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SUMMABLE SUBSEQUENCES IN CONVERGENCE GROUPS

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In this note, we present two examples which are related to solutions of problems concerning convergence groups which were put forth by J. Novák in [5].

In Problem 14 of [5], Novák asks for an example of a convergence group which has a sequence $\{x_i\}$ which converges to o but for which the sum $\sum_{i=1}^{\infty} x_{k_i}$ fails to exist for any subsequence $\{x_{k_i}\}$ of $\{x_i\}$. This problem has been solved by F. Zanolin ([6]) and by R. Frič and V. Koutník ([2]), but in both cases, the spaces involved in the solution are convergence groups. In Example 1, we present a normed space which contains such a sequence.

In Problem 15 of [5], Novák asks for an example of a convergence group which contains a sequence such that each subsequence contains one subsequence which is summable and another subsequence which is not summable. (A sequence $\{x_i\}$ is summable if the series $\sum_{i=1}^{\infty} x_i$ converges.) In [3], C. Klis has given an example of a normed space which contains such a sequence. (C. Ryll-Nardzewski had previously given such an example using the Continuum Hypothesis.) Following Example 1 we give a discussion concerning the types of convergent sequences which arise in this problem.

Henceforth, let m_0 be the subspace of l^∞ , the vector space of all bounded real sequences, which consists of those sequences $x = \{t_j\}$ with finite range, i.e., $Rx = \{t_j : 1 \leq j < \infty\}$ is finite. Equip both l^∞ and m_0 with the sup-norm, $\|x\| = \|\{t_j\}\| = \sup |t_j|$. Under the sup-norm, m_0 is a dense, proper subspace of l^∞ .

Concerning Problem 14 of [5], we have

Example 1. Let e_j be the sequence in m_0 which has a 1 in the j th coordinate and a zero in the other coordinates. Then the sequence $\{(1/j)e_j\}$ converges to o in m_0 but no sequence is summable to an element of m_0 . (If $\{(1/k_j)e_{k_j}\}$ is any subsequence of $\{(1/j)e_j\}$, then $\sum_{j=1}^{\infty} (1/k_j)e_{k_j}$ will not converge to an element of m_0 since each element of m_0 has finite range.)

Concerning Problem 15 of [5], we introduce the following terminology. Let G be an Abelian topological group (or convergence group). A sequence $\{x_i\}$ in G is said

to be \mathcal{H} convergent if each subsequence of $\{x_i\}$ has a subsequence $\{x_{k_i}\}$ such that the series $\sum_{i=1}^{\infty} x_{k_i}$ converges to an element of G . (These sequences are named in honor of Katowice, Poland, where the members of the Katowice Branch of the Mathematics Institute have introduced and studied many such properties of convergent sequences; an equivalent definition in metric linear spaces was introduced by S. Mazur and W. Orlicz in their study of the uniform boundedness principle, [4], Axiom II, p. 169.) A sequence $\{x_i\}$ in G is said to be \mathcal{N} convergent if $\{x_i\}$ has a subsequence $\{x_{k_i}\}$ such that each subsequence of $\{x_{k_i}\}$ is summable to an element of G . (Such sequences were introduced by L. S. Sobolev in Novosibirsk.)

Clearly, any sequence which is \mathcal{N} convergent is also \mathcal{H} convergent. Problem 15 of [5] asks for an example of a sequence which is \mathcal{H} convergent but not \mathcal{N} convergent. Klis has given an example of such a sequence in a normed space ([3]).

If G is a complete normed group, then a sequence $\{x_i\}$ in G is \mathcal{H} convergent iff it is \mathcal{N} convergent. (If $\{x_i\}$ is \mathcal{H} convergent, then $x_i \rightarrow o$ so there is a subsequence $\{x_{k_i}\}$ of $\{x_i\}$ such that $\sum |x_{k_i}| < \infty$, where $|\cdot|$ is the quasi-norm in G . The completeness of G then implies that any subseries of $\sum x_{k_i}$ is summable to an element of G . In fact, this argument shows that any sequence which converges to o is \mathcal{H} convergent.) The following question then naturally arises concerning the converse of the statement above.

Problem 2. Suppose the Abelian topological group (or convergence group) G has the property that a sequence in G is \mathcal{H} convergent iff it is \mathcal{N} convergent. Then is G necessarily complete? That is, does the converse of the statement in the paragraph above hold?

We show in Corollary 4 below that this conjecture is false.

Again, the example involves the space m_0 .

We first state a lemma concerning \mathcal{H} convergent sequences in m_0 . The lemma in both proof and content is modeled after Theorem 1 of [1] so we omit the proof. We introduce the following notation. If A is a subset of m_0 , $[A]$ denotes the linear span of A .

Lemma 3. *If $\{x_n\}$ is \mathcal{H} convergent in m_0 (in sup-norm), then $[x_n; n \in \mathbb{N}]$ is finite dimensional.*

Concerning the problem posed above, we have

Corollary 4. *A sequence in m_0 is \mathcal{H} convergent iff it is \mathcal{N} convergent.*

Proof. Let $\{x_i\}$ be \mathcal{H} convergent in m_0 . Then $\{x_i\}$ is contained in a finite dimensional subspace F of m_0 by Lemma 3. Thus, if a subsequence $\{x_{k_i}\}$ of $\{x_i\}$ is chosen such that $\sum_{i=1}^{\infty} \|x_{k_i}\| < \infty$, then $\sum x_{k_i}$ is subseries convergent in F by the completeness of F . That is, $\{x_i\}$ is \mathcal{N} convergent.

Since \mathcal{N} convergence always implies \mathcal{H} convergence, the result follows.

Thus, m_0 provides an example of an incomplete normed space with the property that any sequence in m_0 is \mathcal{K} convergent iff it is \mathcal{N} convergent. This gives a negative solution to Problem 2.

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