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Georges Hansoul

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BOOLEAN ALGEBRAS WITH A UNARY OPERATOR

GEORGES HANSOUL, Liège

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Introduction. Closure algebras have been intensively studied since 1944 ([5]) and have proved to be of great importance, mainly in their relationship with logic. In 1972, Pierce introduces the more general concept of topological Boolean algebra ([6]), see also [5], p. 108), which is much useful in a classification of countable Boolean algebras.

In the first part of this paper, we show that the properties of the congruence lattice of a closure algebra remain valid for a wider class of Boolean algebras with a unary operator that includes all topological Boolean algebras.

Also, it is well-known ([1]) that the structure of the free closure algebra on one generator is quite complex. If we abandon the closure axioms for the operator, the structure becomes much more complicated. To have some idea of the complexity reached, we study in the second part simple equations or inequations involving the operator only.

1. THE CONGRUENCE LATTICE

1.1. Definition. *A Boolean algebra with a unary operator* (abbreviated UBA) is an algebra $\mathscr{B} = (B; \vee, \wedge, {}^c, 0, 1, f)$ where $(B, \vee, \wedge, {}^c, 0, 1)$ is a Boolean algebra and f is a unary operator on B which is assumed to be *additive* $(f(a \vee b) = f(a) \vee f(b))$ and *normal* $(f(0) = 0)$. Among the UBA are the *closure algebras* $(a \leqq f(a)$ and $ff(a) \leqq f(a))$ and Pierce's *topological Boolean algebras* $(ff(a) \leqq f(a))$.

Recall that an element a of a UBA $\mathscr{B}$ is *closed* if $f(a) \leqq a$. The lattice $\mathscr{L}$ of all closed elements forms a $\{0, 1\}$-sublattice of $\mathscr{B}$. In case $\mathscr{B}$ is a closure algebra, $\mathscr{L}$ is dually relatively pseudocomplemented ([2]). This property remains true (see 1.2) in the class $\mathfrak{C}$ of all UBA satisfying the following axiom:

$$\text{(c)} \quad \forall x\,, \quad \exists n \in \mathbb{N}\,, \quad f^{(n+1)}(x) \leqq \bigvee_{i=0}^{n} f^{(i)}(x)\,,$$

where $\mathbb{N}$ denotes the set of natural numbers and $f^{(i)}: B \to B$ is defined inductively by $f^{(0)}(x) = x$ and $f^{(i+1)}(x) = f(f^{(i)}(x))$.

1.2. Definition. Axiom (c) enables to associate with each $\mathscr{B} \in \mathfrak{C}$ a closure algebra $\mathrm{Cl}(\mathscr{B})$ whose properties are a first approximation of those of $\mathscr{B}$. The Boolean part $\mathrm{Cl}(\mathscr{B})$ is the same as that of $\mathscr{B}$ and $\mathrm{Cl}(\mathscr{B}) = (B; \vee, \wedge, {}^c, 0, 1, \mathrm{Cl}(f))$ where $\mathrm{Cl}(f)$, the *closure operator associated with* f, is defined by

$$\mathrm{Cl}(f)(x) = \bigvee_{i=0}^{\infty} f^{(i)}(x)$$

(axiom (c) makes possible this definition). It is clear that $\mathrm{Cl}(\mathscr{B})$ has the same closed elements as $\mathscr{B}$. To study the congruence lattice of $\mathscr{B}$ (denoted by Con $\mathscr{B}$), we recall that, due to the additivity of f, $f(x \wedge y^c) \geqq f(x) \wedge (f(y))^c$. Hence,

1.3. Lemma. *Let $\mathscr{B}$ be a UBA and suppose θ is an equivalence on B. Then $\theta \in$ $\in \mathrm{Con}\,\mathscr{B}$ if and only if its kernel $I = \{x \in B \mid x\ \theta\ \mathrm{o}\}$ is an ideal satisfying "$x \in I$ implies $f(x) \in I$".*

1.4. Proposition. *Let $\mathscr{B} \in \mathfrak{C}$. Then* $\mathrm{Con}\, B = \mathrm{Con}\,\mathrm{Cl}(\mathscr{B})$.

Proof. It is clear that $\mathrm{Con}\,\mathscr{B} \subseteq \mathrm{Con}\,\mathrm{Cl}(\mathscr{B})$. If $\theta \in \mathrm{Con}\,\mathrm{Cl}(\mathscr{B})$, then its kernel is an ideal I satisfying "$x \in I$ implies $f(x) \leqq \mathrm{Cl}(f)(x) \in I$", whence the conclusion by 1.3.

1.5. Corollary. *Let $\mathscr{B} \in \mathfrak{C}$. Then $\mathscr{B}$ is subdirectly irreducible if and only if the sublattice $\mathscr{L}$ of its closed elements has a smallest non zero element (in other words, if $\mathscr{L}$ is subdirectly irreducible as a dual Heyting algebra),*

1.6. Corollary. *Let $\mathscr{C}$ be a lattice. Then $\mathscr{C}$ is isomorphic to* $\mathrm{Con}\,\mathscr{B}$ *for some $\mathscr{B} \in \mathfrak{C}$ if and only if $\mathscr{C}$ is a distributive algebraic lattice in which the compact elements form a dually relatively pseudocomplemented $\{0, 1\}$-sublattice.*

This result is folklore for closure algebras. In fact, a congruence on $\mathscr{B} \in \mathfrak{C}$ is compact if and only if its kernel is a principal ideal generated by a closed element. Note this is no longer true if $\mathscr{B}$ does not belong to $\mathfrak{C}$. Take for instance the Boolean algebra $\mathscr{B}$ of all finite and cofinite subsets of $\mathbb{N}$, together with the operator f defined by $f(E) = \{e + 1 \mid e \in E\}$ for $E \in B$. Then the congruence θ_n $(n \in \mathbb{N})$ whose kernel is the set of all finite subsets of $\{m \in \mathbb{N} \mid m \geqq n\}$ is compact.

2. EQUATIONAL CLASSES OF BOOLEAN ALGEBRAS WITH A UNARY OPERATOR

As announced in the introduction, we shall consider special equational classes of UBA. More precisely, for $n \in \mathbb{N}$ and $j \in \mathbb{N}$, we shall consider the equational class $\mathfrak{C}(n, j)$ defined by the following axiom:

$$C(n, j) : f^{(n)}(x) \leqq f^{(j)}(x) .$$

Note that the class of all closure algebras is $\mathfrak{C}(0, 1) \cap \mathfrak{C}(2, 1)$ while $\mathfrak{C}(2, 1)$ is the class of all topological Boolean algebras. Also, if $n > j$, then $\mathfrak{C}(n, j) \subseteq \mathfrak{C}$.

To determine the structure of these equational classes when n and j run through $\mathbb{N}$, we need to solve a problem of elementary arithmetic.

2.1. Definition. Let $(n, j, m, k) \in \mathbb{N}^4$. We say that n, j, m, k are *well related*, or that k is well related to (n, j, m), if

(w) there exist $l, \alpha_1, \beta_1, \ldots, \alpha_l, \beta_l$ in $\mathbb{N}$ such that $m = \alpha_1 n + \beta_1, \ldots, \alpha_t j + \beta_t = \alpha_{t+1} n + \beta_{t+1}, \ldots, \alpha_l j + \beta_l = k$.

In order to shorten proofs, we shall interpret – rather naturally – condition (w) particularized to the case $l = 0$ as $m = k$. We let $W(n, j, m)$ to be set of all k that are well related to (n, j, m). It is possible to have a more workable equivalent definition. In what follows, $E(x)$ denotes the greatest natural number p such that $p \leqq x$ (for x a non negative real).

2.2. Proposition. *Let $(n, j, m, k) \in \mathbb{N}^4$. Then the following are equivalent:*

(1) *n, j, m, k are well related;*

(2) *if $m < n$, then $k = m$,*
if $m \geqq n$ and $j \geqq n$, there exists $\alpha \in \mathbb{N}$ with $k - m = \alpha(j - n)$,
if $m \geqq n$ and $j < n$, there exists $\alpha \in \mathbb{N}$ with $k - m = \alpha(j - n)$ and $\alpha \leqq E((m - j)/(n - j)$.

Proof. The cases $m < n$ and $n = 0$ are easy to handle and we may assume that $m \geqq n > 0$. We define a sequence $(E_l \mid l \in \mathbb{N})$ of natural numbers by

$$E_0 = 0, \quad \text{and} \quad E_{l+1} = E\left(\frac{m}{n} + \frac{j}{n} E_l\right).$$

Proposition 2.2 follows from the next two lemmas.

Lemma 1. *Let $l \in \mathbb{N}$. Then the following are equivalent:*

(1) *there exists $\alpha_1, \beta_1, \ldots, \alpha_l, \beta_l$ in $\mathbb{N}$ such that $m = \alpha_1 n + \beta_1, \ldots, \alpha_t j + \beta_t = \alpha_{t+1} n + \beta_{t+1}, \ldots, \alpha_l j + \beta_l = k$;*

(2) *there exists α in $\mathbb{N}$ such that $k - m = \alpha(j - n)$ and $\alpha \leqq E_l$.*

Lemma 2. *The sequence $(E_l \mid l \in \mathbb{N})$ is an increasing sequence which converges to $+\infty$, if $j \geqq n$, and to $E((m - j)/(n - j))$, if $j < n$.*

Proof of Lemma 1. As interpreted after 2.1, Lemma 2 is true for $l = 0$. Suppose now $l \geqq 0$ and Lemma 1 is true for $t = 0, \ldots, l$. We prove Lemma 1 for $l + 1$.

(1) $\Rightarrow$ (2). Let $k_l = \alpha_l j + \beta_l (= \alpha_{l+1} n + \beta_{l+1})$. By the induction hypothesis, there exists $\alpha' \in \mathbb{N}$ with $m - k_l = \alpha'(n - j)$ and $\alpha' \leqq E_l$. Moreover, since $\alpha_{l+1} j + \beta_{l+1} = k$, we have $k_l - k = \alpha_{l+1}(n - j)$. Obviously, $\alpha_{l+1} \leqq E(l/n)$. Let $\alpha = \alpha' + \alpha_{l+1}$. Then $m - k = (m - k_l) + (k_l - k) = \alpha(n - j)$ and $\alpha \leqq \alpha' + E(k_l/n) = \alpha' + E(m/n - \alpha'(n - j)/n) = E(m/n + \alpha'(j/n) \leqq E_{l+1}$.

(2) $\Rightarrow$ (1). Since (E_l) is clearly increasing, we may suppose by the induction hypothesis that $k - m = \alpha(j - n)$ with $\alpha = E_l + \alpha_{l+1}$ and $0 < \alpha_{l+1} \leqq E_{l+1} - E_l$. Let $k_l = m - E_l(n - j)$. Hence $m - k_l = E_l(n - j)$ and there exists $\alpha_1, \beta_1, \ldots$

$\ldots, \alpha_l, \beta_l$ in $\mathbb{N}$ such that $m = \alpha_1 n + \beta_1, \ldots, \alpha_l j + \beta_l = k_l$. Moreover, $k_l - k = m - k - E_l(n - j) = \alpha_{l+1}(n - j)$. Therefore $k_l - \alpha_{l+1}n = k - \alpha_{l+1}j (= \beta_{l+1})$. Since $k_l = \alpha_{l+1}n + \beta_{l+1}$ and $\alpha_{l+1}j + \beta_{l+1} = k$, it remains to prove $\beta_{l+1} \geqq 0$. Now $\beta_{l+1} = k_l - \alpha_{l+1}n \geqq m - E_l(n - j) - (E_{l+1} - E_l)\, n = n(m/n + (j/n)\, E_l - E_{l+1}) \geqq 0$.

Proof of Lemma 2. If $j \geqq n$, then $E_{l+1} = E(m/n + (j/n)\, E_l) = 1 + E_l + E(m + n)/n + ((j - n)/n)\, E_l) > E_l$, and the conclusion follows. Let us suppose $j < n$. It suffices to prove

(1) $E_l \leqq E((m - j)/(n - j))$ and

(2) if $E_l < E((m - j)/(n - j))$, then $E_l < E_{l+1}$.

This is done by induction. Assertion (1) is true for $l = 0$. Suppose now (1) and (2) are true for $t = 0, \ldots, l - 1$ and (1) is true for $t = l$.

a) (2) is true for $t = l$. Indeed, $E_l \leqq E((m - j)/(n - j)) - 1 \leqq (m - n)/(n - j)$. Hence $((n - j)/n)\, E_l \leqq (m - n)/n$, that is $E_l + 1 \leqq m/n + (j/n)\, E_l$, which implies $E_l < E_{l+1}$.

b) (1) is true for $t = l + 1$. Let us first suppose that $E_l < E((m - j)/(n - j))$. As above, we have $E_l \leqq (m - n)/(n - j)$ and $m/n + (j/n)\, E_l \leqq (m - j)/(n - j)$, whence $E_{l+1} \leqq E((m - j)/(n - j))$.

Suppose now $E_l = E((m - j)/(n - j)$. Then $E_l > (m - n)/(n - j)$ and $n\, E_l + n > m + jE_l$. Consequently, $E_l + 1 > m/n + (j/n)\, E_l \geqq E_{l+1}$. This shows $E_l \leqq E_{l+1} \leqq E_l = E((m - j)/(n - j)$.

To handle axiom $C(n, j)$, we use a duality for finite UBA's first desctibed by Jónsson and Tarski in [4] (see also [3] and [6]).

2.3. Definition. A *binary system* is an ordered pair $\mathscr{A} = (A, R)$ where A is a set and R a binary relation on A.

Let $\mathscr{B}$ be a finite UBA. Its *dual binary system* At $(\mathscr{B})$ is the set of all atoms of $\mathscr{B}$ together with the binary relation R on it defined by pRq if and only if $p \leqq f(q)$. Conversely, if $\mathscr{A}$ is a finite binary system, its *dual UBA*, denoted by $\mathfrak{P}(\mathscr{A})$, is the Boolean algebra of all subsets of A, together with the operator f defined by $f(X) = \{p \in A \mid pRq \text{ for some } q \in X\}$ for $X \subseteq A$.

The duality theorem of Jónsson and Tarski states the following ([4], p. 933).

2.4. Recall. *If $\mathscr{B}$ is any finite* UBA *and $\mathscr{A}$ any finite binary system, then $\mathscr{B}$ is isomorphic with* $\mathfrak{P}(\mathrm{At}(\mathscr{B}))$ *and $\mathscr{A}$ is isomorphic with* $\mathrm{At}(\mathfrak{P}(\mathscr{A}))$.

We wish to translate axiom $C(n, j)$ in the dual language.

2.5. Definition. Let $\mathscr{A}$ be a binary system and let $a \in A$, $b \in A$. A *path* from a to b is a sequence $(x_0, \ldots, x_n)$ such that $x_0 = a$, $x_n = b$ and $x_{i-1}Rx_i$ for $i = 1, \ldots, n$. The natural number n is the *length of the path* $(x_0, \ldots, x_n)$. A *n-path* is a path of length n. If there exists a n-path from a to b, we say that b is *n-related to a* (for instance b is o-related to a if and only if $b = a$).

2.6. Proposition. *Let $\mathscr{B}$ be a finite* UBA. *Then $\mathscr{B}$ satisfies axiom $C(n, j)$ if and only if its dual* At $(\mathscr{B})$ *satisfies*:

$P(n, j)$: *if b is n-related to a, then b is j-related to a.*

Proof. Suppose first $\mathscr{B}$ satisfies $C(n, j)$ and let $x_0, \ldots, x_n$ be a n-path. Then $x_0 \leqq \leqq f(x_1) \leqq \ldots \leqq f^{(n)}(x_n) \leqq f^{(j)}(x_n)$. By additivity, there exists $y_1 \in \mathrm{At}(\mathscr{B})$ such that $x_0 \leqq f(y_1)$ and $y_1 \leqq f^{(j-1)}(x_n)$ (supposing $j > 0$). An easy induction shows that x_n is j-related to x_0.

To prove the converse, that is, At $(\mathscr{B})$ satisfies $P(n, j)$ implies $f^{(n)}(x) \leqq f^{(j)}(x)$, we may suppose $x \in \mathrm{At}(\mathscr{B})$. Let $x_0 \in \mathrm{At}(\mathscr{B})$ such that $x_0 \leqq f^{(n)}(x)$. Then x_0 is n-related to x, hence j-related, which proves $x_0 \leqq f^{(j)}(x)$. The rest of the proof follows from additivity.

We are now in position to state and prove our main result. In what follows, $\mathfrak{C}_f(n, j)$ is the class of all finite members of $\mathfrak{C}(n, j)$.

2.7. Theorem. *Let $(n, j, m, k) \in \mathbb{N}^4$. Suppose that $(n, j) \neq (1, 0)$. Then the following are equivalent*:

(1) $\mathfrak{C}(n, j) \subseteq \mathfrak{C}(m, k)$;

(2) $\mathfrak{C}_f(n, j) \subseteq \mathfrak{C}_f(m, k)$;

(3) *n, j, m, k are well related.*

Proof. It is clear that (1) $\Rightarrow$ (2). An inductive argument on parameter l in definition 2.1 shows that (3) $\Rightarrow$ (1). To prove (2) $\Rightarrow$ (3) let us assume that k is not well related to (n, j, m). By 2.6, it suffices to build a finite binary system $\mathscr{A}$ that satisfies axiom $P(n, j)$ but not axiom $P(m, k)$. We distinguish several cases, according to Characterization 2.2 of well related 4-uples.

(α) $m < n$. Let m^+ be the binary system on $\{0, \ldots, m\}$ defined by uRv if and only if $v = u + 1$. Then m^+ satisfies $P(n, j)$ but not $P(m, k)$ since the only path from 0 to m is of length m. Note that is example also covers the case $j = n$.

(β) $m \geqq n > 0$ and $j = 0$. Let n^0 be the binary system on $\{0, \ldots, n - 1\}$ defined by uRv if and only if $v = u + 1$ or ($u = n - 1$ and $v = 0$). Then n^0 satisfies $P(n, j)$ but not $P(m, k)$ (in fact, if v is m-related to u, then v is never k-related to u). (Note this is false for $n = 1$).

(γ) $m \geqq n$ and $n \neq j \neq 0$. We first introduce a general construction of binary systems.

Let $\mathscr{A}$ be a finite binary system and $B \subseteq A$. Denote by $\mathfrak{F}(\mathscr{A}, B)$ the set $\{(a, b) \mid$ there exists a n-path from a to b that meets $B\}$. Then $\mathscr{A}^+B = (A^+B, R^+B)$ where

$A^+B = A \cup \{\alpha_1(a, b), \ldots, \alpha_{j-1}(a, b) \mid (a, b) \in \mathfrak{F}(\mathscr{A}, B)\}$, and

$R^+B = R \cup \{(a, \alpha_1(a, b)), (\alpha_1(a, b), \alpha_2(a, b)), \ldots, (\alpha_{j-1}(a, b), b) \mid (a, b) \in \mathfrak{F}(\mathscr{A}, B)\}$,

where all $\alpha_t(a, b)$ are distinct and distinct from the elements of A. Roughly speaking, we add a j-path from a to b whenever there exists a n-path from a to b meeting B. Note that, if $j = 1$, the carrier of $\mathscr{A}^+B$ is A but R^+B may be different from R, since $R^+B = R \cup \mathfrak{F}(\mathscr{A}, B)$.

Finally, let $(\mathscr{A}_l \mid l \in \mathbb{N})$ be the sequence of binary systems defined by
$\mathscr{A}_0 = \emptyset$,
$\mathscr{A}_1 = m^+$ (see α)),
$\mathscr{A}_{l+1} = \mathscr{A}_l^+(A_l - A_{l-1})$.

Consider first the case $j < n$. Then the maximal length of the paths in $\mathscr{A}_l$ meeting $A_l - A_{l-1}$ is strictly decreasing as l increases. Therefore $\mathfrak{F}(\mathscr{A}_l, A_l - A_{l-1})$ turns out eventually to be empty, say for $l = l_0$. It is clear that $\mathscr{A}_{l_0}$ satisfies axiom $P(n, j)$. Moreover, the length of any path from 0 to m is well-related to (n, j, m) and $\mathscr{A}_{l_0}$ does not satisfies $P(m, k)$.

It remains to consider the case $j > n$. Since the minimal length of the paths from 0 to m in A_l meeting $A_l - A_{l-1}$ is strictly increasing, we cannot hope that $A_l - A_{l-1}$ become empty. Nevertheless, let l_0 be great enough to ensure that all paths from 0 to m meeting $A_{l_0} - A_{l_0-1}$ are strictly greater than k. We build $\mathscr{A}$ from $\mathscr{A}_{l_0}$ by adding the relations aRa for each $a \in A_{l_0} - A_{l_0-1}$. This ensures that $\mathscr{A}$ satisfies $P(n, j)$. Moreover, all paths from 0 to m have length k' with $k' > k$ or k' well related to (n, j, m). Hence $\mathscr{A}$ does not satisfies $P(m, k)$ and the proof is over.

To complete Theorem 2.7, we consider the case $(n, j) = (1, 0)$.

2.8. Theorem. *Let $(m, k) \in \mathbb{N}^2$. Then the following are equivalent:*
(1) $\mathfrak{C}(1, 0) \subseteq \mathfrak{C}(m, k)$,
(2) $\mathfrak{C}_f(1, 0) \subseteq \mathfrak{C}_f(m, k)$,
(3) $m \neq 0$ *or* $k = 0$.

Proof. As in Theorem 2.7, one has (1) ⇒ (2). To prove (2) ⇒ (3), we assume (2) and $k \neq 0$. Let B be the two element Boolean algebra endowed with the operator f taking the constant value 0. Then $B \in \mathfrak{C}_f(1, 0)$ but $B \notin \mathfrak{C}(0, k)$.

It remains to show (3) ⇒ (1). In fact, if $x \leqq f(x)$ holds, then f is entirely determined by $f(1)$ as shown by the formula $f(x) = x \wedge f(1)$. Indeed, $x \wedge f(1) = x \wedge (f(x) \vee \vee f(x^c)) = (x \wedge f(x)) \vee (x \wedge f(x^c)) = f(x)$. Hence, $f^m(x) = f^k(x)$ for all $m, k \neq 0$ (and also obviously for $m = k = 0$).

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Author's address: Institut de Mathématique, Université de Liège, Belgique.