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WHITEHEAD PROPERTY OF MODULES

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INTRODUCTION

In the present note, we study relations between the structure of associative rings and extension properties of modules. Let \( R \) be an associative ring with unit and \( R\)-mod the category of unitary left \( R \)-modules. A module \( N \in R\)-mod is said to have the Whitehead property \((WP)\) if either \( N \) is injective or, for all \( M \in R\)-mod, \( \text{Ext}_R(M, N) = 0 \) implies \( M \) is projective.

A given module may or need not have WP according to the extension of ZFC we work in (this happens e.g. if \( R \) is a countable Dedekind domain and \( N = R \) — see [7] and [4] — or if \( R \) is a simple countable non-completely reducible von Neumann regular ring and \( N \) is any countable \( R \)-module — see Section 2 below). Nevertheless, if we require all \( R \)-modules to have WP, we get results on the structure of the ring \( R \), proved in ZFC. Hence, this requirement seems more appropriate for our aims.

Recall that by [2, Appendix A], a ring \( R \) such that every left \( R \)-module has WP is called a left \( T \)-ring. By [9] we know that every left \( T \)-ring is either left artinian or von Neumann regular. While we have a full description e.g. of left nonsingular left artinian left \( T \)-rings (see [9, 4.4 and 6.1]), only little is known about the regular ones. By [10], if \( R \) is a simple countable regular ring, then \( \text{Ext}_R(M, N) = 0 \) for all countably generated \( R \)-modules \( M, N \) such that \( M \) is non-projective and \( N \) is non-injective. Moreover, assuming \( V = L \), every countable \( R \)-module has WP (see [10, III.6]).

The present note is divided into three sections. In Section 1, we show that in spite of the facts mentioned above, if \( R \) is a simple non-completely reducible regular ring of cardinality \( < 2^{\aleph_0} \), then there is an \( R \)-module which does not have WP. Hence, \( R \) is not a left \( T \)-ring. In Section 2, we show that in some models of ZFC, even no countable \( R \)-module has WP. Hence, the assertion of [10, III.6] is independent of ZFC. In Section 3, we use the solution of Artin’s problem ([6] and [3]) to construct a ring \( R \) which is not a left \( T \)-ring, but every cyclic \( R \)-module has WP.
PRELIMINARIES

In what follows, an ordinal is identified with the set of its predecessors and a cardinal is an ordinal which is not equipotent with any of its predecessors. Let \( \kappa \) be an infinite cardinal and \( E \subseteq \kappa \). Then \( E \) is cofinal in \( \kappa \) if sup \( E = \kappa \). Further, \( E \) is closed in \( \kappa \) if sup \( F \in E \cup \{ \kappa \} \), for every non-empty subset \( F \subseteq E \). We say that \( E \) is stationary in \( \kappa \) if \( E \cap F \neq \emptyset \) for every closed and cofinal subset \( F \) of \( \kappa \).

Let \( G \) be a filter over \( \kappa \). Then \( G \) is \( \kappa \)-complete if \( G \) is closed with respect to intersections of less than \( \kappa \) elements of \( G \). Further, \( G \) is normal if for any \( g_x \in G \), \( x < \kappa \), the set \( \{ \alpha < \kappa \mid \alpha \in \bigcap g_\beta \} \) belongs to \( G \).

In what follows, all rings are associative with unit. If \( S \) and \( T \) are rings, then \( S \oplus T \) denotes the ring direct sum of \( S \) and \( T \). If \( S \) is a ring, \( n \) is a natural number, \( n \geq 1 \), and \( \kappa \) is a cardinal, \( \kappa \geq 1 \), then \( \text{RFM}_{n \times \kappa}(S) \) denotes the set of all row finite matrices of type \( n \times \kappa \) over \( S \).

If \( S \) is a ring, then \( S\text{-mod} \) denotes the category of unitary left \( S \)-modules. A unitary left \( R \)-module is simply called a module. Let \( R \) be a left hereditary ring, \( \kappa \) an infinite cardinal and \( M \in R\text{-mod} \). Then \( M \) is \( \kappa \)-free if every submodule of \( M \) which is generated by less than \( \kappa \) elements is projective. Moreover, \( M \) is strongly \( \kappa \)-free if every submodule \( A \) of \( M \) which is generated by less than \( \kappa \) elements is contained in a projective submodule \( A' \) such that \( A' \) is generated by less than \( \kappa \) elements and \( M/A' \) is \( \kappa \)-free (see [4, §18]). If \( N \) is a module, then \( I(N) \) denotes the injective hull of \( N \) and \( \text{Soc}(N) \) denotes the left socle of \( N \). A ring \( R \) is said to be completely reducible if \( \text{Soc}(R) = \emptyset \). If \( N \in R\text{-mod} \) and \( x \in N \), then \( \text{Ann}_R(x) \) denotes the left annihilator of \( x \) in \( R \).

A module \( N \) is said to have a socle sequence if there are an ordinal \( \sigma \) and a sequence \( S_\nu, \nu \leq \sigma \) of submodules of \( N \) such that \( S_0 = 0 \), \( S_{\nu+1}/S_\nu = \text{Soc}(N/S_\nu) \neq 0 \) for all \( \nu < \sigma \), \( S_\nu = \bigcup S_\mu \), \( \mu < \nu \) for all limit \( \nu \leq \sigma \) and \( S_\sigma = N \). Clearly, if \( N \) has a socle sequence, then \( \sigma \) and \( S_\nu, \nu \leq \sigma \), are unique.

A sum (direct sum) of submodules is denoted by \( \sum \) (by \( \bigoplus \), respectively). If \( \kappa \) is a cardinal, \( \kappa \geq 1 \) and \( N \in R\text{-mod} \), then \( N^{(\kappa)} \) and \( N^\kappa \) denote the direct sum and the direct product of \( \kappa \) copies of \( N \), respectively.

Further concepts and notation can be found e.g. in [1] and [4].

1. REGULAR RINGS AND WP

By [10], the only candidates for non-completely reducible regular left \( T \)-rings are rings of the form \((S \oplus) R\), where \( S \) is a completely reducible ring and \( R \) is a simple regular ring having all left ideals countably generated. Here, in 1.5, we show that, moreover, \( \text{card} \ R \geq 2^{\aleph_0} \). Thus, in 1.6, we obtain a full description of left non-singular left \( T \)-rings of cardinality \( < 2^{\aleph_0} \).

1.1. Let \( R \) be a non-completely reducible regular ring. Let \( A \) be a non-empty set
of countably generated left ideals of $R$. For $N \in R$-mod let $f \in \text{Hom}_R(N, N^{N_0}/N^{(N_0)})$ such that $nf = (n_i + N^{(N_0)} | i < N_0)$, where $n_i = n$ for all $i < N_0$. Define a sequence $S_v, v \leq N_1$ of submodules of $N^{N_0}$ by

(i) $S_0 \equiv N^{(N_0)}$ and $S_0/N^{(N_0)} = (N)f$,
(ii) $S_{v+1} = \langle \{ n \in N^{N_0} \mid \exists I \in A : In \subseteq S_v \rangle \rangle_R$,
(iii) $S_v = \bigcup S_{u}, u < v$ for $v$ limit.

Put $\tilde{N} = S_{N_1}/N^{(N_0)}$.

**Lemma.** $N$ is isomorphic to a submodule of $\tilde{N}$ and, for all $I \in A$, $\text{Ext}_R(R/I, \tilde{N}) = 0$.

**Proof.** Obviously, $N \cong (N)f \subseteq \tilde{N}$. The assertion is clear if $I$ is finitely generated. Let $g \in \text{Hom}_R(I, \tilde{N})$, where $I = \sum \inf R e_j, j < N_0$, and $\{ e_j | j < N_0 \}$ is a set of pairwise orthogonal idempotents of $R$ (see [5, \S 2]). Let $e_i g = (s_i^j + N^{(N_0)} | i < N_0)$, where $e_i s_i^j = s_i^j$ for all $i, j < N_0$. Let $v < N_1$ be the smallest ordinal such that $e_i g \in S_{v}/N^{(N_0)}$, for all $j < N_0$. Define an $n = (n_i | i < N_0) \in N^{N_0}$ by $n_i = s_i^0 + \ldots + s_i^v, i < N_0$. It is easy to see that $n \in S_{v+1}$ and $e_i g = e_in + N^{(N_0)}$ for all $j < N_0$, whence $\text{Ext}_R(R/I, \tilde{N}) = 0$.

**1.2. Lemma.** Let $R$ be a simple regular ring. Let $N_1, i < N_0$, be a sequence of modules such that $N_i$ is a proper submodule of $N_i+1$ for all $i < N_0$. Put $N = \bigcup N_i$, $i < N_0$, and let $I$ be a countably infinitely generated left ideal of $R$. Then $\text{Ext}_R(R/I, N) \neq 0$.

**Proof.** We have $I = \sum \inf R e_i, i < N_0$, where $\{ e_i | i < N_0 \}$ is a set of pairwise orthogonal idempotents of $R$. Since $R$ is simple, there is $n_i \epsilon (e_i N_i+1 - N_i)$, for each $i < N_0$. Now, $g \in \text{Hom}_R(I, N)$, defined by $e_i g = n_i$, is not a restriction of an element of $\text{Hom}_R(R, N)$.

**1.3. Proposition.** Let $R$ be a regular left T-ring. If $N \in R$-mod, then $I(N)/N$ has a socle sequence of length $\sigma \leq N_1$, where either $\sigma = N_1$ or $\sigma$ is non-limit. Hence, if $M, N \in R$-mod and $N$ is essential in $M$, then $M/N$ has a socle sequence of length $\leq N_1$.

**Proof.** By [10, II.4], we can use 1.1 with $A$ — the set of all maximal left ideals of $R$. With regard to 1.2, there is an ordinal $\alpha \leq N_1$ such that either $\alpha = N_1$ or $\alpha$ is non-limit, and $S_v + N^{(N_0)}/S_0 + N^{(N_0)}, v \leq \sigma$ is a socle sequence of $\tilde{N}/(N)f$. The rest is clear.

**1.4. Lemma.** Let $R$ be a left primitive ring, $J$ a simple faithful module and $K = \text{End}_R(J)$. Then $R$ is a dense subring of $\text{End}_K(J)$ and the following conditions are equivalent:

(i) all simple modules are isomorphic,
(ii) $\dim_K(\cap \ker s, s \in I) = 1$, for each maximal left ideal $I$ of $R$.

**Proof.** The density of $R$ is well-known. Assume (i) and let $I$ be a maximal left ideal of $R$. There is a $j \in J$ with $\text{Ann}_R(j) = I$, i.e. $JK \subseteq \cap \ker s, s \in I$. By the density, for each $k \in (J - JK)$ there is an $r \in R$ with $rk \neq 0$ and $rj = 0$, whence $k \notin \cap \ker s$.
\( s \in I \). Assume (ii). Let \( I \) and \( L \) be maximal left ideals of \( R \) and \( jK = \bigcap \Ker s, s \in I \); \( kK = \bigcap \Ker s, s \in L \). By the density, there is an \( r \in R \) with \( rk = j \). Hence, \( r \notin L \) and \( Ir \leq L \), and \( \Hom_R(R/I, R/L) \neq 0 \).

1.5. Theorem. Let \( R \) be a simple regular ring such that \( R \) is not completely reducible. Let \( J \) be a simple module and \( K = \text{End}_R(J) \). Assume \( \dim K < 2^{\aleph_0} \) (this holds e.g. if \( \text{card } R < 2^{\aleph_0} \)). Then there are a non-projective cyclic module \( M \) and a non-injective module \( N \) such that \( \Ext_R(M, N) = 0 \).

Proof. We prove the theorem in two steps.

Step I. Let \( 2 = \{0, 1\} \) and for \( x \in 2^{\aleph_0} \), \( x = (x_0, \ldots, x_n) \) put \( \ln(x) = n \). For \( x_i \in 2 \) denote by \( x_i \) the binary complement of \( x_i \). By induction, we define a set \( \{e_x \mid x \in 2^{\aleph_0}\} \) such that

(i) for each \( n < \aleph_0 \), \( \{e_x \mid x \in 2^{\aleph_0} \& \ln(x) = n\} \) is a complete set of pairwise orthogonal idempotents of \( R \);

(ii) if \( x, y, z \in 2^{\aleph_0} \), \( x = (x_0, \ldots, x_n) \), \( y = (x_0, \ldots, x_n, 0) \), \( z = (x_0, \ldots, x_n, 1) \), then \( e_x + e_y = e_x \).

Put \( e_0 = e, e_1 = 1 - e \), where \( e \in R, e^2 = e \notin \{0, 1\} \). Then (i) holds for \( n = 0 \).
Assume \( e_x \) are defined for each \( x \in 2^{\aleph_0} \) with \( \ln(x) \leq m \) and (i) holds for each \( n \leq m \) and (ii) for each \( n \leq m - 1 \). Let \( x, y, z \in 2^{\aleph_0} \), \( x = (x_0, \ldots, x_m) \), \( y = (x_0, \ldots, x_m, 0) \), \( z = (x_0, \ldots, x_m, 1) \). Since \( R \) is simple, the rings \( R \) and \( e_xR e_x \) are Morita equivalent and there are orthogonal idempotents \( e_y, e_z \in e_xR e_x \) with \( e_y + e_z = e_x \) and \( e_y \neq \pm e_x \neq e_z \). Then (i) holds for \( m \leq n + 1 \) and (ii) for \( m \leq n \). Further, for \( u \in 2^{\aleph_0} \), \( u = (u_i \mid i < \aleph_0) \) put \( w_0 = u_0 \) and \( w_n+1 = (u_0, \ldots, u_n, u_{n+1}) \), \( n < \aleph_0 \). Let \( I_u \) be a maximal left ideal of \( R \) containing the set \( \{e_{w_n} \mid n < \aleph_0\} \). If \( u^0, \ldots, u^n \) are different elements of \( 2^{\aleph_0} \), let \( i < \aleph_0 \) be the smallest index such that for each \( 0 < k \leq m \) there is a \( j \leq i \) with \( u^0_j \neq u^i_j \). By (i) and (ii), we have \( (e_{w_0} + \ldots + e_{w,\infty}) \in I_u, 0 \), and for each \( 0 < k \leq m \), \( 1 \in (e_{w_0} + \ldots + e_{w,\infty}) + I_u k \).

Step II. Assume that, for each cyclic non-projective module \( M \) and each non-injective module \( N \), \( \Ext_R(M, N) = 0 \). In particular, \( \Ext_R(S, N) = 0 \) and \( \Hom_R(S, (N)/N) = 0 \) for each simple module \( S \). Hence, \( (N)/N \) has a socle sequence with factors isomorphic to direct powers of \( S \). Thus, all simple modules are isomorphic. By 1.4, for each \( u \in 2^{\aleph_0} \) there is a \( j_u \in J \) with \( j_u K = \bigcap \Ker x, x \in I_u \). We shall show that \( P = \{j_u \mid u \in 2^{\aleph_0}\} \) is an independent subset of the right \( K \)-module \( J \). On the contrary, let \( \{j_{u^0}, \ldots, j_{u^m}\} \) be a dependent subset of \( P \) with a smallest number of elements. We have \( j_{u^0} k_0 + \ldots + j_{u^m} k_m = 0 \) for some \( 0 \neq k \in K, n = 0, \ldots, m \). By Step I, \( 0 = (e_{w_0} + \ldots + e_{w,\infty})(j_{u^0} k_0 + \ldots + j_{u^m} k_m) = j_{u^1} k_1 + \ldots + j_{u^m} k_m \), a contradiction. Hence, \( \dim K(J) \geq 2^{\aleph_0} \), a contradiction.

1.6. Theorem. Let \( R \) be a ring of cardinality \( < 2^{\aleph_0} \). Then the following conditions are equivalent:

(i) \( R \) is a left non-singular left \( T \)-ring;

(ii) either \( R = S \) or \( R = T \) or \( R = S \boxplus T \), where \( S \) is a completely reducible
ring of cardinality $< 2^\aleph_0$ and there is a division ring $D$ of cardinality $< 2^\aleph_0$ such that $T$ is Morita equivalent to the upper triangular matrix ring of degree 2 over $D$.

**Proof.** By [9, 4.4 and 6.1], [10, II.4] and 1.5.

## 2. INDEPENDENCE FOR COUNTABLE MODULES

In this section, we use a combinatorial principle due to S. Shelah to prove independence of WP for countable modules over simple countable non-completely reducible regular rings (various examples of such rings can be found e.g. in [5]).

**2.1.** For $E \subseteq \mathfrak{N}_1$, consider the assertion: $(A_E)$ Let $(n_v | v \in E)$ be a sequence of strictly increasing $\aleph_0$-sequences such that for each limit $v \in E : \sup_i n_i(i) = v$. Let $(h_v | v \in E)$ be a sequence of functions from $\aleph_0$ to $\aleph_0$. Then there is a function $f : \mathfrak{N}_1 \to \mathfrak{N}_0$ such that for each limit $v \in E : \exists j < \aleph_0 \forall i : (n_i(i))f = (i)h_v$.

**Lemma.** If ZFC is consistent, then ZFC + GCH + "$\exists E \subseteq \mathfrak{N}_1 : E$ stationary in $\mathfrak{N}_1 \& (A_E)$" is consistent.

**Proof.** Let $E$ be a stationary subset in $\mathfrak{N}_1$ such that $\mathfrak{N}_1 - E$ is stationary in $\mathfrak{N}_1$, too. Take $D$ - a normal $\mathfrak{N}_1$-complete filter over $\mathfrak{N}_1$ such that $(\mathfrak{N}_1 - E) \in D$ - and use [8, 2.1].

**2.2.** Let $R$ be a non-completely reducible regular ring. Let $I$ be a countably infinitely generated left ideal of $R$. By [5, § 2], $I = \sum_i e_i$, $i < \aleph_0$, where $e_i$, $i < \aleph_0$ are pairwise orthogonal idempotents of $R$. Let $E$ be a stationary subset in $\mathfrak{N}_1$ and $F$ the set of limit ordinals from $E$. Clearly, $F$ is stationary in $\mathfrak{N}_1$, too. Take a $v \in F$.

Then either there is a strictly increasing sequence $v_i$, $i < \aleph_0$ of limit ordinals less than $v$ with $\sup_i v_i = v$, or there is a limit ordinal $\mu < v$ with $v = \mu + \aleph_0$. In the former case, put $n_i(i) = v_i + i + 1$, $i < \aleph_0$ and in the latter put $n_i(i) = \mu + i + 1$, $i < \aleph_0$. Further, for $\alpha < \mathfrak{N}_1$ denote by $\pi_\alpha$ the $\alpha$-th canonical projection $R^{(\mathfrak{N}_1)} \to R$.

Now, for $v \in F$, denote by $g_v$ the element of $R^{(\mathfrak{N}_1)}$ with $\pi_{v_\alpha}(g_v) = e_i$, $\pi_{\tau}(g_v) = -e_i$, and $\pi_{\tau}(g_v) = 0$ otherwise. Let $M'_E = \sum_{i < \aleph_0} g_i$, $i < \aleph_0$, $v \in F$ and put $M_E = R^{(\mathfrak{N}_1)} / M_E$.

**Theorem.** $M_E$ is a strongly $\mathfrak{N}_1$-free, non-projective module. Moreover, $(A_E)$ implies $\text{Ext}_R(M_E, N) = 0$ for each countable $N \in R$-mod.

**Proof.** For $\alpha < \mathfrak{N}_1$ let $t_\alpha$ be the element of $R^{(\mathfrak{N}_1)}$ with $\pi_\alpha(t_\alpha) = 1$ and $\pi_\beta(t_\alpha) = 0$ otherwise. Put $M_0 = 0$ and for $0 < \mu < \mathfrak{N}_1$ let $M_\mu = \sum R(t_\alpha + M_\alpha)$, $\alpha < \mu$. Hence, for each limit $\mu < \mathfrak{N}_1$: $M_\mu = \bigcup M_v$, $v < \mu$. Further, for each $0 < \mu < \mathfrak{N}_1$: $M_\mu = \sum R_{\sigma, \alpha} \alpha < \mu$, where

(i) $v_\sigma = (1 - e_i) t_\sigma + M'_E$ and $Rv_\sigma \simeq R(1 - e_i)$ provided there are $i < \aleph_0$ and $\sigma \in F$, $\sigma < \mu$ with $\alpha = n_\sigma(i)$,

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(ii) \( v_a = t_a + M_E \) and \( R_{v_a} \simeq R \) otherwise.

Hence, for each \( \mu < \aleph_1 \), \( M_\mu \) is projective. Moreover, for \( v < \mu < \aleph_1 \), \( M_\mu/M_v \simeq \simeq \sum_i I_{a_i} v \leq \alpha < \mu \), where

(i) \( I_a = R(1 - e_i) \) provided there are \( i < \aleph_0 \) and \( \sigma \in F \), \( \sigma < \mu \) with \( \alpha = n_a(i) \),

(ii) \( I_a = R \sum_{i \in A_a} R e_i \) provided \( \alpha \in F \), \( v \leq \alpha < \mu \) and \( A_a = \{ i \mid n_a(i) < v \} \),

(iii) \( I_a = R \) otherwise.

Now, if \( v \notin F \), then for all \( \mu \) with \( v < \mu < \aleph_1 \), all the sets \( A_a \), \( \alpha \in F \), \( v \leq \alpha < \mu \) are finite and hence \( M_\mu/M_v \) is projective. Thus \( M_E = \bigcup M_v \), \( v < \aleph_1 \) is strongly \( \aleph_1 \)-free. On the other hand, if \( v \in F \), then \( M_{v+1}/M_v \simeq R/I \) is not projective. By [4, 5.1 and § 18], \( M_E \) is not projective. To prove the rest, let \( N \) be a countable module and \( r : N \rightarrow \aleph_0 \) an injective mapping. Let \( p \in \text{Hom}_R(M_E, N) \). Assume \( (A_E) \). Then also \( (A_F) \), for \( (n_v \mid v \in F) \) defined as above and for \( h_v : \aleph_0 \rightarrow \aleph_0 \) defined by \( h_v = (g_v) pr, i < \aleph_0 \), \( v \in F \). Note that \( (g_v) pr \in (e_i N) r \) for all \( i < \aleph_0 \), \( v \in F \).

Hence, there is a function \( f : \aleph_1 \rightarrow \aleph_0 \) such that for each \( v \in F \) there is a \( j_v < \aleph_0 \) with \( n_v(i) f r^{-1} = (g_v) p \), for all \( j_v < i < \aleph_0 \). For each \( \alpha \in F \) and each \( i \leq j_v \) put \( \delta_{i_v} = n_v(i) f r^{-1} \) if there is a \( \beta \in F \) such that \( j_\beta < i \) and \( n_\alpha(i) = n_\beta(i) \), and \( \delta_{i_v} = 0 \) otherwise. Define a \( q \in \text{Hom}_R(R^{(\aleph_1)}, N) \) by

(i) \( t_{\alpha}q = (e_\alpha) r^{-1} \) provided there are \( v \in F \) and \( i < \aleph_0 \) such that \( j_v < i \) and \( \alpha = n_v(i) \),

(ii) \( t_{\alpha}q = \sum_{i=0}^{j_v} (\delta_{i_v} - (g_{i_v}) p) \) provided \( \alpha \in F \),

(iii) \( t_{\alpha}q = 0 \) otherwise.

Then, for each \( i < \aleph_0 \), \( v \in F \), we have \( (g_v) q = e_i (t_{n_v(i)} q - t_{\alpha} q) = (g_v) p \), whence \( \text{Ext}_R(M_E, N) = 0 \).

2.3. Theorem. Assume GCH + "\( \exists E \subseteq \aleph_1 \): \( E \) stationary in \( \aleph_1 \) & \( (A_E) \)". Let \( R \) be a simple countable non-completely reducible regular ring. Then no non-zero countably generated module has WP.

Proof. By 2.2 and [10, III.2 and III.4].

2.4. Theorem. Let \( R \) be a simple countable non-completely reducible regular ring. Then the assertion "every countably generated module has WP" is independent of ZFC.

Proof. By [10, III.6] (or by [10, III.4] and [4, 21.6]), the assertion holds if \( V = L \) is assumed. The rest follows from 2.1 and 2.3.

3. ARTIN'S PROBLEM AND WP

Recently (see [6] and [3]), Artin's problem for skew field extensions has been solved: for each pair of cardinals \( (\alpha, \beta) \) with \( \alpha > 1, \beta > 1 \), there are division rings \( S \) and \( T \) such that \( T \) is a subring of \( S \), the left dimension of \( S \) over \( T \) is \( \alpha \) and the right
dimension is $\beta$. Here, in 3.2, we use this fact to construct a matrix ring $R$ such that $R$ is not a left $T$-ring, but each cyclic module has WP. Our result was announced in [9, 5.4].

Let $m$ be a natural number, $m \geq 1$, $n = m + 1$, and let $S, T$ be division rings such that $T$ is a subring of $M_{m \times m}(S)$. If $\kappa$ is a cardinal, $\kappa \geq 1$, we shall shortly write $M_\kappa$ and $M_\kappa^+$ instead of $RFM_{m \times m}(S)$ and $RFM_{n \times n}(S)$, respectively. Note that $M_\kappa$ ($M_\kappa^+$) is a left $M_m$ ($M_n^+$, respectively)-module. For a matrix $a \in M_\kappa^+$, let $a' \in M_\kappa$ be such that $a'_{ij} = a_{i+1,j+1}$ for all $0 \leq i, j < m$. Let $R = U(m, S, T)$ be the subring of $M_\kappa^+$ formed by the set of matrices $a \in M_\kappa^+$ with $a_{i0} = \ldots = a_{m0} = 0$ and $a' \in T$. Let $e \in R$ be such that $e_{00} = 1$ and $e_{ij} = 0$ otherwise and put $f = 1 - e$. It is easy to see that $\{e, f\}$ is a basic set of primitive idempotents of $R$, whence $R$ is a basic ring. Further properties of $R$ can be found e.g. in [9, 5.1].

If $\kappa$ is a cardinal, $\kappa \geq 1$ and $X$ ($Y$) is a subset of $M_m$ ($M_\kappa$, respectively), we put

$$X \cdot Y = \left\{ \sum_{i=0}^{k} x_i y_i : k < \aleph_0, x_i \in X, y_i \in Y \text{ for all } i = 0, \ldots, k \right\}.$$ 

3.1. Lemma. Let $\kappa$ be a cardinal, $\kappa \geq 1$. Then the following conditions are equivalent:

(i) there are a non-projective module $M$ and a non-injective module $N$ such that $\dim(\text{Soc}(N)) = \kappa$ and $\text{Ext}_R(M, N) = 0$,

(ii) there are a finitely generated right $T$-submodule $X$ of $M_m$ and a proper left $T$-submodule $Y$ of $M_\kappa$ such that $X \cdot Y = M_\kappa$.

Proof. Denote by $A$ the module $R/\text{Soc}(R)$. Let $N$ be a non-injective module. Using [9, 5.1], it is easy to see that $I(N)/\text{Soc}(N)$, and thus $I(N)/N$, is a direct sum of copies of $A$. Further, if $M$ is any module, then by [9, 5.1.(i)], there is a projective cover $(P, p)$ of $M$. By [1, 28.13], there are cardinals $\alpha, \beta, \gamma, \delta$ such that $P = (R\text{e})^{(\alpha)} + (R\text{f})^{(\beta)}$ and $\text{Ker} p \cong (R\text{e})^{(\gamma)} + (R\text{f})^{(\delta)}$. Since $\text{Ker} p$ is superfluous in $P$, we have $\delta = 0$ and $\text{Ker} p \subseteq (R\text{f})^{(\delta)}$.

Assume (i). Let $x \in \text{Ker} p$ be such that $Rx$ is a direct summand of $\text{Ker} p$ and $\text{Ann}_R(x) = Rf$. Since $\text{Ext}_R(P/\text{Ker} p, N) = 0$, we have $\text{Ext}_R(P/Rx, N) = 0$. Let $q$ be the smallest natural number such that $x \in (R\text{f})^{(\delta)}$, i.e. $x = (x_0, \ldots, x_{q-1})$, where $0 \neq x_k \in \text{Soc}(R\text{f})$ for all $k < q$. Put $G = (R\text{f})^{(\delta)}/R\text{f}$. Then $G$ is not projective and $\text{Ext}_R(G, N) = 0$. By [9, 5.1.(ii)], we may assume that $\text{Hom}_R(A, N) = 0$. Hence, by [9, 5.1], we have $I(N) = M_\kappa$ and

$$\text{Soc}(N) = \text{Soc}(M_\kappa) = \{a \in M_\kappa^+ : a_{ij} = 0 \text{ for all } 0 < i < m \text{ and } 0 \leq j < \kappa\}.$$ 

Now, put $Y = \{a' : a \in N\}$. By [9, 5.1.(vi)], $Y$ is a proper left $T$-submodule of $M_\kappa$. Further, for $0 \leq i < m$ and $0 \leq k < q$, let $z_k \in M_m$ be such that $(z_k)_{ij} = (x_k)_{0,j+1}$ for all $0 \leq j < m$ and $(z_k)_{ij} = 0$ otherwise. Let $X$ be the right $T$-submodule of $M_m$ generated by $\{z_k : 0 \leq i < m \text{ and } 0 \leq k < q\}$. We shall prove that $X \cdot Y = M_\kappa$. 

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Take \( u \in M_\kappa \) and let \( u_i \) be the \( i \)-th row of \( u \), hence \( u_i \in S(\kappa) \) for all \( 0 \leq i < m \). Clearly, for each \( 0 \leq i < m \), there are \( v^i_k \in M_\kappa \), \( 0 \leq k < q \), such that \( \sum_{k=0}^{q-1} x_k v^i_k = u_i \). Let \( w^i_k \in M_\kappa^+ \) be such that \( (w^i_k)' = v^i_k \), \( 0 \leq i < m \) and \( 0 \leq k < q \). Since \( \sum_{k=0}^{q-1} x_k w^i_k \in \text{Soc} (N) \) and \( \text{Ext}_R (G, N) = 0 \), there are \( t^i_k \in M_\kappa^+ \), \( 0 \leq i < m \) and \( 0 \leq k < q \), with \( \sum_{k=0}^{q-1} x_k t^i_k = 0 \) and \( t^i_k + N = w^i_k + N \) for all \( 0 \leq i < m \) and \( 0 \leq k < q \). Now, put \( y^i_k = (w^i_k - t^i_k)' \), \( 0 \leq i < m \) and \( 0 \leq k < q \). Then \( y^i_k \in Y \), for all \( 0 \leq i < m \) and \( 0 \leq k < q \), and \( \sum_{k=0}^{q-1} x_k y^i_k = u_i \), for all \( 0 \leq i < m \), whence \( \sum_{i=0}^{m-1} \sum_{k=0}^{q-1} y^i_k = u \).

Assume (ii). Let \( N \) be a submodule of \( M_\kappa^+ \) such that \( \text{Soc} (N) = \{ a \in M_\kappa^+ \mid a_{ij} = 0 \text{ for all } 0 \leq i < m \text{ and } 0 \leq j < \kappa \} \) and \( Y = \{ a' \mid a \in N \} \). Clearly, \( N \) is not injective and \( I(N) = M_\kappa^+ \). Since \( \text{Soc} (N) = \text{Soc} (M_\kappa^+) \), [9, 5.1] implies \( \dim (\text{Soc} (N)) = \kappa \). Let \( \{ z_k \mid 0 \leq k < q \} \) be a finite set of generators of the right \( T \)-module \( X \). For each \( 0 \leq k < q \), let \( x_k \in \text{Soc} (Rf) \) be such that the 0-th row of \( x_k \) equals the 0-th row of \( z_k \).

Then \( \sum_{k=0}^{q-1} x_k N = \text{Soc} (N) \). Let \( x = (x_0, \ldots, x_{q-1}) \in \sum_{k=0}^{q-1} Rf_k \), where \( f_k = f \) for all \( 0 \leq k < q \), and put \( M = \sum_{k=0}^{q-1} Rf_k / R \). We shall prove that \( \text{Ext}_R (M, N) = 0 \).

Take \( g \in \text{Hom}_R (M, I(N)/N) \). Then \( (f_k + R \times) g = u_k + N \), for all \( 0 \leq k < q \), where \( u_k \in M_\kappa^+ \), \( 0 \leq k < q \). Since \( \sum_{k=0}^{q-1} x_k u_k \in \text{Soc} (N) \), there exist \( n_k \in N \), \( 0 \leq k < q \), such that \( \sum_{k=0}^{q-1} x_k (u_k - n_k) = 0 \). Hence, if \( h \in \text{Hom}_R (M, I(N)) \) is defined by \( (f_k + R \times) h = u_k - n_k \), \( 0 \leq k < q \), then \( g = h \pi \), where \( \pi: I(N) \to I(N)/N \) is the canonical projection, whence \( \text{Ext}_R (M, N) = 0 \).

3.2. Theorem. Let \( S, T \) be division rings such that \( T \) is a subring of \( S \), the left dimension of \( S \) over \( T \) is two and the right dimension is infinite. Let \( R = U(1, S, T) \).

Then \( \text{Ext}_R (M, N) = 0 \) for each non-projective module \( M \) and each cyclic non-injective module \( N \), but \( R \) is not a left \( T \)-ring.

Proof. By [9, 5.3], \( R \) is not a left \( T \)-ring (in fact, the proof of [9, 5.3] shows that there are a non-projective 2-generated module \( M \) and a non-injective module \( N \) such that \( \text{Ext}_R (M, N) = 0 \)). Further, for \( \kappa = 1 \), we have \( M_\kappa = S \) and hence \( X \cdot Y \neq S \), for any finitely generated right \( T \)-submodule \( X \) of \( S \) and any proper left \( T \)-submodule \( Y \) of \( S \). Now, it is easy to see that each cyclic module is a direct sum of modules \( N \) with \( \dim (\text{Soc} (N)) = 1 \), and it suffices to apply 3.1.
References


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