') are reduced to the conditions

$$\int_t^\infty p(s) \frac{g(s)}{s} \, ds = \infty$$

and

$$\lim_{t \to \infty} \sup_t \int_t^\infty p(s) \frac{g(s)}{s} \, ds > 1.$$ 

Hence Corollary 3 generalizes the first part of Theorem 1 in [3].

If instead of (5) we suppose (6) we obtain the following theorem.

Theorem 4. Let the conditions (4), (6), (7), (19) and (32) be satisfied. Further, let
the conditions (20) or (20') and (33) or (33') be fulfilled for \( l = n - 2, n - 4, \ldots, 2 \) for \( n \) even \((l = n - 2, n - 4, \ldots, 3, 1 \) for \( n \) odd). Finally, let the condition

\[
\int_0^\infty G\left(t, \frac{c P_{n-1}[g(t), a]}{p_0[g(t)]}\right) dt = \infty
\]

be fulfilled for each \( c > 0 \) and each sufficiently great \( a \).

Then the equation (1) has property B.

The proof of this theorem proceeds in the same way as that of Theorem 3. Comparing Definition 3 with Lemma 3 yields that the only thing which remains to show is that in the case \( l = n \),

\[
\lim_{t \to \infty} \delta L_{n-1} u(t) = \infty.
\]

Hence let \( l = n \). Then by (6), \( \delta L_{n-1} u \) is nondecreasing in a neighbourhood of \( \infty \) and hence, by statement (c) in Lemma 3, there exists a constant \( c > 0 \) such that

\[
\delta L_{n-1} u(t) \geq c > 0
\]

in \( (a, \infty) \). Using the same lemma we obtain by repeated integration of (40) that

\[
\delta L_{n-j} u(t) \geq c I_{j-1}(t, a; p_{n-j+1}, \ldots, p_{n-1}) , \quad t \geq a , \quad j = 1, \ldots, n
\]

and thus

\[
\delta L_0 u(t) \geq c P_{n-1}(t, a) , \quad t \geq a.
\]

This implies that

\[
|L_n u(t)| = |f(t, u[g(t)])| \geq G\left(t, \frac{|L_0 u[g(t)]|}{p_0[g(t)]}\right) \geq G\left(t, \frac{c P_{n-1}[g(t), a]}{p_0[g(t)]}\right), \quad t \geq a
\]

and in view of (38), (39) follows.

**Corollary 4.** Let the conditions (4), (6), (7), (30) be satisfied. Further, let the conditions (31) or (31') and (37) or (37') be fulfilled for \( l = n - 2, n - 4, \ldots, 2 \) when \( n \) is even \((l = n - 2, n - 4, \ldots, 3, 1 \) when \( n \) is odd). Finally, let the condition

\[
\int_0^\infty \alpha(t) \frac{P_{n-1}[g(t), a]}{p_0[g(t)]} dt = \infty
\]

be fulfilled for all sufficiently great \( a \).

Then the equation (1) has property B.

Theorem 2 does not say anything about the case \( l = 0 \). In a special case of the deviating argument the answer is given by

**Theorem 5.** Let the conditions (4), (5), (7), (19) and (20) or (20') for \( l = 0 \) (the conditions (4), (6), (7), (19) and (20) or (20') for \( l = 0 \) be satisfied. Let \( n \) be odd \((n \) even). For the function \( g \) let there exist an increasing sequence of points \( \{t_k\}_{k=1}^\infty \) with \( \lim k \to \infty t_k = \infty \) such that

\[
g(\langle t_k, \infty \rangle) \subset \langle t_k, \infty \rangle, \quad k = 1, 2, \ldots.
\]
In particular, (42) is satisfied when \( g(t) \geq t \) for \( t \geq t_0 \). Let there exist a function 
\( H = H(t, y): D \to (0, \infty) \) which is continuous, nondecreasing in \( y \) for each fixed \( t \) 
and such that 
\[
(43) \quad |f(t, y)| \leq H(t, |y|)
\]
for every \((t, y) \in D\), and for any sufficiently great number \( a \) there exists a \( k_0, 0 < k_0 < 1 \) 
such that for each \( k > 0 \),
\[
(44) \quad \int_{s}^{\infty} p_n^{-1}(s) Q_1(s, t) \frac{H(s, k \frac{p_o^{-1}[g(s)]}{k})}{p_o[g(s)]} \, ds \leq k_0
\]
for all \( s \geq a \).

Then there is no nonoscillatory solution \( u \) of the equation (1) with property \( P_0 \).

**Proof.** Let \( u \) be an arbitrary nonoscillatory solution of (1) with property \( P_0 \). 
Since all assumptions of Theorem 1 for \( l = 0 \) are satisfied, \( c_0 = 0 \) in (22) or (22') 
and hence
\[
|L_0 u(t)| = \int_{s}^{\infty} p_n^{-1}(s) Q_1(s, t) f\left(s, \frac{L_0 u[g(s)]}{p_o[g(s)]}\right) \, ds
\]
and in view of (43) we have
\[
(45) \quad |L_0 u(t)| \leq \int_{s}^{\infty} p_n^{-1}(s) Q_1(s, t) H\left(s, \frac{L_0 u[g(s)]}{p_o[g(s)]}\right) \, ds.
\]
By Lemma 3, \(|L_0 u|\) is a nonincreasing function which converges to 0 as \( t \to \infty \). 
Thus for any \( \varepsilon > 0 \) there exists a \( t_k = a \) satisfying (42) and such that
\[
(46) \quad |L_0 u(t)| \leq |L_0 u(t_k)| \leq \varepsilon \quad \text{for all } t \geq a.
\]
Putting (46) into (45), on the basis of (44) we come to the inequality
\[
(47) \quad |L_0 u(t)| \leq \int_{s}^{\infty} p_n^{-1}(s) Q_1(s, t) H(s, \varepsilon \frac{p_o^{-1}[g(s)]}{p_o[g(s)]}) \, ds \leq k_0 \varepsilon
\]
for all \( t \geq a \). Hence the inequality (46) in \((a, \infty)\) has led to the inequality (47) in 
the same interval. Repeating this process \( p \)-times we get that
\[
|L_0 u(t)| \leq k_0^p \varepsilon
\]
in \((a, \infty)\) which for \( p \to \infty \) implies that \( L_0 u(t) \equiv 0 \) in \((a, \infty)\) which contradicts 
the condition (3). Hence \( u \) with property \( P_0 \) does not exist.

**Corollary 5.** Let the conditions of Theorem 3 be satisfied. Further, if \( n \) is odd, 
let (20) or (20') for \( l = 0 \), (42), (43), (44) be satisfied. Then each solution of the 
equation (1) is oscillatory.

Another sufficient condition for the nonexistence of a nonoscillatory solution \( u \) 
of (1) with property \( P_i \) is given in the next theorem. As usual, let us denote
\[
(48) \quad \gamma(t) = \sup \{s \geq t_0: g(s) \leq t\} \quad \text{for all } t \geq t_0.
\]
With help of this function, we define
\[ m(t) = \max \{ \gamma(t), t \}, \quad t \geq t_0. \]

By virtue of (48) and the continuity of \( g \), for each \( s > \gamma(t) \) we have \( g(s) > t \) and \( g[\gamma(t)] = t \). Hence \( m \) possesses the following properties:

(49) \[ s \geq m(t) \implies g(s) \geq t, \quad m(t) \geq t \quad \text{and, further,} \]
\[ \text{if } g(t) \leq t, \quad \text{then } m(t) = \gamma(t). \]

**Theorem 6.** Let \( 1 \leq l \leq n - 1 \) be an integer. Let the conditions (4), (5), (7), (19), (20) or (20') (the conditions (4), (6), (7), (19), (20) or (20')) be satisfied. Further, let (32) be fulfilled. Then the condition:

For each \( a \) from a neighbourhood of \( \infty \) either

(50) \[ \int_{m(t)}^{\infty} p_{+1}^{-1}(s) Q_{l+1}(s, t) G(s, p_{-1}^{-1}[g(s)]) \, ds = \infty \]

for all sufficiently great \( t \) or

(50') \[ \lim_{t \to \infty} \sup_{t \to \infty} p_{+1}^{-1}(s) Q_{l+1}(s, t) G(s, p_{-1}^{-1}[g(s)]) \, ds > 1 \]

is a sufficient condition that there exists no nonoscillatory solution \( u \) of the equation (1) with property \( P_1 \).

**Proof.** If \( u \) is a nonoscillatory solution of (1) with property \( P_1 \), then similarly as in the proof of Theorem 2 we come to the inequality

(28') \[ |L_0 u(t)| \geq p_{+1}(t, a) \int_{m(t)}^{\infty} p_{+1}^{-1}(s) Q_{l+1}(s, t) G(s, [u[g(s)]) \, ds \geq \]
\[ \geq p_{+1}(t, a) \int_{m(t)}^{\infty} p_{+1}^{-1}(s) Q_{l+1}(s, t) G(s, [u[g(s)]) \, ds, \quad t \geq a. \]

Since \( |L_0 u| \) is increasing, (49) implies that
\[ |u[g(s)]| = |L_0 u[g(s)]| p_{-1}^{-1}[g(s)] \geq |L_0 u(t)| p_{-1}^{-1}[g(s)] \]

for all \( s \geq m(t) \) and the inequality for \( L_0 u \) turns into

\[ |L_0 u(t)| \geq p_{+1}(t, a) \int_{m(t)}^{\infty} p_{+1}^{-1}(s) Q_{l+1}(s, t) G(s, [L_0 u(t)] p_{-1}^{-1}[g(s)]) \, ds \]

which in view of (32) leads to the inequality
\[ 1 \geq p_{+1}(t, a) \int_{m(t)}^{\infty} p_{+1}^{-1}(s) Q_{l+1}(s, t) G(s, p_{-1}^{-1}[g(s)]) \, ds, \quad t \geq a. \]

This contradicts (50) or (50') and thus Theorem 6 is proved.

**Corollary 6.** If the assumptions of Theorem 6 are satisfied but (19) is replaced by (30), (20) or (20') by (31) or (31') and (32) is omitted, then the condition:
Either

\[ \int_{m(t)}^{\infty} p_{n-1}(s) Q_{l+1}(s, t) \alpha(s) p_0^{-1}[g(s)] \, ds = \infty \]

for all \( t \) from a neighbourhood of \( \infty \) or for all sufficiently great \( a \),

\[ \limsup_{t \to \infty} P_{l}(t, a) \int_{m(t)}^{\infty} p_{n-1}(s) Q_{l+1}(s, t) \alpha(s) p_0^{-1}[g(s)] \, ds > 1, \]

is sufficient that there exists no nonoscillatory solution \( u \) of the equation (1) with property \( P_l \).

If instead of Theorem 2 we use Theorem 6 in the proof of Theorem 3, we get

**Corollary 7.** Let the conditions (4), (5), (7), (19), (32) be satisfied. Further, let the conditions (20) or (20') and (50) or (50') be fulfilled for \( l = n - 1, n - 3, \ldots, 1 \) when \( n \) is even \((l = n - 1, n - 3, \ldots, 2 \) when \( n \) is odd).

Then the equation (1) has property \( A \).

The next corollary is a modification of Theorem 4.

**Corollary 8.** Let the conditions (4), (6), (7), (19) and (32) be satisfied. Further, let the conditions (20) or (20') and (50) or (50') be fulfilled for \( l = n - 2, n - 4, \ldots, 2 \) when \( n \) is even \((l = n - 2, n - 4, \ldots, 3, 1 \) when \( n \) is odd). Finally, let the condition (38) be satisfied. Then the equation (1) has property \( B \).

**Remark.** In a similar way Corollaries 3, 4 and 5 can be modified.

**Literature**


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