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OSCILLATORY PROPERTIES OF SOLUTIONS OF NONLINEAR 
DIFFERENTIAL SYSTEMS WITH DEVIATING ARGUMENTS

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INTRODUCTION

In this paper we consider the nonlinear differential system with deviating arguments:

(S) \[ \begin{align*}
    y'_i(t) &= p_i(t) y_{i+1}(t), & i &= 1, 2, \ldots, n - 2, \\
    y'_{n-1}(t) &= p_{n-1}(t) f_{n-1}(y_n(h_n(t))), \\
    y'_n(t) &= -p_n(t) f_n(y_1(h_1(t))).
\end{align*} \]

The following conditions are always assumed to be fulfilled:

(1) (a) \( p_i: [0, \infty) \to [0, \infty) \), \( i = 1, 2, \ldots, n \), are continuous functions and not identically zero on any subinterval of \( [a, \infty) \subset [0, \infty) \); \( \int_0^\infty p_i(t) \, dt = \infty \), \( i = 1, 2, \ldots, n \).

(b) \( h_i: [0, \infty) \to \mathbb{R}, \quad i = 1, n \), are continuous and \( \lim_{t \to \infty} h_i(t) = \infty \);

(c) \( f_i: \mathbb{R} \to \mathbb{R}, \quad i = n - 1, n \), \( f_i(u) > 0 \) for \( u > 0 \) and \( f_i(u) \) are nondecreasing in \( u \).

Definition 1. System (S) is called \((\alpha_{n-1}, \alpha_n)\) superlinear if there are positive numbers \( \alpha_{n-1}, \alpha_n \) such that \( \alpha_n \cdot \alpha_{n-1} > 1 \) and

\[ \frac{|f_i(u)|}{|u|^\alpha_i} = \frac{|f_i(v)|}{|v|^\alpha_i} \quad \text{for} \quad |u| > |v|, \quad u, v > 0, \quad i = n - 1, n. \]

Denote by \( W \) the set of all solutions \( y(t) = (y_1(t), \ldots, y_n(t)) \) of the system (S) which exist on some ray \( [T, \infty) \subset [0, \infty) \) and satisfy \( \sup \{ \sum_{i=1}^n |y_i(t)|; t \geq T \} > 0 \) for \( T \geq T_y \).

Definition 2. A solution \( y \in W \) is called oscillatory if each of its components has arbitrarily large zeros. A solution \( y \in W \) is called nonoscillatory (weakly nonoscillatory) if each of its components (at least one component, respectively) is eventually of a constant sign.
By Lemma 1 [4] it follows that every solution of (S) is either oscillatory or non-oscillatory.

**Definition 3.** We shall say that the system (S) has the property A, if every solution $y \in W$ is oscillatory for $n$ even, while for $n$ odd it is either oscillatory or $y_i \ (i = 1, 2, \ldots, n)$ tend monotonically to zero as $t \to \infty$.

The oscillation properties of two-dimensional nonlinear differential systems with deviating arguments were studied for example by Kitamura and Kusano [2, 3], Sevelo and Varech [5, 6, 7]. The oscillation results for $n$-dimensional systems were obtained by Foltynska and Werbowski, and by the present author [4].

In this paper we extend some results established in [7] to the system (S).

**OSCILLATION THEOREMS**

In what follows we shall use the following notations:

$$h_i^*(t) = \min \{h_i(t), t\}, \quad i = 1, n,$$

$$\gamma_i(t) = \sup \{s \geq 0; t > h_i^*(s)\} \quad \text{for} \quad t \geq 0, \quad i = 1, n,$$

$$\gamma(t) = \max \{\gamma_1(t), \gamma_n(t)\}.$$  

Let $i_k \in \{1, 2, \ldots, n\}, \ k \in \{1, 2, \ldots, n - 1\}, \ t, s \in [a, \infty)$. We define: $I_0 = 1$,

$$I_k(t, s; p_{i_k}, \ldots, p_{i_1}) = \int_s^t p_i(x) I_{k-1}(x, s; p_{i_k-1}, \ldots, p_{i_1}) \, dx.$$  

It is not difficult to verify that the following identities hold:

$$y_i(t) y_i(t) > 0 \quad \text{on} \quad [t_0, \infty) \quad \text{for} \quad i = 1, 2, \ldots, l,$$

$$(-1)^{n+i} y_i(t) y_i(t) > 0 \quad \text{on} \quad [t_0, \infty) \quad \text{for} \quad i = l + 1, \ldots, n.$$

**Lemma 1.** Let (1a)–(1c) hold. Let $y = (y_1, \ldots, y_n) \in W$ be a nonoscillatory solution of (S) on the interval $[a, \infty)$. Then there exist an integer $l \in \{1, 2, \ldots, n\}$, $n \equiv l \ (\mod 2)$, and a $t_0 \geq a$ such that

$$y_i(t) y_i(t) > 0 \quad \text{on} \quad [t_0, \infty) \quad \text{for} \quad i = 1, 2, \ldots, l,$$

$$(-1)^{n+i} y_i(t) y_i(t) > 0 \quad \text{on} \quad [t_0, \infty) \quad \text{for} \quad i = l + 1, \ldots, n.$$

**Lemma 2.** Let (1a)–(1c) hold. Let $y = (y_1, \ldots, y_n) \in W$ be a solution on the interval $[a, \infty)$. Then the following relations hold:

$$y_i(t) = \sum_{j=0}^{m} (-1)^j y_{i+j}(s) I_j(s, t; p_{i+j}, \ldots, p_i) +$$

$$+ (-1)^{m+1} \int_t^s y_{i+m+1}(x) p_{i+m}(x) I_m(x, t; p_{i+m}, \ldots, p_i) \, dx.$$  

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for $0 \leq m \leq n - i - 2,$ $1 \leq i \leq n - 2,$ $t, s \in [a, \infty);$

\[
y_i(t) = \sum_{j=0}^{n-i-1} (-1)^j y_{i+j}(t) I_j(t, s; p_{i+j-1}, \ldots, p_i) +
+ (-1)^{n-i} \int_s p_{n-1}(x) f_{n-1}(y_n(h_n(x))) I_{n-i-1}(x, s; p_{n-2}, \ldots, p_i) \, dx
\]

for $i = 1, 2, \ldots, n - 1,$ $t, s \in [a, \infty).$

The proofs of Lemma 1 and Lemma 2 are found in the paper [4].

**Lemma 3.** Let \((1a)-(1c)\) hold. Let \(y = (y_1, \ldots, y_n) \in \mathbb{W}\) be a nonoscillatory solution of \((S)\) on the interval \([a, \infty)\) with \(y_1(t) > 0\) for \(t \geq a.\)

Then there exist an integer \(l \in \{1, 2, \ldots, n\}, l \equiv n \, (\text{mod} \, 2),\) and a \(t_0 \geq a\) such that \((4), (5)\) hold,

\[
y_i(t) \geq \int_{t_0}^t H_{l, l-1}(s, t_0) p_{n-1}(s) f_{n-1}(y_n(h_n(s))) \, ds,
\]

for \(l \in \{2, 3, \ldots, n\}, i = 1, 2, \ldots, l - 1, t \geq t_0,

where

\[
H_{l, l-1}(s, t_0) = \int_{t_0}^s I_{l-1}(t, x; p_l, \ldots, p_{l-1}) I_{l-1}(x) \times
\times I_{n-1}(s, x; p_{n-2}, \ldots, p_l) \, dx,
\]

\(l \in \{2, 3, \ldots, n-1\}, s \geq t_0,

\[
H_{l, n-1}(s, t_0) = I_{n-1}(t, s; p_l, \ldots, p_{n-2}), \quad l = n, \quad t_0 \leq s \leq t.
\]

**Proof.** We put \(m = l - i - 1, s = t_0\) in \((6)\) and use \((3), (4).\) Then we have

\[
y_i(t) = \sum_{j=0}^{n-i-1} (-1)^j y_{i+j}(t) I_j(t, t_0; p_l, \ldots, p_{l+j-1}) +
+ \int_{t_0}^t y_i(u) p_{l-1}(u) I_{l-1}(t, u; p_l, \ldots, p_{l-2}) \, du \geq
\]

\[
\int_{t_0}^t y_i(u) p_{l-1}(u) I_{l-1}(t, u; p_l, \ldots, p_{l-2}) \, du \quad \text{for} \quad i = 1, 2, \ldots, l - 1, \quad t \geq t_0.
\]

On the other hand, we put \(i = l, s = u\) in \((7)\) and using \((5)\) for \(t \geq u\) we then have

\[
y_i(u) = \sum_{j=0}^{n-l-1} (-1)^j y_{i+j}(t) I_j(t, u; p_{i+j-1}, \ldots, p_i) +
+ (-1)^{n-l} \int_u^t p_{n-1}(x) f_{n-1}(y_n(h_n(x))) I_{n-1}(x, u; p_{n-2}, \ldots, p_l) \, dx \geq
\]

\[
\int_u^t p_{n-1}(x) f_{n-1}(y_n(h_n(x))) I_{n-1}(x, u; p_{n-2}, \ldots, p_l) \, dx.
\]
Substituting (12) into (11), we get

\[ y_i(t) \geq \int_0^t \left( p_{t-1}(u) I_{t-1}(t, u; p_1, \ldots, p_{t-2}) \int_u^t p_{n-1}(x) f_{n-1}(y_n(h_n(x))) \right) \times \]
\[ \times I_{n-1}(x, u; p_{n-2}, \ldots, p_1) \, dx \, du = \]
\[ = \int_0^t H_{i, t-1}(x, t_0) p_{n-1}(x) f_{n-1}(y_n(h_n(x))) \, dx . \]

Let \( l = n \). Put \( t = t_0, s = t \) in (7) and use (3) and (4). We get

\[ y_i(t) \geq \int_0^t p_{n-1}(x) I_{t-1}(t, x; p_1, \ldots, p_{n-2}) f_{n-1}(y_n(h_n(x))) \, dx \quad \text{for} \quad t \geq t_0 . \]

The proof of the lemma is complete.

Let us denote

\[ \phi_n(t) = \int_t^\infty p_n(s) \, ds, \]
\[ J_{k,n}(t, t_0) = I_{n-1}(t, t_0; p_k, \ldots, p_{n-1}), \]
\[ J_{k,l}(t, t_0) = \int_{t_0}^t H_{k, l-1}(s, t_0) p_{n-1}(s) \, ds \quad \text{for} \quad l = 1, 2, \ldots, n - 1 . \]

**Theorem 1.** Let there exist a continuous nondecreasing function \( g \) on \([a, \infty)\) such that

(13) \[ h_n(t) \leq g(t), \quad g(h_1(t)) \leq t . \]

Let (14) i) \( f_n(u, v) \geq K f_n(u) f_n(v) \) (0 < \( K = \text{const.} \));

ii) \( \int_0^\infty \frac{dx}{f_n(f_{n-1}(x))} < \infty, \quad \int_0^{-a} \frac{dx}{f_n(f_{n-1}(x))} < \infty \)

for every constant \( \alpha > 0 \);

(15) \[ \int_U^\infty p_n(t) f_n(J_{1,l}(h_1(t), T)) \, dt = \infty \quad \text{for} \quad l = 2, 3, \ldots, n . \]

If \( n \) is odd, suppose in addition that for every constant \( L > 0 \),

(16) \[ \int_T^\infty p_{n-1}(t) I_{n-2}(L, \phi_n(h_n(t))) \, dt = \infty . \]

Then the system (S) has the property A.

**Proof.** Let \( y = (y_1, \ldots, y_n) \in W \) be a nonoscillatory solution of (S). Without loss of generality we may suppose that \( y_1(t) > 0, y_1(h_1(t)) > 0 \) for \( t \geq t_1 \geq a \). Then the \( n \)-th equation of (S) implies that \( y_n'(t) \leq 0 \) for \( t \geq t_1 \) and it is not identically zero on any subinterval of \([t_1, \infty)\). Because \( y_1(t) > 0, y_1'(t) \leq 0 \) for \( t \geq t_1 \), then by Lemma 3, for \( t \geq t_2 \geq t_1 \) (4), (5) and (8) hold.
Let \( l \in \{2, 3, \ldots, n\} \). For \( i = 1 \), \( t_0 = t_2 \), using the monotonicity of \( y_n, f_{n-1} \), (13) and (3), we obtain from (8) that

\[
y_1(t) \geq \int_{t_2}^{t} H_{1,l-1}(s, t_2) p_{n-1}(s) f_{n-1}(y_n(h_s(s))) \, ds \geq f_{n-1}(y_n(g(t))) J_{1,l}(t, t_2), \quad t \geq t_2.
\]

Putting (17) into the \( n \)-th equation of (S) and then using (13), (14i), we get

\[
y'_n(t) \geq -p_n(t) f_n(y_n(h_1(t))) \leq -p_n(t) f_n(f_{n-1}(y_n(g(h_1(t)))) J_{1,l}(h_1(t), t_2)) \leq -p_n(t) f_n(f_{n-1}(y_n(t))) J_{1,l}(h_1(t), t_2) \leq -K p_n(t) f_n(f_{n-1}(y_n(t))) f_n(J_{1,l}(h_1(t), t_2))
\]

for \( t \geq t_3 = \gamma(t_2), l = 2, 3, \ldots, n \).

Dividing (18) by \( f_n(f_{n-1}(y_n(t))) \) and then integrating from \( t_3 \) to \( u(\geq t_3) \), we get

\[
\int_{t_3}^{u} \frac{y'_n(t)}{f_n(f_{n-1}(y_n(t)))} \, dt \leq -K \int_{t_3}^{u} p_n(t) f_n(J_{1,l}(h_1(t), t_2)) \, dt.
\]

From (19) for \( u \to \infty \) we obtain

\[
K \int_{t_3}^{\infty} p_n(t) f_n(J_{1,l}(h_1(t), t_2)) \, dt \leq \int_{0}^{p_n(t_3)} \frac{dx}{f_n(f_{n-1}(x))} < \infty,
\]

which contradicts (15).

Let \( l = 1 \) \((n \) is odd). Then \( y_1(t) \downarrow k \) as \( t \uparrow \infty \), where \( k \geq 0 \). We suppose that \( k > 0 \). If we put \( i = 1, s = t_2 \) in (7) and use (5), we have

\[
y_1(t_2) \geq \int_{t_2}^{t} p_{n-1}(x) f_{n-1}(y_n(h_1(x))) I_{n-2}(x, t_2; p_{n-2}, \ldots, p_1) \, dx \quad \text{for} \quad t \geq t_2.
\]

Integrating the \( n \)-th equation of (S) from \( t \) to \( \infty \) and using \( y_1(t) \geq k \) for \( t \geq t_2 \), we get

\[
y_n(t) \geq f_n(k) \int_{t}^{\infty} p_n(s) \, ds = L \phi_n(t), \quad \text{where} \quad L = f_n(k) \neq 0.
\]

Then in view of the monotonicity of \( y_n, f_{n-1} \) and (13), the inequality (20) yields

\[
y_1(t_2) \geq \int_{t_2}^{t} p_{n-1}(x) I_{n-2}(x, t_2; p_{n-2}, \ldots, p_1) f_{n-1}(L \phi_n(h_1(x))) \, dx,
\]

which contradicts (16) for \( t \to \infty \).

Therefore \( \lim_{t \to \infty} y_i(t) = 0 \) for \( i = 1, 2, \ldots, n \).

Remark. Theorem 1 extends the results of the author [4; Theorem 3], Kitamura and Kusano [3; Theorem 6], Sevelo and Varech [7; Theorem 1].

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Theorem 2. Suppose that (14), (16) hold and
\[ h_n(t) \leq t, \quad h_1(t) \geq t \quad \text{on} \quad [a, \infty). \]
If
\[ \int_a^\infty p_n(t) f_n(J_1, t(t, T)) \, dt = \infty \quad \text{for} \quad l = 2, 3, \ldots, n, \]
then the system (S) has the property A.

Proof. Let \( y = (y_1, \ldots, y_n) \in W \) be a nonoscillatory solution of (S) such that \( y_1(h_1(t)) > 0 \) for \( t \geq t_1 \). Proceeding in the same way as in the proof of Theorem 1 we get (4), (5), (7) and (8) for \( t \geq t_2 \geq t_1 \).

I. Let \( l \in \{2, 3, \ldots, n\} \). For \( i = 1, t_0 = t_2 \), using (21) and the monotonicity of \( y_n, f_{n-1} \), we obtain from (8) that
\[ y_j(t) \geq f_n(y_j(t)) J_1, J(t, t_2), \quad t \geq t_2. \]
If we put the last inequality into the \( n \)-th equation, we get
\[ y_j(t) \leq -p_n(t) J_n(y_j(t)) \leq -K p_n(t) f_n(f_{n-1}(y_j(t))) J_n(J_1, t(t, t_2)) \quad \text{for} \quad l = 2, 3, \ldots, n, \quad t \geq t_2. \]
Dividing (23) by \( f_n(f_{n-1}(y_j(t))) \) and then integrating from \( t_2 \) to \( \tau \to \infty \) we get a contradiction to (22).

II. If \( l = 1 \) \((n \text{ is odd})\) we proceed in the same way as in the case II of the proof of Theorem 1.

Theorem 3. Let the system (S) be \( (\sigma_{n-1}, \sigma_n) \) superlinear. Let
\[ g_1(t) \leq \min \{h_1(t), t\}, \quad h_n(t) \leq t \quad \text{on} \quad [a, \infty), \]
where \( g_1 \) is an increasing function on \([a, \infty)\) and \( \lim_{t \to \infty} g_1(t) = \infty \).

Let
\[ \int_a^\infty p_n(t) \, dt < \infty, \]
\[ \int_a^\infty J_2, t(g_1(t), a) p_1(g_1(t)) g_1'(t) f_{n-1}(K \phi_n(t)) \, dt = \infty \]
for any constant \( K > 0 \), \( l = 3, 4, \ldots, n \).

In addition we suppose that a) for \( n \) even,
\[ \int_a^\infty p_1(g_1(t)) g_1'(t) \int_t^\infty p_{n-1}(x) f_{n-1}(K \phi_n(x)) l_{n-3}(x, g_1(t); p_{n-2}, \ldots, p_2) \, dx \, dt = \infty \]
for any \( K > 0 \);

b) for \( n \) odd, (16) holds.

Then the system (S) has the property A.
Proof. Let \( y = (y_1, \ldots, y_n) \in W \) be a nonoscillatory solution of (S). Proceeding in the same way as in the proof of Theorem 1, we get (4), (5), (7) and (8). We suppose that \( y_1(t) > 0, y_1(h_1(t)) > 0 \) for \( t \geq T_1 \). Integrating the \( n \)-th equation of (S) from \( t(\geq T_1) \) to \( \tau \), we get
\[
y_n(\tau) - y_n(t) = - \int_t^\tau p_n(s) f_n(y_1(h_1(s))) \, ds ,
\]
and then for \( \tau \to \infty \) we have
\[
y_n(t) \geq \int_t^\infty p_n(s) f_n(y_1(h_1(s))) \, ds , \quad t \geq T_1 .
\]

I. Let \( l \geq 2 \). Then \( y_1 \) is nondecreasing and therefore \( y_1(h_1(t)) \geq c \) for some \( c > 0 \) and \( t \geq T_2 \geq T_1 \). Using the fact that the system (S) is superlinear, we obtain
\[
f_n(y_1(h_1(t))) \geq \frac{f_n(c)}{c^{\alpha_n}} (y_1(h_1(t))) = c^{-\alpha_n} f_n(c) (y_1(h_1(t)))^{\alpha_n} \quad \text{for} \quad t \geq T_3 \geq T_2 .
\]

Combining (29) with (28) we get
\[
y_n(t) \geq c^{-\alpha_n} f_n(c) \int_t^\infty p_n(s) (y_1(h_1(s)))^{\alpha_n} \, ds , \quad t \geq T_3 .
\]

Because \( y_1(h_1(t)) \geq c \) for \( t \geq T_2 \), (28) implies
\[
y_n(g_1(t)) \geq f_n(c) \int_{g_1(t)}^\infty p_n(s) \, ds = M \phi_n(g_1(t)) , \quad \text{where} \quad M = f_n(c) .
\]

In view of (30), (24) and the monotonicity of \( y_n \) we have
\[
y_n(g_1(t)) \geq y_n(t) \geq c^{-\alpha_n} M \int_t^\infty p_n(s) (y_1(h_1(s)))^{\alpha_n} \, ds .
\]

Using the superlinearity of \( f_{n-1} \) and (31), we get
\[
f_{n-1}(y_n(g_1(t))) \geq \frac{f_{n-1}(M \phi_n(t))}{(M \phi_n(t))^{\alpha_{n-1}}} (y_n(g_1(t)))^{\alpha_{n-1}} .
\]

a) Let \( l \geq 3 \). We put \( i = 2, T_3 = t_0 \) in (8) and using the monotonicity of \( f_{n-1}, y_n \) and (24), we obtain
\[
y_{2}(t) \geq \int_{T_1}^{t} H_{2,1-1}(s, T_3) p_{n-1}(s) f_{n-1}(y_n(h_n(s))) \geq f_{n-1}(y_n(t)) J_{2,1}(t, T_3) \quad (l = 3, 4, \ldots, n - 1) ,
\]
and
\[
y_{2}(t) \geq \int_{T_3}^{t} I_{n-3}(t, s; p_2, \ldots, p_{n-2}) p_{n-1}(s) f_{n-1}(y_n(h_n(s))) \, ds \geq f_{n-1}(y_n(t)) J_{2,1}(t, T_3) .
\]
Substituting (33) and (32) in (34), we get
\[
\frac{y_2(g_1(t))}{y_1(g_1(t))} \geq f_{n-1}(y_n(g_1(t))) J_{2,1}(g_1(t), T_3) \geq \frac{f_{n-1}(M \phi_n(t))}{(M \phi_n(t))^{\alpha-1}} \left( M c^{-\alpha} \int_t^\infty p_n(s) (v_1(g_1(s)))^\alpha ds \right)^{\alpha-1} J_{2,1}(g_1(t), T_3) \geq \frac{f_{n-1}(M \phi_n(t)) c^{-\alpha}(y_1(g_1(t)))^\alpha J_{2,1}(g_1(t), T_3)}{\alpha},
\]
where \( \alpha = \alpha_n \alpha_{n-1} > 1, \ l = 3, 4, \ldots, n. \)

Multiplying the last inequality by \( p_1(g_1(t)) (y_1(g_1(t)))^{-\alpha} g_1'(t) \) and using the first equation of (S), we get
\[
(35) \quad \frac{y_1'(g_1(t)) g_1'(t)}{(y_1(g_1(t)))^\alpha} \geq c^{-\alpha} f_{n-1}(M \phi_n(t)) J_{2,1}(g_1(t), T_3) p_1(g_1(t)) g_1'(t).
\]

Integrating (35) from \( T_4 = \gamma(T_3) \) to \( \tau \), we obtain
\[
\frac{c^\alpha}{\alpha - 1} \left[ y_1(g_1(T_3)) \right]^{1-\alpha} \geq \int_{T_4}^{\tau} J_{2,1}(g_1(t), T_3) p_1(g_1(t)) g_1'(t) f_{n-1}(M \phi_n(t)) dt,
\]
which contradicts (26) as \( \tau \to \infty \).

Let \( L = 2 \). We put \( i = 2 \) in (7) and use (5), obtaining
\[
(36) \quad y_2(t) \geq \int_t^\tau p_{n-1}(x) f_{n-1}(y_n(h_n(x))) I_{n-3}(x, t; p_{n-2}, \ldots, p_2) dx \quad \text{for} \quad \tau \geq t.
\]

Using the superlinearly of \( f_{n-1} \), (24) and (30), we obtain
\[
y_2(g_1(t)) \geq \int_{g_1(t)}^\tau p_{n-1}(x) f_{n-1}(M \phi_n(x)) (y_n(x))^{\alpha-1}.
\]
\[
\cdot I_{n-3}(x, g_1(t); p_{n-2}, \ldots, p_2) dx, \quad t \geq T_3.
\]

Multiplying the last inequality by \( p_1(g_1(t)) g_1'(t) \) and using the first equation of (S), (32) and (24), we get
\[
(37) \quad y_1'(g_1(t)) g_1'(t) \geq p_1(g_1(t)) g_1'(t) \int_t^\tau p_{n-1}(x) f_{n-1}(M \phi_n(x)) c^{-\alpha}(y_1(g_1(x)))^\alpha.
\]
\[
\cdot I_{n-3}(x, g_1(t); p_{n-2}, \ldots, p_2) dx \geq \geq c^{-\alpha}(y_1(g_1(t)))^\alpha p_1(g_1(t)) g_1'(t) \int_t^\tau p_{n-1}(x) f_{n-1}(M \phi_n(x)).
\]
\[
\cdot I_{n-3}(x, g_1(t); p_{n-2}, \ldots, p_2) dx, \quad t \geq T_3.
\]

Let \( g_1(t) \geq T_3 \) for \( t \geq T_4 \). Multiplying (37) by \( c^\alpha y_1(g_1(t))^{-\alpha} \) and then integrating from \( T_4 \) to \( u \), we get
\[
\frac{c^\alpha}{\alpha - 1} \left( y_1(g_1(T_4)) \right)^{1-\alpha} \geq \int_{T_4}^{u} \left( p_1(g_1(t)) g_1'(t) \int_t^\tau p_{n-1}(x) f_{n-1}(x) f_{n-1}(M \phi_n(x)) \cdot I_{n-3}(x, g_1(t); p_{n-2}, \ldots, p_2) dx \right) dt,
\]
which contradicts (27) as \( u \to \infty, \tau \to \infty. \)
II. Let \( l = 1 \) (\( n \) is odd). Then we proceed in the same way as in the proof of Theorem 1. This completes the proof of the theorem.

References


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