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ON COMMON EXTENSIONS OF TWO QUASI-MEASURES

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1. INTRODUCTION

Throughout the paper $X$ stands for an arbitrary (non-empty) set and $M$ and $N$ stand for algebras of subsets of $X$. We denote by $S(M)$ the linear space spanned by the characteristic functions $1_M$, $M \in M$. We note that

\begin{equation}
S(M) \cap S(N) = S(M \cap N).
\end{equation}

The closure of $S(M)$ in the Banach space of all real-valued bounded functions on $X$ with the supremum norm $\| \cdot \|$ is denoted by $B(M)$. The dual of $B(M)$ can be identified with the Banach space $ba(M)$ of all real-valued quasi-measures, i.e., bounded additive set functions, on $M$ with the total variation norm also denoted by $\| \cdot \|$ (see, e.g., [7], Corollary 4.7.5). The unique element of $B(M)^*$ corresponding to $\mu \in ba(M)$ is denoted by $I_{\mu}$ ([7], Theorem 4.7.4).

We note that

\begin{equation}B'(M) \cap B'(N) = B(M \cap N)
\end{equation}
and the equality holds in case $M$ and $N$ are $\sigma$-algebras, but not in general. Indeed, for $M$ and $N$ as specified in Example 1 below the identity function on $[0, 1]$ is a counter-example.

We are concerned with the following problem:

Given $\mu \in ba(M)$ and $\nu \in ba(N)$ which are consistent, i.e.,

$$\mu \big| M \cap N = \nu \big| M \cap N,$$

when does there exist $\varphi \in ba(F)$, where $F$ stands for the algebra generated by $M \cup N$, with $\varphi \big| M = \mu$ and $\varphi \big| N = \nu$ (called in the sequel a common extension of $\mu$ and $\nu$)?

This problem has been suggested by the papers by Guy [1] and Pták [6]. The former gave a complete solution to a version of the problem with $\mu$, $\nu$ and $\varphi$ positive (see also [2], [4] and [7], Theorem 3.6.1). The latter dealt with the general question of extending simultaneously two continuous linear functionals defined on subspaces of a locally convex space. We also note that, with the boundedness condition dropped, the problem admits an easy affirmative solution ([4], Corollary 2.1, [7], Theorem 3.6.2).
We shall present below two negative examples and three affirmative partial solutions to the problem*). The first two solutions (Propositions 1 and 2) are of global character, i.e., they involve assumptions on \( M \) and \( N \) only, while the third (Corollary) imposes some strong conditions on one of the quasi-measures to be extended. The global solutions are related to some results of [6] (see Remark 2 below). Reasonable necessary and sufficient conditions in order that the answer to the problem be affirmative individually, i.e., in terms of \( \mu \) and \( \nu \), seem hard to find. The proofs of Propositions 1 and 2 are based on the Hahn-Banach theorem.

2. NEGATIVE RESULTS

The following examples show that the condition that \( M \cap N = \{ \emptyset, X \} \) is not sufficient even in the case when \( \mu \) and \( \nu \) are positive. In the first example \( \mu \) and \( \nu \) are additionally two-valued, while in the second \( M \) and \( N \) are \( \sigma \)-algebras generated by countable partitions and \( \mu \) and \( \nu \) are measures.

Example 1. Suppose \( M \cap N = \{ \emptyset, X \} \) and the following condition holds:
(3) There exist \( M_n \in M \) and \( N_n \in N \) with \( \emptyset \neq M_1 < N_1 < M_2 < N_2 < \ldots < X \) (e.g., \( X = [0, 1) \)) and \( M \) and \( N \) are generated by the families
\[
\{ [0, a); 0 < a < 1 \text{ is rational} \},
\]
\[
\{ [0, a); 0 < a < 1 \text{ is irrational} \},
\]
respectively. Extend \( \{ M_n: n = 1, 2, \ldots \} \) to a maximal ideal \( I \) in \( M \) and put
\[
\mu(M) = 0 \text{ if } M \in I \text{ and } \mu(M) = 1 \text{ if } M \notin I.
\]
Choose \( x \in N_1 \) and for \( N \in N \) put
\[
\nu(N) = 0 \text{ if } x \notin N \text{ and } \nu(N) = 1 \text{ if } x \in N.
\]
Observe that every common additive extension \( \varphi \) of \( \mu \) and \( \nu \) to \( F \) is unbounded. Indeed, \( N_n \setminus M_n \) are pairwise disjoint and
\[
\varphi(N_n \setminus M_n) = \nu(N_n) - \mu(M_n) = 1.
\]

Example 2. Let \( X \) be the set of all natural numbers and let \( M \) and \( N \) be the \( \sigma \)-algebras of subsets of \( X \) generated by the partitions
\[
\{ 1 \}, \{ 2, 3 \}, \ldots, \{ 2n - 2, 2n - 1 \}, \ldots,
\]
\[
\{ 1, 2 \}, \{ 3, 4 \}, \ldots, \{ 2n - 1, 2n \}, \ldots,
\]
respectively. Clearly, \( M \cap N = \{ \emptyset, X \} \). Let \( (a_n) \) and \( (b_n) \) be sequences of positive

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* Some of these results were announced at the 13th Winter School on Abstract Analysis, Srní (in the Šumava Mountains), 1985; see Suppl. Rend. Circ. Mat. Palermo (2), to appear.
real numbers such that
\[ a_n < 0, \quad \sum_{n=1}^{\infty} a_n = \infty, \quad b_{n+1} > a_n - a_{n+1} \quad \text{and} \quad \sum_{n=1}^{\infty} b_n < \infty \]
(e.g., \( a_n = 1/n, \ b_{n+1} = 1/n^2 \)). Put
\[
\mu(\{1\}) = b_1 \quad \text{and} \quad \mu(\{2n - 2, 2n - 1\}) = b_n,
\]
\[
v(\{1, 2\}) = b_1 + a_1 \quad \text{and} \quad v(\{2n - 1, 2n\}) = b_n + (a_n - a_{n-1}).
\]
Clearly, \( \mu \) and \( v \) extend uniquely to (positive) measures on \( M \) and \( N \), respectively, which we also denote by \( \mu \) and \( v \). Moreover, \( \mu(X) = v(X) \) since \( a_n \to 0 \). Let \( \varphi \) be a common additive extension of \( \mu \) and \( v \) to \( F \). We have
\[
\varphi(\{2n\}) = v(\{1, \ldots, 2n\}) - \mu(\{1, \ldots, 2n - 1\}) = a_n.
\]
Hence \( \varphi \) is unbounded.

3. AFFIRMATIVE RESULTS AND COMMENTS

We say that \( M \) and \( N \) are weakly independent if, given two partitions \( \{M_1, \ldots, M_m\} \subset M \) and \( \{N_1, \ldots, N_n\} \subset N \) of \( X \) into non-empty sets, the set
\[
\{(i,j): 1 \leq i \leq m, 1 \leq j \leq n \text{ and } M_i \cap N_j \neq \emptyset\}
\]
contains a row \( \{(i_0, j): 1 \leq j \leq n\} \) and a column \( \{(i, j_0): 1 \leq i \leq m\} \) of the matrix \( \{(i, j): 1 \leq i \leq m, 1 \leq j \leq n\} \).

Clearly, this condition implies \( M \cap N = \{0, X\} \). Moreover, it is implied by the independence of \( M \) and \( N \) in the sense of Marczewski ([5], p. 220), i.e., the condition that for every pair of non-empty sets \( M \in M \) and \( N \in N \) we have \( M \cap N \neq \emptyset \). The latter implication cannot be reversed as shown by the following simple

Example 3. Let \( X \) be the set of all natural numbers and let \( M \) and \( N \) be algebras of subsets of \( X \) generated by the even and the odd singletons, respectively. Then \( M \) and \( N \) are weakly independent but not independent.

Lemma 1. Let \( M \) and \( N \) be weakly independent and let \( g \in S(M) \) and \( h \in S(N) \). Then there exists a real number \( c \) such that
\[
3\|g + h\| \geq \|g - c\| + \|h + c\|.
\]
Proof. Let \( a_i, i = 1, \ldots, m, \) and \( b_j, j = 1, \ldots, n, \) be all the values of \( g \) and \( h \), respectively. Let \( i_0, j_0 \) be such that
\[
g^{-1}(a_{i_0}) \cap h^{-1}(b_{j_0}) = \emptyset \quad \text{and} \quad g^{-1}(a_i) \cap h^{-1}(b_{j_0}) = \emptyset
\]
for \( j = 1, \ldots, n \) and \( i = 1, \ldots, m \). Then \( |b_j + a_{i_0}| \leq \|g + h\| \) and
\[
|a_i - a_{i_0}| \leq |a_i + b_{j_0}| + |a_{i_0} + b_{j_0}| \leq 2\|g + h\|.
\]
Thus we may take \( c = a_{i_0} \).

The following is a partial extension of a result of Marczewski ([5], Theorem 1).
Proposition 1. Let $M$ and $N$ be weakly independent and let $\mu \in ba(M)$ and $\nu \in ba(N)$ be consistent. Then there exists $\varphi \in ba(F)$ which is a common extension of $\mu$ and $\nu$ and satisfies $\|\varphi\| \leq 3 \max(\|\mu\|, \|\nu\|)$.

Proof. By (1), $S(M) \cap S(N)$ consists of constant functions only. Hence the formula $J(g + h) = I_{\mu}(g) + I_{\nu}(h)$ defines unambiguously a linear functional $J$ on $S(M) + S(N)$. In view of Lemma 1,

$$\|J\| \leq 3 \max(\|\mu\|, \|\nu\|).$$

Hence, by the Hahn-Banach theorem, $J$ extends to a continuous linear functional $K$ on $S(F)$ with $\|K\| = \|J\|$. Then $\varphi$ defined on $F$ by $\varphi(F) = K(1_F)$ is as desired.

It follows from [3], Example 1, that the constant "3" in Proposition 1 is best possible even in the case where $\mu$ and $\nu$ are two-valued.

Lemma 2. If $f \in B(M)$ has finite range, then $f \in S(M)$.

Proof. Clearly, it is enough to prove that if $Z_j$, $j = 1, \ldots, n$, are non-empty and pairwise disjoint and $Z_j \notin M$ for some $j$, then

$$\left\| \sum_{j=1}^{n} b_j 1_{Z_j} + g \right\| \geq \frac{1}{2} \min \{|b_j|, |b_k - b_l|: 1 \leq j, k, l \leq n; k \neq l\}$$

whenever $b_j$, $j = 1, \ldots, n$, are (non-zero distinct) real numbers and $g \in B(M)$. We may and do assume that $g \in S(M)$. Accordingly, let $g = \sum_{i=1}^{m} a_i 1_{M_i}$, where $M_i \in M$, $i = 1, \ldots, m$, are non-empty and pairwise disjoint. Denote by $\delta$ the right-hand side of the above inequality.

Suppose, to get a contradiction, that

$$\left\| \sum_{j=1}^{n} b_j 1_{Z_j} + \sum_{i=1}^{m} a_i 1_{M_i} \right\| < \delta.$$

Since $|b_j| \geq \delta$, we have

(a) $Z_j \subset \bigcup_{i=1}^{m} M_i$ for $j = 1, \ldots, n$.

Moreover,

(b) $M_i \cap Z_j = \emptyset$ implies $M_i \subset Z_j$.

Indeed, first observe that $M_i \cap Z_k = \emptyset$ for all $k \neq j$. Otherwise $|b_j + a_i|, |b_k + a_i| < \delta$, which implies $|b_j - b_k| < 2\delta$, a contradiction with the definition of $\delta$. If $M_i \cap Z_j = \emptyset$, it follows that $|a_i| < \delta$. Since, moreover, $|b_j + a_i| < \delta$, we get $|b_j| < 2\delta$, which contradicts the definition of $\delta$.

From (a) and (b) we infer that for each $j$

$$Z_j = \bigcup_{i \in T_j} M_i,$$

where $T_j = \{1 \leq i \leq n: M_i \cap Z_j = \emptyset\},$

which contradicts the assumption that $Z_j \notin M$ for some $j$.

The following is a partial generalization of Proposition 3 of [3].

Proposition 2. Let $N$ be finite and let $\mu \in ba(M)$ and $\nu \in ba(N)$ be consistent. Then there exists $\varphi \in ba(F)$ which is a common extension of $\mu$ and $\nu$.  

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Proof. In view of (2) and Lemma 2,

\[ B(M) \cap B(N) = B(M \cap N). \]

Hence the formula \( J(g + h) = I_\mu(g) + I_\nu(h) \) defines unambiguously a linear functional on \( B(M) + B(N) \). Clearly, the restrictions of \( J \) to \( B(M) \) as well as to \( B(N) \) are continuous. Since \( B(M) \) is complete and \( B(N) \) is finite-dimensional, it is not hard to see that \( J \) itself is continuous. Now, applying the Hahn-Banach theorem as in the proof of Proposition 1, we get the assertion.

We shall need the following notation. For \( v \in ba(N) \) we put

\[ \mathcal{N}(v) = \{ N \in N: \nu(S) = 0 \text{ for all } N \supseteq S \in N \}. \]

In case \( \mathcal{N}(v) \) is a hereditary family of subsets of \( X \), \( v \) is called (Lebesgue) complete.

**Corollary.** Let \( \mu \in ba(M) \) and \( v \in ba(N) \) be consistent and let \( v \) be complete and have finite range. Then there exists \( \varphi \in ba(F) \) which is a common extension of \( \mu \) and \( v \).

**Proof.** First we note that if \( N \in \mathcal{N}(v) \) and \( N \supseteq M \in M \), then \( \mu(M) = 0 \). (Indeed, \( M \in \mathcal{N}(v) \), and so \( v(M) = 0 \).) Hence, by [3], Proposition 1, \( \mu \) extends to a real-valued quasi-measure \( \mu' \) on \( M' \), where \( M' \) stands for the algebra generated by \( M \cup \cup \mathcal{N}(v) \), such that

\[ \mu'(M \downarrow Z) = \mu(M) \quad \text{for all } M \in M \quad \text{and} \quad Z \in \mathcal{N}(v). \]

We claim that \( \mu' \) and \( v \) are consistent. Indeed, if \( M \downarrow Z = N \) with \( M \in M \), \( Z \in \mathcal{N}(v) \) and \( N \in N \), then \( M = N \downarrow Z \). Hence \( \mu(M) = v(N) \), and so \( \mu'(N) = \nu(N) \).

Let \( N' \) be a finite subalgebra of \( N \) such that for every \( N \in N \) there exists \( N' \in N' \) with \( N \downarrow N' \in \mathcal{N}(v) \). Then \( F \) coincides with the algebra generated by \( M' \cup N' \). Put \( v' = v \mid N' \). Clearly, \( \mu' \) and \( v' \) are consistent, whence, by Proposition 2, there exists \( \varphi \in ba(F) \) which is a common extension of \( \mu' \) and \( v' \). Since \( \mathcal{N}(v) \subseteq \mathcal{N}(\varphi) \) and \( \varphi \mid N' = v' \), we have \( \varphi \mid N = v \).

We note that both the above assumptions on \( v \) are essential as is shown by Examples 1 and 2, respectively.

**Remark 1.** Condition (3) of Example 1 admits the following strengthening:

\[ (\forall M \in M, M \neq X) (\exists N \in N, N \neq \emptyset) \]

\[ (\forall N \in N, N \neq X) (\exists M \in M, M \neq \emptyset) \quad [M \cap N = \emptyset]. \]

The latter might be called the total dependence of \( M \) and \( N \). Unfortunately, it is much stronger than just the negation of the weak independence of \( M \) and \( N \). This sheds some light on the dimension of the gap which exists between the negative Example 1 and the affirmative Proposition 1.

We shall present another strengthening of condition (3).

**Proposition 3.** If \( B(M) \cap B(N) \) contains a non-constant function \( f \), then (3) holds.

**Proof.** Fix \( a \in f(X) \). We first show that given \( \varepsilon > 0 \), we can find \( M \in M \) such that

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\[ |a - f(x)| < 2\varepsilon \quad \text{for all} \quad x \in M, \quad |a - f(x)| > \varepsilon \quad \text{for all} \quad x \in X \setminus M. \]

Indeed, let \( S_1, \ldots, S_n \in M \) be a partition of \( X \) with
\[ |f(x) - f(y)| < \varepsilon \quad \text{whenever} \quad x, y \in S_i, \quad i = 1, \ldots, n \]
(see [7], Proposition 4.7.2). Put
\[ T = \{ 1 \leq i \leq n : |a - f(x)| \leq \varepsilon \text{ for some } x \in S_i \} \quad \text{and} \quad M = \bigcup_{i \in T} S_i. \]

Fix \( \varepsilon > 0 \) with \( 2^{-2\varepsilon} < \sup \{ |a - f(x)| : x \in X \} \). By what we have just proved, there exist \( M_n \in M \) and \( N_n \in N \) with \( |a - f(x)| < 2^{-2\varepsilon} \) for all \( x \in M_n \), \( |a - f(x)| > 2^{-(2n+1)\varepsilon} \) for all \( x \in X \setminus M_n \), \( |a - f(x)| < 2^{-(2n+1)\varepsilon} \) for all \( x \in N_n \), \( |a - f(x)| > 2^{-(2n+2)\varepsilon} \) for all \( x \in X \setminus N_n \), \( n = 1, 2, \ldots \). Then \( M_1 \neq X \) and \( f^{-1}(a) \subset M_n \). Moreover, \( (X \setminus M_n) \cap N_n = \emptyset \), and so \( N_n \subset M_n \). Analogously, \( M_{n+1} \subset N_n \). Thus \( X \setminus M_n \) and \( X \setminus N_n \) satisfy (3).

Remark 2 (H. Weber). The existence of a common extension \( \varphi \in ba(F) \) for every consistent pair \( \mu \in ba(M) \) and \( \nu \in ba(N) \) is equivalent to the conjunction of the conditions:

(i) \( B(M) \cap B(N) \subset B(M \cap N) \),

(ii) \( B(M) + B(N) \) is closed in \( B(F) \).

This follows from [6], Theorems 2.1 and 2.4, and (1). In case \( M \cap N = \{0, X\} \), the necessity of (i) also follows from Proposition 3 and Example 1 above. Finally, note that (i) \( \not\Rightarrow \) (ii) (see Example 2).


References


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