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FREDHOLM RANDOM OPERATORS AND RANDOM SUBSPACES IN BANACH SPACES

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0. INTRODUCTION

The theory of random operators was initiated in the 1950's by the Prague School of mathematicians around Špacek and Hanš. Such "operators" are simply measurable (in some sense) families of operators parametrized by points of a probability space. Various problems were studied in this theory: random inverses and generalized inverses, random solutions of random equations, random fixed points, etc. (see, for example, [1, 2, 4] and references in these papers). The theory covers many important classes of random operators such as ordinary and partial differential operators with random coefficients, integral operators with random kernels etc. But it should be noted that there are some examples forcing us to extend the notion of random operator (see [9]). Nonetheless, the present paper deals with the usual random operators only. Its main aim is to prove existence of random regularizers to Fredholm random operators. This is closely related to the work of H. W. Engl and M. Z. Nashed [4]. Our approach is based on examining finite (co)dimensional random subspaces of Banach spaces and is partially inspired by works of S. G. Krein and others on continuous and analytic families of operators (see the survey [7]).

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1. PRELIMINARIES

Let E and F be separable Banach spaces (real or complex) and let $L(E, F)$ be the space of all bounded linear operators from E into F. Let $(\Omega, \mathscr{A}, \mu)$ be a probability space, i.e. Ω is a set with a σ-algebra $\mathscr{A}$ and with a measure μ such that $\mu(\Omega) = 1$. (In fact we may suppose the measure μ only to be σ-finite.) We will suppose μ to be a complete measure. This assumption may be omitted if we replace "all ω" by "almost all ω" in our results.

A family $A(\omega) \in L(E, F)$, $\omega \in \Omega$, is called a *random operator* if for any $u \in E$ the F-valued function $A(\omega) u$ is μ-measurable. A family $M(\omega) \subset E$, $\omega \in \Omega$, of subsets is called a *random subset* in E if for any open $U \subset E$ the set $\{\omega \in \Omega \mid M(\omega) \cap U \neq \emptyset\}$ is measurable (in [4] the term "measurable family" is used, but here these words have another meaning). It is known [4] that for any random operator $A(\omega) \in L(E,F)$ the families $\ker A(\omega)$, $\operatorname{im} A(\omega)$ and $\overline{\operatorname{im} A(\omega)}$ are random subspaces. As usual we denote by $\ker A$ and $\operatorname{im} A$ the kernel (or null-space) and the image (or range) of A, respectively.

Later we will study Fredholm random operators. One says that $A(\omega)$ is a *Fredholm random operator* (or Noether in terms of [8]) if $A(\omega)$ is a Fredholm operator for any $\omega \in \Omega$, i.e. $\dim \ker A(\omega) < \infty$ and $\operatorname{codim} \operatorname{im} A(\omega) < \infty$.

We denote by $G(E)$ the set of all closed subspaces in the space E. There is a metric on $G(E)$ called an *opening* or a *gap* [5]. It is defined by the formula

$$\mathrm{d}(M, N) = \max \left[\sup_{u \in S_M} \operatorname{dist}(u, S_N), \sup_{v \in S_N} \operatorname{dist}(v, S_M) \right],$$

where $M, N \in G(E)$ and S_M, S_N are the unit spheres in M and N, respectively. The metric space $(G(E), \mathrm{d})$ is complete.

Taking the annihilator $M \mapsto M^{\perp}$ we obtain the isometric homeomorphism $\perp : G(E) \to^{\sim} G(E^*)$, where the asterisk stands for the dual Banach space. Denote by $G_f(E)$ or $G^f(E)$ the set of all subspaces in E having a finite dimension or codimension, respectively. For any $n \in \boldsymbol{Z}_+$ set

$$G_n(E) = \{M \in G(E) \mid \dim M = n\},$$
$$G^n(E) = \{M \in G(E) \mid \operatorname{codim} M = n\}.$$

They all are open-closed subsets in $G(E)$. The same is true for

$$G_f(E) = \bigcup_{n \in \boldsymbol{Z}_+} G_n(E),$$
$$G^f(E) = \bigcup_{n \in \boldsymbol{Z}_+} G^n(E).$$

The map $\perp$ induces homeomorphisms $G_n(E) \to^{\sim} G^n(E^*)$ and $G^n(E) \to^{\sim} G_n(E^*)$.

The following statement is well-known.

Lemma. *The spaces $G_n(E)$, $n \in \boldsymbol{Z}_+$, and $G_f(E)$ are separable. If E^* is separable, then $G^n(E)$, $n \in \boldsymbol{Z}_+$, and $G^f(E)$ are separable as well.*

Proof. Let $R_n(E) \subset \underbrace{E \times \ldots \times E}_{n} = E^n$ be the set of all linearly independent n-tuples $(e_1, \ldots, e_n)$, $e_j \in E$ (*n-frames*). The map

(1) $$R_n(E) \to G_n(E),$$
$$(e_1, \ldots, e_n) \mapsto \operatorname{span}(e_1, \ldots, e_n)$$

is continuous and onto. As $R_n(E)$ is separable, this implies separability of $G_n(E)$. The second statement is obvious in view of the homeomorphism $\perp$. □

Any family of subspaces $M(\omega) \subset E$, $\omega \in \Omega$, induces a map $\tilde{M}: \Omega \to G(E)$. We say that $M(\omega)$ is *measurable* if the map $\tilde{M}$ is measurable with respect to the canonical Borel structure on $G(E)$ induced by the metric d. It is easy to see that any measurable family is a random subspace. The family $M(\omega)$ is called *finite-dimensional* (or *having a finite codimension*), if $\dim M(\omega) < \infty$ ($\operatorname{codim} M(\omega) < \infty$, respectively) for any $\omega \in \Omega$.

Theorem 1. *Assume that $M(\omega) \subset E$ is a finite-dimensional random subspace (or has a finite codimension and E^* is separable). Then the integer-valued function* $\dim M(\omega)$ ($\operatorname{codim} M(\omega)$, *respectively*) *is measurable.*

Proof. In view of duality $M \leftrightarrow M^{\perp}$ it is sufficient to prove the dim-assertion only.

For any n-frame $f = (f_1, \ldots, f_n) \in R_n(E)$ we define $r(f)$ as the radius of the maximal ball in E^n with the center f which is contained in $R_n(E)$, i.e. $B(f; r(f)) \subset R_n(E)$. Here and in the sequel,

$$B(f; r) = \{g \mid \|f - g\| < r\}$$

is the open ball centered at f with the radius r, E^n is endowed with the norm

$$\|u\|_{E^n} = \max\{\|u_j\|_E, \ u = (u_1, \ldots, u_n)\} .$$

Let us consider $\{f^l\}_{l \in Z_+} = \{(f_1^l, \ldots, f_n^l)\}_{l \in Z_+}$, a countable dense subset in $R_n(E)$, and set $r_l = r(f^l)$. Define the following (evidently, measurable) sets

$$\Omega_{n,j}^l = \{\omega \in \Omega \mid M(\omega) \cap B(f_j^l, r_l) \neq \emptyset\} ,$$

$$\Omega_n^l = \bigcap_j \Omega_{n,j}^l , \quad \Omega_n = \bigcup_l \Omega_n^l ,$$

where $l \in Z_+$ and $j = 1, \ldots, n$. Then

$$\Omega_n = \{\omega \in \Omega \mid \dim M(\omega) \geqq n\} .$$

Actually, if $\omega \in \Omega_n$, then $\omega \in \Omega_n^l$ for some $l \in Z_+$. Hence

$$B(f_j^l, r_l) \cap M(\omega) \neq \emptyset , \quad j = 1, \ldots, n ,$$

and we see that there is an n-frame in $M(\omega)$ lying in $B(f^l; r_l)$. Thus

$$\dim M(\omega) \geqq n . \tag{2}$$

Conversely, suppose that (2) is fulfilled. Then there is an n-frame $f \subset M(\omega)$. For any given sufficiently small $\varepsilon > 0$ we may find f^l such that

$$\|f - f^l\|_{E^n} < \varepsilon .$$

For $r = r(f)$ it is easy to see that

$$B(f; r - \varepsilon) \cup B(f^l; r_l - \varepsilon) \subset B(f; r) \cap B(f^l; r_l)$$

and, as a consequence, $|r - r_l| < \varepsilon$. Therefore, if $\varepsilon < r/2$, then $r_l > r - \varepsilon > > r/2 > \varepsilon$. Hence $f \in B(f^l; r_l)$ and $\omega \in \Omega_n^l$. The proof is complete. □

Theorem 2. *Let $M(\omega)$ be a random subspace in E. If $M(\omega)$ is finite-dimensional, then it is measurable. If E^* is separable and $M(\omega)$ is of finite codimension, then it is measurable as well.*

Proof. As above it is sufficient to prove the first statement only. By Theorem 1, without loss of generality we may assume that $\dim M(\omega) \equiv n$.

As the canonical map $R_n(E) \to G_n(E)$ defined by (1) is continuous, it is sufficient to construct a measurable n-frame

$$f(\omega) = (f_1(\omega), \ldots, f_n(\omega)) \subset M(\omega) \tag{3}$$

(measurable in the sense that $f\colon \Omega \to R_n(E)$ is a measurable map). To select $f(\omega)$ we use the following recurrent procedure.

The family $M(\omega)$ and, as a consequence,

$$S_{M(\omega)} = S \cap M(\omega)$$

are measurable multivalued maps with closed values. Here S is the unit sphere of E. Therefore, by the fundamental theorem of K. Kuratowski and C. Ryll-Nardzewski [6, 3] there is a measurable selection $f_1(\omega) \in S_{M(\omega)}$. Now we suppose that a measurable k-frame

$$(f_1(\omega), \ldots, f_k(\omega)) \subset M(\omega)\,, \quad 1 \leqq k < n\,,$$

has been constructed. In addition we may suppose that $\|f_j(\omega)\| = 1$, $j = 1, \ldots, k$. Denote by $U_r(\omega)$ the r-neighbourhood ($r < 1$) of $\operatorname{span}(f_1(\omega), \ldots, f_k(\omega))$. It is not hard to verify that $U_r(\omega)$ and

$$V_r(\omega) = S \cap (M(\omega) \setminus U_r(\omega))$$

are measurable multivalued maps and the latter has closed values. By the Kuratowski and Ryll-Nardzewski theorem there is a measurable selection $f_{k+1}(\omega) \in V_r(\omega)$. Obviously

$$(f_1(\omega), \ldots, f_{k+1}(\omega)) \subset M(\omega)$$

is a unit $(k + 1)$-frame. Thus we can construct the required n-frame and the proof is complete. □

The following statement is our key step.

Theorem 3. *Suppose E^* is separable and $M(\omega)$ is a random subspace in E of a finite dimension or codimension. Then there is a random direct complement to $M(\omega)$.*

Proof. It is sufficient to consider the case of finite dimension only. By Lemma there is a countable dense subset $\{N_l\}_{l \in \mathbf{Z}_+}$ in $G^f(E)$. Set

$$\Omega_l = \{\omega \in \Omega \mid M(\omega) \oplus N_l = E\}\,.$$

The set of all direct complements to a fixed subspace is open in $G(E)$ (stability of direct complements). In view of Theorem 2 this implies that all Ω_l are measurable. Further, stability of direct complements implies that $\Omega = \bigcup_l \Omega_l$. Hence we may construct a random direct complement $N(\omega)$ to $M(\omega)$ by the formula

$$N(\omega) = N_l, \quad \omega \in \Omega_l \setminus \left(\bigcup_{j=1}^{l-1} \Omega_j\right)$$

and the proof is complete. □

Remark 1. Theorem 3 implies existence of a random projector onto $M(\omega)$.

Remark 2. V. E. Ljance suggested another proof of Theorem 3. We may suppose that $\dim M(\omega) \equiv n$. Let $(f_1(\omega), \ldots, f_n(\omega))$ be a measurable basis for $M(\omega)$. By a standard argument we can prove the random version of the Hahn-Banach theorem. Hence we may construct the biorthogonal n-frame $(f^1(\omega), \ldots, f^n(\omega)) \subset E^*$. It is easy to see that the subspace

$$N(\omega) = \{u \in E \mid \langle f^j(\omega), u \rangle = 0,\ j = 1, \ldots, n\}$$

is a random direct complement to $M(\omega)$.

3. RANDOM FREDHOLM OPERATORS

Now we consider random Fredholm operators $A(\omega)$ from E into F. If we assume F^* to be separable, then Theorem 1 implies that the integer-valued functions

$$\dim \ker A(\omega),$$

$$\operatorname{codim} \operatorname{im} A(\omega) = \dim \operatorname{coker} A(\omega),$$

and

$$\operatorname{index} A(\omega) = \dim \ker A(\omega) - \dim \operatorname{coker} A(\omega)$$

are measurable.

Now we consider the problem of existence of random regularizers to random Fredholm operators. Recall that a linear operator B is a *regularizer* to a linear operator A, if both the composition AB and BA are of the form "identity plus a compact operator". In addition, if B is invertible, we say that B is an *equivalent regularizer*. It is well-known that A has a regularizer if and only if it is Fredholm. An equivalent regularizer to A exists if and only if A is a Fredholm operator of zero index. For the general theory of Fredholm operators including regularizers see, for example, [8]. Up to the end of the paper we shall assume both the spaces E^* and F^* to be separable.

Theorem 4. *Suppose E^* and F^* are separable. Then any random Fredholm operator from E into F has a random regularizer.*

Proof. Such a regularizer may be constructed in the following way. Let $M(\omega)$ and $N(\omega)$ be random direct complements to $\operatorname{im} A(\omega)$ and $\ker A(\omega)$, respectively. They exist in view of Theorem 3. Let $P(\omega)$ be the projector onto $\operatorname{im} A(\omega)$ along $M(\omega)$

and let $\tilde{A}(\omega)$ be the restriction $A(\omega)|_{N(\omega)}$ considered as a map from $N(\omega)$ into im $A(\omega)$. Then

(4) $$B(\omega) = (\tilde{A}(\omega))^{-1} P(\omega)$$

is a random operator by [4, Theorem 5.9]. It is evident that $B(\omega)$ is a regularizer to $A(\omega)$ for any $\omega \in \Omega$. □

Remark 3. For the regularizers constructed by the formula (4) we see that both $A(\omega) B(\omega)$ and $B(\omega) A(\omega)$ are of the form "identity plus a random finite-dimensional operator".

Theorem 5. *Assume E^* and F^* to be separable. Then any random Fredholm operator from E into F with zero index has an equivalent regularizer.*

Proof. We use the notations as in the proof of Theorem 4. By Theorem 1, without loss of generality we may assume that $\dim \ker A(\omega) = \dim M(\omega) \equiv n$. Choose measurable bases $(e_1(\omega), \ldots, e_n(\omega))$ and

$$(f_1(\omega), \ldots, f_n(\omega)) \quad \text{in} \quad \ker A(\omega) \quad \text{and} \quad M(\omega),$$

respectively. Consider the map $T(\omega)\colon M(\omega) \to \ker A(\omega)$ defined by

$$T(\omega)\colon f_j(\omega) \mapsto e_j(\omega), \quad j = 1, \ldots, n.$$

It is not hard to verify that

$$B_1(\omega) = T(\omega)(I - P(\omega))$$

is a random operator. Then

$$\tilde{B}(\omega) = B(\omega) + B_1(\omega),$$

where $B(\omega)$ defined by (4) is the required equivalent regularizer. □

Remark 4. As usual (see [8]), Theorem 5 implies that under our separability assumption on E^* and F^* any random Fredholm operator $A(\omega)$ with index $A(\omega) = 0$ has the form

$$A(\omega) = V(\omega) + K(\omega),$$

where the random operators $V(\omega)$ and $K(\omega)$ are invertible and compact, respectively.

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