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# WEAKLY REGULAR ALGEBRAS IN VARIETIES WITH PRINCIPAL COMPACT CONGRUENCES

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An algebra $A$ with a nullary operation 0 is *weakly regular* if every two congruence $\theta$, $\Phi$ on $A$ coincide whenver $[0]_\theta = [0]_\Phi$. Varieties of such algebras were characterized by many authors, see [5] or [6] and references therein.

It is an interesting problem to find weakly regular algebras in varieties which are not varieties of weakly regular algebras. One can find such attempts e.g. in [3].

An algebra $A$ with a nullary operation 0 has 0-*transferable principal congruences* (briefly 0-TPC) if for each $a, b \in A$ there exists an element $c$ of $A$ such that $\theta(a, b) = \theta(0, c)$. Varieties of such algebras were characterized in [1]. It is easy to prove that every variety $\mathscr{V}$ with a nullary operation 0 whose all members have 0-TPC is a variety of wekly regular algebras. There is a natural question: under which condition on $\mathscr{V}$ also the weak regularity of algebras of $\mathscr{V}$ implies 0-TPC. The aim of this paper is to pick out some broad class of varieties whose members have this property.

An algebra $A$ has *Principal Compact Congruences* if every compact congruence on $A$ is principal, i.e. if for every elements $a_i, b_i \in A$ $(i = 1, \ldots, n)$ there exist elements $a, b$ of $A$ such that

$$\theta(a_1, b_1) \vee \ldots \vee \theta(a_n, b_n) = \theta(a, b)$$

in the lattice Con $A$. Varieties of such algebras were characterized in [9], [8], [7], in the case of permutable varieties also in [2]. This conpcet can be modified in the following way:

**Definition.** Let $A$ be an algebra with a nullary operation 0. $A$ has 0-*Principal Compact Congruences* if for every elements $a_1, \ldots, a_n \in A$ there exists an element $a \in A$ such that

$$\theta(0, a_1) \vee \ldots \vee \theta(0, a_n) = \theta(0, a)$$

in Con $A$. A variety $\mathscr{V}$ with a nullary operation 0 has 0-*Principal Compact Congruences* if every $A \in \mathscr{V}$ has this property.

First we characterize such varieties by a Mal'cev type condition:

**Theorem 1.** *Let $\mathscr{V}$ be a variety with a nullary operation* 0. *The following conditions are equivalent*:

(1) *$\mathscr{V}$ has* 0-*Principal Compact Congruences*;

(2) *there exist a binary polynomial $p$ and 4-ary polynomials $r_1, \ldots, r_n, s_1, \ldots, s_m$ such that*

$p(0, 0) = 0,$
$0 = r_1(0, p(x, y), x, y),$
$r_i(p(x, y), 0, x, y) = r_{i+1}(0, p(x, y), x, y)$ *for* $i = 1, \ldots, n - 1,$
$x = r_n(p(x, y), 0, x, y),$
$0 = s_1(0, p(x, y), x, y),$
$s_j(p(x, y), 0, x, y) = s_{j+1}(0, p(x, y), x, y)$ *for* $j = 1, \ldots, m - 1,$
$y = s_m(p(x, y), 0, x, y).$

Proof. (1) $\Rightarrow$ (2): Let $\mathscr{V}$ be a variety with a nullary operation 0 which has 0-Principal Compact Congruences. Let $A = F_2(x, y)$ be a free algebra of $\mathscr{V}$ with two free generators $x, y$. Then there exists an element $a \in A$ such that

$$(*) \qquad \theta(0, x) \vee \theta(0, y) = \theta(0, a) .$$

Since $a \in F_2(x, y)$, there exists a binary polynomial $p$ of $\mathscr{V}$ such that $a = p(x, y)$. Then $(*)$ implies

$$\langle 0, x\rangle \in \theta(0, p(x, y)) ,$$
$$\langle 0, y\rangle \in \theta(0, p(x, y)) .$$

By Theorem 1 in [4], there exist 4-ary polynomials $r_1, \ldots, r_n, s_1, \ldots, s_m$ such that

$$0 = r_1(0, p(x, y), x, y) ,$$
$$r_i(p(x, y), 0, x, y) = r_{i+1}(0, p(x, y), x, y) \text{ for } i = 1, \ldots, n - 1,$$
$$x = r_n(p(x, y), 0, x, y) ,$$
$$0 = s_1(0, p(x, y), x, y) ,$$
$$s_j(p(x, y), 0, x, y) = s_{j+1}(0, p(x, y), x, y) \text{ for } j = 1, \ldots, m - 1,$$
$$y = s_m(p(x, y), 0, x, y) .$$

Let us inspect the factor algebra $A/\theta$, where $\theta = \theta(0, x) \vee \theta(0, y)$. Since $A/\theta \in \mathscr{V}$, the condition

$$\theta(0, x) \vee \theta(0, y) = \theta(0, p(x, y))$$

gives in $A/\theta$

$$\omega = \theta(0, 0) = \theta(0, 0) \vee \theta(0, 0) = \theta(0, p(0, 0)) ,$$

whence $p(0, 0) = 0$.

(2) $\Rightarrow$ (1): Let $\mathscr{V}$ be a variety with a nullary operation 0 satisfying (2). Let $A \in \mathscr{V}$ and let $a, b$ be elements of $A$. Then, by (2) and Theorem 1 of [4],

$$\langle 0, a\rangle \in \theta(0, p(a, b)) ,$$
$$\langle 0, b\rangle \in \theta(0, p(a, b)) ,$$

thus

$$\theta(0, a) \vee \theta(0, b) \subseteq \theta(0, p(a, b)) .$$

Further,

$$\langle 0, a\rangle \in \theta(0, a) \vee \theta(0, b) ,$$

$$\langle 0, b\rangle \in \theta(0, a) \vee \theta(0, b) ,$$

whence

$$\langle 0, p(a, b)\rangle = \langle p(0, 0), p(a, b)\rangle \in \theta(0, a) \vee \theta(0, b)$$

which implies the converse inclusion, thus

$$\theta(0, a) \vee \theta(0, b) = \theta(0, p(a, b)) .$$

By induction, we obtain (1). □

There exist varieties with 0-Principal Compact Congruences which have no Principal Compact Congruences:

Example 1. Every variety of lattices with the least element 0 has 0-Principal Compact Congruences.

We can put $n = m = 1$, $p(x, y) = x \vee y$ and $r_1(a, b, c, d) = a \wedge c$, $s_1(a, b, c, d) = a \wedge d$. Then $p(0, 0) = 0$,

$$r_1(0, p(x, y), x, y) = 0 \wedge x = 0 ,$$

$$r_1(p(x, y), 0, x, y) = (x \vee y) \wedge x = x ,$$

$$s_1(0, p(x, y), x, y) = 0 \wedge y = 0 ,$$

$$s_1(p(x, y), 0, x, y) = (x \vee y) \wedge y = y .$$

Example 2. The variety of all $\vee$-semilattices with the least element 0 has 0-Principal Compact Congruences.

We can put $n = m = 2$, $p(x, y) = x \vee y$,

$$r_1(a, b, c, d) = a , \quad r_2(a, b, c, d) = b \vee c ,$$

$$s_1(a, b, c, d) = a , \quad s_2(a, b, c, d) = b \vee d .$$

Then $p(0, 0) = 0$,

$$r_1(0, p(x, y), x, y) = 0 ,$$

$$r_1(p(x, y), 0, x, y) = x \vee y = x \vee (x \vee y) = r_2(0, p(x, y), x, y) ,$$

$$r_2(p(x, y), 0, x, y) = 0 \vee x = x ,$$

$$s_1(0, p(x, y), x, y) = 0 ,$$

$$s_1(p(x, y), 0, x, y) = x \vee y = (x \vee y) \vee y = s_2(0, p(x, y), x, y) ,$$

$$s_2(p(x, y), 0, x, y) = 0 \vee y = y .$$

The 0-principality of compact congruences can be characterized also in another way similar to that of B. Csákány [5] for regularity:

**Theorem 2.** *Let $\mathscr{V}$ be a variety with a nullary operation* 0. *The following conditions are equivalent*:
(1) *$\mathscr{V}$ has* 0-*Principal Compact Congruences*;
(2) *there exists a binary polynomial $b(x, y)$ of $\mathscr{V}$ such that*

$$b(x, y) = 0 \quad \textit{if and only if} \quad x = 0 \quad \textit{and} \quad y = 0 .$$

Proof. Let $\mathscr{V}$ be a variety of algebras with a nullary operation 0. Suppose $\mathscr{V}$ has 0-Principal Compact Congruences and let $F_2(x, y) \in \mathscr{V}$ be a free algebra with generators $x, y$. Then there exists a binary polynomial $b(x, y)$ such that

$$(**) \qquad \theta(0, x) \vee \theta(0, y) = \theta(0, b(x, y)) .$$

By the same argument in $F_2(x, y)/\theta$ for $\theta = \theta(0, x) \vee \theta(0, y)$ as in the proof of Theorem 1 we obtain $b(0, 0) = 0$. Conversely, suppose $b(x, y) = 0$.

Then $\theta(0, b(x, y)) = \theta(0, 0) = \omega$, thus $(**)$ implies

$$\theta(0, x) \vee \theta(0, y) = \omega .$$

Hence $\theta(0, x) = \omega$, $\theta(0, y) = \omega$ which gives $x = 0$ and $y = 0$.

Thus (1) $\Rightarrow$ (2) is true. Prove (2) $\Rightarrow$ (1): Clearly

$$\langle 0, x \rangle \in \theta(0, x) \vee \theta(0, y) ,$$
$$\langle 0, y \rangle \in \theta(0, x) \vee \theta(0, y)$$

gives

$$\langle 0, b(x, y) \rangle = \langle b(0, 0), b(x, y) \rangle \in \theta(0, x) \vee \theta(0, y) ,$$

thus $\theta(0, b(x, y)) \subseteq \theta(0, x) \vee \theta(0, y)$ for every $x, y \in A \in \mathscr{V}$, where $\mathscr{V}$ satisfies (2). Further, consider the factor algebra $A/\phi$ for $\phi = \theta(0, b(x, y))$. Then

$$[0]_\phi = [b(x, y)]_\phi = b([x]_\phi, [y]_\phi) .$$

Since $A/\phi \in \mathscr{V}$, by (2) also

$$[x]_\phi = [0]_\phi \quad \text{and} \quad [y]_\phi = [0]_\phi , \quad \text{i.e.} \quad \langle 0, x \rangle \in \phi \quad \text{and} \quad \langle 0, y \rangle \in \phi .$$

Hence $\theta(0, x) \subseteq \theta(0, b(x, y))$, $\theta(0, y) \subseteq \theta(0, b(x, y))$, thus $\theta(0, x) \vee \theta(0, y) \subseteq$ $\subseteq \theta(0, b(x, y))$. The condition (1) is evident. □

Example 3. For a variety of lattices with the least element 0 we can put $b(x, y) =$ $= x \vee y$. The same polynomial $b(x, y)$ can be chosen also for the variety of all $\vee$-semilattices with 0.

Example 4. By the same argument as in the previous example, every variety of p-algebras or Heyting algebras has 0-Principal Compact Congruences. A variety of all Boolean algebras has 0-Principal Compact Congruences.

Example 5. Every variety of loops has 0-Principal Compact Congruences (0 is the unit element).

We can put $b(x, y) = x \setminus y$.

Example 6. Although varieties of rings need not have Principal Compact Congruences, see [7], [9], every variety of rings has 0-Principal Compact Congruences. Clearly we can take $b(x, y) = x - y$.

Now, we can formulate our characterization of weakly regular algebras in varieties with 0-Principal Compact Congruences.

**Theorem 3.** *Let $\mathscr{V}$ be a variety with a nullary operation* 0. *Let $\mathscr{V}$ has* 0-*Principal Compact Congruences. The following conditions are equivalent for $A \in \mathscr{V}$:*

(1) *$A$ is weakly regular;*

(2) *$A$ has* 0-TPC.

Proof. (1) $\Rightarrow$ (2). Let $A$ be weakly regular and let $a$, $b$ be elements of $A$. Denote $N = [0]_{\theta(a,b)}$. Let $\theta(B)$ be the least congruence on $A$ such that

$$x, y \in B \Rightarrow \langle x, y \rangle \in \theta(B),$$

and let $\theta[B, C]$ be the least congruence on $A$ such that

$$x \in B, \quad y \in C \Rightarrow \langle x, y \rangle \in \theta[B, C]$$

for $B \subseteq A$, $C \subseteq A$.

Clearly $N$ is the congruence class of $\theta(N)$ and, by (1), $\theta(N) = \theta(a, b)$. Clearly $\theta(N) = \theta[\{0\}, N]$, thus

$$\theta(a, b) = \theta[\{0\}, N],$$

i.e. $\langle a, b \rangle \in \theta[\{0\}, N]$. This implies the existence of a finite subset $F \subseteq N$ with $\langle a, b \rangle \in \theta[\{0\}, F]$. Denote $F = \{c_1, \ldots, c_n\}$. Then we have

$$\langle a, b \rangle \in \theta(0, c_1) \vee \ldots \vee \theta(0, c_n).$$

Since $A$ has 0-Principal Compact Congruences, there exists an element $c$ of $A$ with

$$\theta(0, c_1) \vee \ldots \vee \theta(0, c_n) = \theta(0, c),$$

thus $\langle a, b \rangle \in \theta(0, c)$. Hence $\theta(a, b) \subseteq (0, c)$. Since $c_i \in N$, we have $\theta(0, c_i) \subseteq \subseteq \theta[\{0\}, N] = \theta(a, b)$, thus

$$\theta(0, c) = \theta(0, c_1) \vee \ldots \vee \theta(0, c_n) \subseteq \theta(a, b),$$

whence

$$\theta(a, b) = \theta(0, c),$$

i.e. $A$ has 0-TPC.

(2) $\Rightarrow$ (1): Let $A$ have 0-TPC and let $\theta \in \operatorname{Con} A$. Denote $N = [0]_\theta$. To prove (1) we only need to prove $\theta = \theta(N)$, i.e. that every congruence on $A$ is determined by its congruence class containing 0.

(i) Suppose $\langle a, b \rangle \in \theta$ and denote $N_{ab} = [0]_{\theta(a,b)}$. Evidently, $\theta(N_{ab}) \subseteq \theta(a, b)$. By (2), there exists an element $c \in N_{ab}$ such that $\theta(a, b) = \theta(0, c)$. However, $c \in N_{ab}$ implies $\theta(0, c) \subseteq \theta(N_{ab})$. Thus $\theta(a, b) = \theta(N_{ab})$.

(ii) Clearly $N_{ab} \subseteq N$ for all $\langle a, b\rangle \in \theta$. Hence

$$\theta(N_{ab}) \subseteq \theta(N) \subseteq \theta$$

and, by (i), we obtain

$$\theta = \bigvee\{\theta(a, b); \langle a, b\rangle \in \theta\} = \bigvee\{\theta(N_{ab}); \langle a, b\rangle \in \theta\} \subseteq \theta(N) \subseteq \theta\,,$$

i.e. $\theta = \theta(N)$ which completes the proof. □

Example 7. A lattice $L$ with the least element 0 is weakly regular if and only if all its congruences are of the form $\theta(0, x)$ for $x \in L$. E.g. the lattices in Fig. 1 have this property.

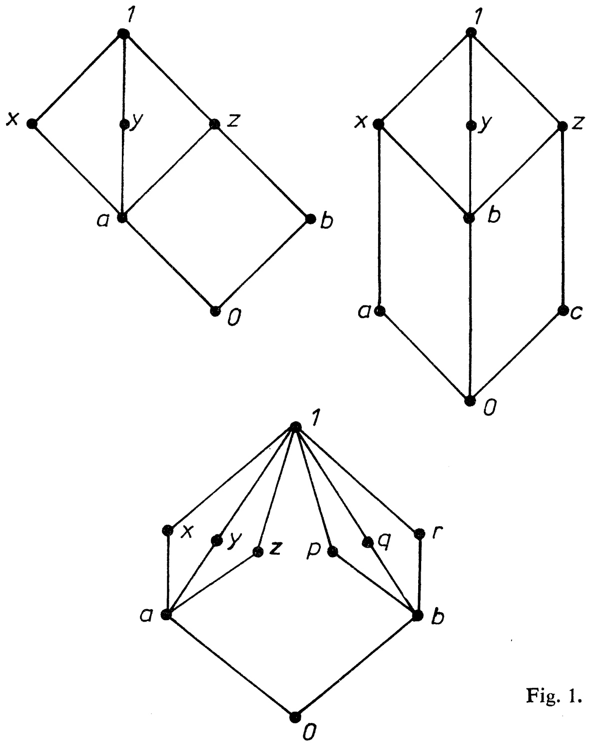


Fig. 1.

**Corollary.** *Join semilattice S with the least element* 0 *is weakly regular if and only if it has at most two elements.*

Proof. If $S$ has one or two elements, the assertion is trivial. Suppose $S$ has at least three elements. Then $S$ contains a three element chain.

$$0 < a < b\,.$$

Clearly $[0]_{\theta(a,b)} = \{0\}$, thus there exists no element $c \in S$ with $\theta(a, b) = \theta(0, c)$. Thus $S$ is not weakly regular.

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