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FINE CONVERGENCE IN FREE GROUPS

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Dedicated to Professor Josef Novák on the occasion of his 80th birthday.

Free topological groups have been investigated for over four decades (see [HR]). The study of sequential convergence in free commutative groups was initiated by F. Zanolin in [Z_1] (see also [Z_2], [Z_3]). The noncommutative sequential case was first considered by J. Novák in [N_2]. He investigated the so-called pointed free group, i.e., given an infinite set Y equipped with a convergence of sequences, a point $p \in Y$ is singled out and the free group G over $Y \setminus \{p\}$ is equipped with a compatible convergence of sequences subjected to the following restriction: if a sequence of points of $Y \setminus \{p\}$ converges to a point x in Y, then the sequence converges in G to x provided $x \neq p$ and to the neutral element e of G provided $x = p$ (elements of $Y \setminus \{p\}$ are considered as one-letter words in G). In fact, he considered the case when p is the only nonisolated point of Y.

In the present paper we consider all four types of free groups: commutative, noncommutative, pointed commutative and pointed noncommutative. For each type we construct the finest of all compatible convergences of sequences extending the original convergence for the alphabet set. The idea of the construction is used in [FZ_1] and [FZ_2]. The general theory of free sequential convergence groups is outlined in [FZ_3].

In sequential convergence group terminology we generally follow [N_1]. Concerning sequences of points, however, the following notation serves better our purpose. Let X be a set. Sequences of points of X (i.e. mappings of $\mathbb{N}$ into X) will usually be denoted by capital letters S, T, U. Letter $\mathcal{S}$ denotes the set of all increasing mappings of $\mathbb{N}$ into $\mathbb{N}$. If $S \in X^N$ and $s \in \mathcal{S}$, then $S \circ s$ denotes the subsequence of S the n-th term of which is the point $S(s(n)) \in X$. If X is a group and $S, T \in X^N$, then $(ST)(n) = S(n)\, T(n)$ and $S^{-1}(n) = S(n)^{-1}$ for all $n \in \mathbb{N}$ (in the additive notation, $(S + T)(n) = S(n) + T(n)$ and $(-T)(n) = -T(n)$).

Recall that a sequential convergence group is a group G equipped with a sequential convergence $\mathfrak{G} \subset G^N \times G$ satisfying the axioms (L_0)-uniqueness of sequential limits, (L_1)-constant sequences converge, (L_2)-subsequences of convergent sequences converge, (L_3)-Urysohn condition, (S^*G)-compatibility of the sequential convergence and the group operation (i.e., if $(S, x), (T, y) \in \mathfrak{G}$, then $(ST^{-1}, xy^{-1}) \in \mathfrak{G}$). It is well-known (cf. [N_1]) that, in order to equip a group G with a compatible sequential convergence, it suffices to start with $\mathfrak{G} \subset G^N \times G$ satisfying axioms (L_0), (L_1), (L_2) and

(SG) if $(S, x), (T, y) \in \mathfrak{G}$, then for some $\mathscr{s} \in \mathscr{S}$ we have
$$((S \circ \mathscr{s})(T^{-1} \circ \mathscr{s}), xy^{-1}) \in \mathfrak{G}.$$

Then the Urysohn modification $\mathfrak{G}^*$ of $\mathfrak{G}$ (i.e. $(S, x) \in \mathfrak{G}^*$ whenever for each $\mathscr{s} \in \mathscr{S}$ there exists $\mathscr{t} \in \mathscr{S}$ such that $(S \circ \mathscr{s} \circ \mathscr{t}, x) \in \mathfrak{G}$) is a sequential convergence for G satisfying all five axioms for sequential convergence groups. Further, it is known (cf. [Z_1]) that in the definition of a sequential convergence group axiom (L_0) can be replaced by the following weaker one:

(L_{00}) if $(S, x) \in \mathfrak{G}$ and $S(n) = e$ for all $n \in \mathbb{N}$, then $x = e$.

1. THE COMMUTATIVE CASE

Consider an infinite set Y equipped with a sequential convergence $\mathfrak{L} \subset Y^N \times Y$ satisfying axioms (L_0), (L_1), (L_2) and (L_3). Choose a point $p \in Y$. Let G be the free commutative group over $X = Y \setminus \{p\}$. Recall that G can be represented as the set of all mappings g of X into the group $\mathbb{Z}$ of integers such that $g(y) = 0$ for all but finitely many $y \in X$, the group operation of G being the usual pointwise addition. It is more convenient to represent each element g of G as $\sum_{i=1}^{k} \lambda_i y_i$, where $\lambda_i \in \mathbb{Z} \setminus \{0\}$, $y_i \in X$, and $y_i \neq y_j$ for $i \neq j$. For $k > 0$ such linear combinations will be called *canonical*. For $k = 0$, we get the neutral element e of G. A fundamental role in our construction will be played by sequences in G^N of the form $\sum_{i=1}^{k} \lambda_i S_i$, where $k > 0$, $\lambda_i \in \mathbb{Z} \setminus \{0\}$, $S_i \in X^N$, and $S_i(n) \neq S_j(n)$ for all $n \in \mathbb{N}$ whenever $i \neq j$. Such linear combinations will also be called *canonical*. We are going to equip G with a sequential convergence such that G becomes a pointed free group and the convergence is the finest possible. If p is an isolated point of Y (i.e. no one-to-one sequence converges to p in Y), then our construction yields the usual free convergence group (cf. (FZ_2], [FZ_3]).

First, define $\mathfrak{G} \subset G^N \times G$ as follows:

(PCC) $(S, g) \in \mathfrak{G}$ if either

(PCC_1) S is the constant sequence generated by the neutral element e (i.e. $S(n) = e$ for all $n \in \mathbb{N}$) and $g = e$;

or

(PCC_2) S is a canonical linear combination of $\mathfrak{L}$-convergent sequences $S_i \in X^N$, $i = 1, \ldots, k$, and g is the reduction of the corresponding linear combination of their pointed $\mathfrak{L}$-limits (i.e. $(S_i, x_i) \in \mathfrak{L}$, and $S = \sum_{i=1}^{k} \lambda_i S_i$, where $\lambda_i \in \mathbb{Z} \setminus \{0\}$

and $S_i(n) \neq S_j(n)$ for all $n \in \mathbb{N}$ whenever $i \neq j$, and (with the same λ_i as above) $g = \sum_{i=1}^{k} \lambda_i y_i$, where $y_i = e$ for $x_i = p$ and $y_i = x_i$ otherwise).

Note that if $(S, g) \in \mathfrak{G}$ by (PCC_2), then the canonical linear combination $S = \sum_{i=1}^{k} \lambda_i S_i$ in (PCC_2) is not unique. E.g., if $(T, x) \in \mathfrak{L}$ and T is a one-to-one sequence, define a sequence S as follows: $S(n) = T(2n - 1) + T(2n)$. By (PCC_2) we have $(S, 2x) \in \mathfrak{G}$, but the sequences S_1 and S_2 can be chosen in infinitely many ways. For each pair of mappings $s_1, s_2 \in \mathscr{S}$ such that $\{s_1(n), s_2(n)\} = \{2n - 1, 2n\}$ for all $n \in \mathbb{N}$ it suffices to put $S_i = T \circ s_i$, $i = 1, 2$, and $S = S_1 + S_2$. Nevertheless, the next lemma shows that this type of difference is the only possible.

Lemma. *Let* $S = \sum_{i=1}^{k} \lambda_i S_i$ *and* $R = \sum_{i=1}^{l} \mu_i R_i$ *be two canonical linear combinations of sequences in* X. *If* $S(n) = R(n)$ (*in* G) *for all* $n \in \mathbb{N}$, *then* $k = l$ *and there is a permutation* a *of the set* $\{1, \ldots, k\}$ *such that for each* $i \in \{1, \ldots, k\}$ *we have*

(1) $\lambda_i = \mu_{a(i)}$;

(2) $S_i(n) = R_{a(i)}(n)$ *for infinitely many* $n \in \mathbb{N}$.

Proof. Since $S(1) = \lambda_1 S_1(1) + \ldots + \lambda_k S_k(1)$ and $R(1) = \mu_1 R_1(1) + \ldots + \mu_l R_l(1)$ are canonical linear combinations and $S(1) = R(1)$, necessarily $k = l$ and there is a permutation b of $\{1, \ldots, k\}$ such that $\lambda_i = \mu_{b(i)}$ for all $i \in \{1, \ldots, k\}$.

For a fixed $n \in \mathbb{N}$, consider ordered k-tuples $(\lambda_1 S_1(n), \ldots, \lambda_k S_k(n))$ and $(\mu_1 R_1(n), \ldots, \mu_k R_k(n))$. We are looking for a permutation a of $\{1, \ldots, k\}$ such that $(\lambda_1 S_1(n), \ldots, \lambda_k S_k(n)) = (\mu_{a(1)} R_{a(1)}(n), \ldots, \mu_{a(k)} R_{a(k)}(n))$ for infinitely many $n \in \mathbb{N}$. Even though $\sum_{i=1}^{k} \lambda_i S_i(n) = \sum_{i=1}^{k} \mu_{b(i)} R_{b(i)}(n)$ and $\lambda_i = \mu_{b(i)}$ for all $i \in \{1, \ldots, k\}$, for $\lambda_i = \lambda_j$, $i \neq j$, it can happen that $R_{b(i)}(n) = S_j(n)$ and $R_{b(j)}(n) = S_i(n)$. Denote by c_n the permutation of $\{1, \ldots, k\}$ such that $(\lambda_1 S_1(n), \ldots, \lambda_k S_k(n)) = (\mu_{b(1)} R_{c_n(b(1))}(n), \ldots, \mu_{b(k)} R_{c_n(b(k))}(n))$. Since $\mu_{c_n(b(i))} = \mu_{b(i)} = \lambda_i$ and there is a permutation c such that $c = c_n$ for infinitely many $n \in \mathbb{N}$, it suffices to put $a = c \circ b$. This completes the proof.

Second, let $\mathfrak{G}^*$ be the Urysohn modification of $\mathfrak{G}$.

Theorem 1. (The commutative case.) *G equipped with $\mathfrak{G}^*$ is a sequential convergence group.*

Proof. Axioms (L_{00}), (L_1) and (L_2) follow immediately from (PCC) (note that axiom (L_0) follows from Lemma). It remains to prove axiom (SG). Suppose that (S, g), $(T, h) \in \mathfrak{G}$. If any of the two sequences converges according to (PCC_1), axiom (SG) is trivially satisfied. So, suppose that $\mathfrak{G}$-lim $S(n) = g$ and $\mathfrak{G}$-lim $T(n) = h$ according to (PCC_2). Then S and T are canonical linear combinations $\sum_{i=1}^{k} \lambda_i S_i$ and $\sum_{j=1}^{l} \mu_j T_j$ of $\mathfrak{L}$-convergent sequences $S_i \in X^N$ $(i = 1, \ldots, k)$ and $T_j \in X^N$ $(j = 1, \ldots, l)$, respectively, and g and h are the corresponding linear combinations of their pointed $\mathfrak{L}$-limits. We are looking for a mapping $s \in \mathscr{S}$ such that the sequence $(S - T) \circ s$ is a canonical linear combination of $\mathfrak{L}$-convergent sequences in X and $g - h$ is the corresponding linear combination of their pointed $\mathfrak{L}$-limits. Clearly it suffices to show that there is a mapping $s \in \mathscr{S}$ such that for each pair (i, j) of indexes $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, l\}$ we have either $S_i \circ s = T_j \circ s$ or $(S_i \circ s)(n) \neq (T_j \circ s)(n)$ for all $n \in \mathbb{N}$. But this can be easily achieved. The mapping s is a superposition of $k \,.\, l$ elements of $\mathscr{S}$ constructed as follows. Consider the sequences S_1 and T_1. Either for some $t \in \mathscr{S}$ we have $(S_1 \circ t)(n) \neq (T_1 \circ t)(n)$ for all $n \in \mathbb{N}$ and then put $s_1 = t$, or there is a natural number m such that we have $S_1(n) = T_1(n)$ for all $n > m$ and then define $s_1 \in \mathscr{S}$ by $s_1(n) = n + m$. Now consider the sequences $S_2 \circ s_1$ and $T_1 \circ s_1$. Similarly as in the preceding step, there is a mapping $s_2 \in \mathscr{S}$ such that either the sequences $S_2 \circ s_1 \circ s_2$ and $T_1 \circ s_1 \circ s_2$ are identical or $(S_2 \circ s_1 \circ \circ s_2)(n) \neq (T_1 \circ s_1 \circ s_2)(n)$ for all $n \in \mathbb{N}$. This way we proceed up to the sequences $S_k \circ s_1 \circ \ldots \circ s_k$ and $T_1 \circ s_1 \circ \ldots \circ s_k$. Further, for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, l - 1\}$ we consider the sequences $S_i \circ s_1 \circ \ldots \circ s_{k.j+i-1}$ and $T_j \circ s_1 \circ \ldots \circ s_{k.j+i-1}$ and choose a mapping $s_{k.j+i} \in \mathscr{S}$ such that either the sequences $S_i \circ s_1 \circ \ldots \circ s_{k.j+i}$ and $T_j \circ \circ s_1 \circ \ldots \circ s_{k.j+i}$ are identical or $(S_i \circ s_1 \circ \ldots \circ s_{k.j+i})(n) \neq (T_j \circ s_1 \circ \ldots \circ s_{k.j+i})(n)$ for all $n \in \mathbb{N}$. Finally, put $s = s_1 \circ \ldots \circ s_{k.l}$. After collecting $\lambda_i(S_i \circ s)$ and $\mu_j(T_j \circ s)$ whenever $S_i \circ s = T_j \circ s$, we get $(S - T) \circ s$ as a canonical linear combination of $\mathfrak{L}$-convergent sequences in Y and $g - h$ as the corresponding linear combination of their pointed $\mathfrak{L}$-limits. This completes the proof.

Let us identify the neutral element e of G and the point $p \in Y$. Then Y can be considered as a subset $X \cup \{e\}$ of G.

Theorem 2. (The commutative case.) *$\mathfrak{G}^*$ restricted to Y coincides with $\mathfrak{L}$ and $\mathfrak{G}^*$ is the finest of all sequential group convergences for G the restriction of which to Y coincides with $\mathfrak{L}$. If p is an isolated point of Y, then X is sequentially closed in G.*

Proof. The assertion is an immediate consequence of (PCC).

2. THE NONCOMMUTATIVE CASE

Let Y be an infinite set equipped with a sequential convergence $\mathfrak{L} \subset Y^N \times Y$ satisfying axioms (L_0), (L_1), (L_2) and (L_3). Choose a point $p \in Y$. Let G be the free group over $X = Y \setminus \{p\}$. Recall that if $x \in G$, then x is either the empty word e, or $x = x_1^{\varepsilon_1} \ldots x_1^{\varepsilon_k}$, where for $i = 1, \ldots, k$ we have $x_i \in X$, $\varepsilon_i \in \{-1, 1\}$, and $\varepsilon_i = \varepsilon_{i+1}$ whenever $x_i = x_{i+1}$; such words are said to be *reduced*. Denote by $\sigma(y_1, \ldots, y_k)$ the product of elements $y_i \in G$, $i = 1, \ldots, k$. We are going to equip G with a a sequential convergence such that G becomes a pointed free group and the convergence is as fine as possible. Again, if p is an isolated point of Y, then our construction yields the usual free convergence group over X (cf. [FZ$_2$], [FZ$_3$]).

Define $\mathfrak{G} \subset G^N \times G$ as follows:

(PC) $(S, g) \in \mathfrak{G}$ *if either*

(PC_1) S is the constant sequence generated by the empty word e and $g = e$; or

(PC_2) S is a reduced product of $\mathfrak{L}$-convergent sequences $S_i \in X^N$, $i = 1, \ldots, k$, and g is the corresponding product of their pointed $\mathfrak{L}$-limit symbols (i.e. $(S_i, x_i) \in \mathfrak{L}$, $\varepsilon_i \in \{-1, 1\}$, $i = 1, \ldots, k$, $S(n) = S_1^{\varepsilon_1}(n) \ldots S_k^{\varepsilon_k}(n)$ is a reduced word for all $n \in \mathbb{N}$, and $g = \sigma(y_1^{\varepsilon_1}, \ldots, y_k^{\varepsilon_k})$ with $y_i = x_i$ for $x_i \neq p$ and $y_i = e$ for $x_i = p$).

Let $\mathfrak{G}^*$ be the Urysohn modification of $\mathfrak{G}$.

Theorem 3. (The noncommutative case.) *G equipped with $\mathfrak{G}^*$ is a sequential convergence group.*

Proof. It suffices to show (cf. [N_1]) that $\mathfrak{G}$ satisfies axioms (L_0), (L_1), (L_2) and (SG). In the noncommutative case all three axioms (L_0), (L_1) and (L_2) follow directly from (PC). It remains to prove (SG). Suppose that (S, g), $(T, h) \in \mathfrak{G}$. Similarly as in the proof of Theorem 1, we consider only condition (PC_2). Let S and T be reduced products $S_1^{\varepsilon_1} \ldots S_k^{\varepsilon_k}$ and $T_1^{\delta_1} \ldots T_l^{\delta_l}$ of $\mathfrak{L}$-convergent sequences $S_i \in X^N$ ($i = 1, \ldots, k$) and $T_i \in X^N$ ($i = 1, \ldots, l$), respectively, and let g and h be the corresponding products of their pointed $\mathfrak{L}$-limits.

Clearly, $(T^{-1}, h^{-1}) \in \mathfrak{G}$ according to (PC_2). Hence it suffices to show that there is a mapping $s \in \mathscr{S}$ such that the sequence $(ST) \circ s$ is a reduced product of $\mathfrak{L}$-convergent sequences in X and $\sigma(g, h)$ is the corresponding product of their pointed $\mathfrak{L}$-limits. Consider the sequences $S_k^{\varepsilon_k}$ and $T_1^{\delta_1}$. If there is a mapping $t \in \mathscr{S}$ such that $(S_k^{\varepsilon_k} \circ t)(n) \neq ((T_1^{\delta_1} \circ t)(n))^{-1}$ for all $n \in \mathbb{N}$, it suffices to put $s = t$. Otherwise there is a natural number m_1 such that $S_k^{\varepsilon_k}(n) = (T_1^{\delta_1}(n))^{-1}$ for all $n > m_1$ and hence $(\mathfrak{L}\text{-lim } S_k(n))^{\varepsilon_k} = (\mathfrak{L}\text{-lim } T_1(n))^{-\delta_1}$. In this case consider sequences S_{k-1} and T_2. Proceeding in this way, in j steps, $0 \leqq j \leqq \min(k, l)$, we get a mapping $s \in \mathscr{S}$ such that $S \circ s = S_1^{\varepsilon_1} \circ s \ldots S_{k-j}^{\varepsilon_{k-j}} \circ s T_{j+1}^{\delta_{j+1}} \circ s \ldots T_l^{\delta_l} \circ s$ is a reduced product and, putting $y_i = \mathfrak{L}\text{-lim } S_i(n)$ provided $p \neq \mathfrak{L}\text{-lim } S_i(n)$ and $y_i = e$ otherwise, or $z_i = \mathfrak{L}\text{-lim } T_i(n)$ provided $p \neq \mathfrak{L}\text{-lim } T_i(n)$ and $z_i = e$ otherwise, by the associativity of the operation σ we get $\sigma(g, h) = \sigma(y_1^{\varepsilon_1}, \ldots, y_{k-j}^{\varepsilon_{k-j}}, z_{j+1}^{\delta_{j+1}}, \ldots, z_l^{\delta_l})$. For $j = k = l$, $(ST) \circ s$ is the constant sequence generated by the empty word. This completes the proof.

Let us identify the neutral element e of G and the point p of Y. Then Y can be considered as a subset $X \cup \{e\}$ of G.

Theorem 4. (The noncommutative case.) *$\mathfrak{G}^*$ restricted to Y coincides with $\mathfrak{L}$ and $\mathfrak{G}^*$ is the finest of all sequential group convergences for G the restriction of which to Y coincides with $\mathfrak{L}$. If p is an isolated point of Y, then X is sequentially closed in G.*

Proof. The assertion is an immediate consequence of (PC).

P. Kratochvíl pointed out in [K] that a free group G (over $X = Y \setminus \{p\}$) equipped with the sequential group convergence introduced in [N_2], denote it by $\mathfrak{M}^*$ (this convergence is actually denoted by $\mathfrak{L}^*$ in [N_2] and [K]), fails to have the properties of a free object (recall that $\mathfrak{M}^*$ is constructed only in case when, in our notation, p is the only nonisolated point of Y and p is identified with $e \in G$). Namely, for each space Y with the only nonisolated point p he constructed a continuous mapping f of Y into the group R of real numbers such that f cannot be extended to a continuous homomorphism of G equipped with $\mathfrak{M}^*$ into R. We present a general construction shedding more light on the nature of $\mathfrak{M}^*$. The following statement describes the relationship between $\mathfrak{M}^*$ and $\mathfrak{G}^*$ and yields another general construction showing that $\mathfrak{M}^*$ fails to be the free convergence.

Theorem 5. *Let Y be an infinite set equipped with a sequential convergence $\mathfrak{L}$ satisfying axioms* (L_0), (L_1), (L_2) *and* (L_3). *Let $p \in Y$ be the only nonisolated point of Y. Then $\mathfrak{G}^*$ is strictly finer than $\mathfrak{M}^*$.*

Proof. Let S be a one-to-one sequence in Y such that $(S, p) \in \mathfrak{L}$. Define a sequence $T \in G^N$ as follows: for $n \in \mathbb{N}$ put $T(n) = S(2^n - 1) \dots S(2^{n+1} - 2)$. From the definitions of $\mathfrak{M}^*$ and $\mathfrak{G}^*$ we get $(T, e) \in \mathfrak{M}^* \setminus \mathfrak{G}^*$. (In fact, observe that the length of the word $T(n)$ tends to infinity.)

Example. Let f be the canonical mapping of Y equipped with $\mathfrak{L}$ into G equipped with $\mathfrak{G}^*$, i.e. $f(p) = e$ and $f(x) = x$ for $x \in X = Y \setminus \{p\}$. Then, by Theorem 4, f is a continuous mapping. The identity mapping of G is the only extension of f to a homomorphism of G into itself. Since it follows from Theorem 5 that the identity mapping of G equipped with $\mathfrak{M}^*$ into G equipped with $\mathfrak{G}^*$ is not continuous, f cannot be extended to a continuous homomorphism of G equipped with $\mathfrak{M}^*$ into G equipped with $\mathfrak{G}^*$.

References

[FZ_1] *R. Frič* and *F. Zanolin:* A sequential convergence group having no completion. (To appear.)

[FZ_2] *R. Frič* and *F. Zanolin:* Remarks on sequential convergence in free groups. (To appear.)

[FZ_3] *R. Frič* and *F. Zanolin:* Sequential convergence in free groups. (To appear.)

[HR] *E. Hewitt* and *K. A. Ross:* Abstract harmonic analysis, Vol. I. Springer-Verlag, Heidelberg, 1963.

[K] *P. Kratochvíl:* Sequential convergence as a partial operation. General Topology and its Relations to Modern Analysis and Algebra V (Proc. Fifth Prague Topological Sympos., 1981). Heldermann Verlag, Berlin, 1983, 442—447.

[N_1] *J. Novák:* On convergence groups. Czechoslovak Math. J. *20* (*95*) (1970), 357—374.

[N_2] *J. Novák:* On a free convergence group. Proc. Convergence Structures, Lawton 1980, Cameron University, Lawton, Oklahoma, 97—102.

[Z_1] *F. Zanolin:* Solution of a problem of Josef Novák about convergence groups. Bollettino Un. Mat. Ital. (5) *14*-A (1977), 375—381.

[Z_2] *F. Zanolin:* A note on convergences in commutative groups. Rend. Ist. Mat. Univ. Trieste *13* (1981), 79—86.

[Z_3] *F. Zanolin:* Example of a convergence commutative group which is not separated. Czechoslovak Math. J. *34* (*109*) (1984), 169—171.

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