Czechoslovak Mathematical Journal

Július Korbaš

Some partial formulae for Stiefel-Whitney classes of Grassmannians

Czechoslovak Mathematical Journal, Vol. 36 (1986), No. 4, 535–540

Persistent URL: http://dml.cz/dmlcz/102114

Terms of use:

© Institute of Mathematics AS CR, 1986

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

Czechoslovak Mathematical Journal, 36 (111) 1986, Praha

SOME PARTIAL FORMULAE FOR STIEFEL-WHITNEY CLASSES OF GRASSMANNIANS

JÚLIUS KORBAŠ, Žilina

(Received April 23, 1985)

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $G_{n,r}$, $n \geqq 2r$, denote the Grassmann manifold of all linear r-subspaces in the real Euclidean n-space R^n. After Hsiang and Szczarba's paper [3] (cf. Theorem 1.1 ibid.) had appeared it became reasonable to ask how to explicitly express the Stiefel-Whitney class $w_k(G_{n,r}) \in H^k(G_{n,r}; Z_2)$ in terms of $w_i(\gamma_{n,r})$, $i = 1, \ldots, r$, the Stiefel-Whitney classes of the canonical r-plane bundle $\gamma_{n,r}$ over $G_{n,r}$.

This question turned out to be difficult but, at least to some extent, manageable: [7] gives an insight into the cases $r = 2$ and $r = 3$: in [8], the class of maximal dimension, $w_{r(n-r)}(G_{n,r})$, is expressed as a determinant; in [2], the classes $w_k(G_{n,r})$, $k \leqq 9$, are computed as polynomials in $w_i(\gamma_{n,r})$, $i = 1, \ldots, r$, and in [4], a more effective method (as compared to that of [2]) of inductive computation of $w_k(G_{n,r})$'s will be described and illustrated by some new complete formulae, e.g. for $w_{16}(G_{n,r})$.

On the other hand, it seems to be impossible to get a complete formula for $w_k(G_{n,r})$ with k, n, r arbitrary.

Nevertheless, we prove the following general result.

(1.1) **Theorem.** *Let w_i abbreviate $w_i(\gamma_{n,r}) \in H^i(G_{n,r}; Z_2)$, and for any positive integer p let p_i be its i-th ($i \geqq 0$) dyadic coefficient. Then we have, for $j \geqq 2$:*

(i) $w_{4j-2}(G_{n,r}) = n_0 w_{4j-2} + (n_1 + r_0)\, w_{2j-1}^2 + (n_1 + r_0)\, w_1^2 w_{2j-2}^2 + \ldots,$

(ii) $w_{4j-1}(G_{n,r}) = n_0 w_{4j-1} + n_0(n_1 + r_0)\, w_1 w_{2j-1}^2 + n_0(n_1 + r_0)\, w_1^3 w_{2j-2}^2 + \ldots,$

(iii) $w_{4j}(G_{n,r}) = n_0 w_{4j} + (n_1 + r_0)\, w_{2j}^2 + (1 + n_1 + r_0) w_1^2 w_{2j-1}^2 + \ldots,$

(iv) $w_{4j+1}(G_{n,r}) = n_0 w_{4j+1} + n_0(n_1 + r_0)\, w_1 w_{2j}^2 + n_0(1 + n_1 + r_0)\, w_1^3 w_{2j-1}^2 + \ldots.$

We hope that these formulae can be extended. But even in the present form they find quite interesting applications. We content ourselves with mentioning just one, namely that to the vector fields problem on $G_{n,r}$.

As is well-known, the Euler-Poincaré characteristic of even-dimensional real grassmannian does not vanish. Therefore the question, what is the maximal number of everywhere linearly independent tangent vector fields on $G_{n,r}$, the so-called span $G_{n,r}$ (cf. [10], for example), is intriguing only when n is even and r is odd.

Writing such an odd r as $2s - 1$, we put:

$$\begin{aligned} a'(n, r) &= \max\{j \in N;\ 2 \leqq j \leqq s - 1,\ 4j < n - r\}, \\ a''(n, r) &= \max\{j \in N;\ 2 \leqq j \leqq s,\ 4j - 2 < n - r\}, \\ a(n, r) &= \max\{4a', 4a'' - 2\}, \\ b(n, r) &= \max\{j \in N;\ 2 \leqq j \leqq s,\ 4j < n - r\}, \end{aligned}$$

where we set $\max \emptyset = 0$.

Of course, these $a(n, r)$ and $b(n, r)$ are non-trivial provided $s \geqq 2$ with $n - r \geqq 7$ and $n - r \geqq 9$, respectively. This, however, covers almost all cases of interest. Indeed, the case $r = 1$ is that of the well-understood projective spaces.

Now Theorem (1.1) yields

(1.2) **Proposition.** *If* $n \equiv 0 \bmod 4$, *then* $w_a(G_{n,r}) \neq 0$ *and if* $n \equiv 2 \bmod 4$, *then* $w_{4b}(G_{n,r}) \neq 0$.

(1.3) **Corollary.** *If* $n \equiv 0 \bmod 4$, *then span* $G_{n,r} \leqq r(n - r) - a(n, r)$ *and if* $n \equiv 2 \bmod 4$, *then span* $G_{n,r} \leqq r(n - r) - 4b(n, r)$.

A little thinking of (1.2) justifies

(1.4) **Conjecture.** *For each* $m \geqq 0$, $m \in N$, *there exists* $k(m) \in N$ *such that* $w_{2k-2}(G_{4k-2m-2,2k-2m-1}) \neq 0$ *for any* $k \geqq k(m)$.

As one sees easily, in fact just the case of m even is still open.

2. PROOFS OF RESULTS

Let $T(G_{n,r})$ denote the tangent bundle of $G_{n,r}$. Since by [3],

$$T(G_{n,r}) \oplus \gamma_{n,r} \otimes \gamma_{n,r} \cong n\gamma_{n,r}, \tag{2.1}$$

we shall begin with establishing some facts about the Stiefel-Whitney classes of the n-fold Whitney sum $n\eta = \eta \oplus \ldots \oplus \eta$ and of the tensor square $\eta \otimes \eta$, η being an r-plane bundle over a paracompact space.

The k-th Stiefel-Whitney class $w_k(n\eta)$ is easily expressible in terms of $w_1^{i(1)} \ldots w_k^{i(k)}$, where $i(1) + 2i(2) + \ldots + ki(k) = k$ and w_i abbreviates $w_i(\eta)$. Namely, it is rather obvious that the coefficient of such a monomial is

$$\binom{n}{i(0)}\binom{n - i(0)}{i(1)} \cdots \binom{n - i(0) - \ldots - i(k-1)}{i(k)} \bmod 2, \tag{2.2}$$

with $i(0) = n - \sum_{j=1}^{k} i(j)$.

To decide whether a binomial coefficient is 0 or 1 mod 2, it is very convenient to use the following classical result

$$(2.3) \qquad \binom{p}{k} \equiv \prod_{i \geqq 0} \binom{p_i}{k_i} \bmod 2\,,$$

where, as always in the paper, p_i means the same as in (1.1).

Equipped with this, one checks at once that the following holds:

(2.4) **Lemma.** *Let w_i abbreviate $w_i(\eta)$ for an r-plane bundle η over a paracompact space. Then*

$$w_{2k}(n\eta) = n_0 w_{2k} + n_1 w_k^2 + 0\,.\,w_1^2 w_{k-1}^2 + \dots\,, \quad \text{for} \quad k \geqq 1\,.$$

At this moment we shall pass to the study of the tensor square. For our purposes here, it will be sufficient to prove

(2.5) **Proposition.** *With the notation of (2.4), for $j \geqq 2$ we have*

$$w_{4j-2}(\eta \otimes \eta) = 0\,.\,w_{4j-2} + r_0 w_{2j-1}^2 + r_0 w_1^2 w_{2j-2}^2 + \dots\,,$$

$$w_{4j}(\eta \otimes \eta) \quad = 0\,.w_{4j} + r_0 w_{2j}^2 + (1 + r_0)\, w_1^2 w_{2j-1}^2 + \dots\,.$$

Moreover, $w_k(\eta \otimes \eta) = 0$ for any odd k.

Proof. Let $\sigma_1, \dots, \sigma_r$ denote the elementary symmetric functions in variables $x_1, \dots, x_r$. Then (cf. [9]) there is a unique element Φ_r in the ring $Z_2[x_1, \dots, x_r]$ of polynomials over the integers modulo 2, having the property

$$(2.6) \qquad \Phi_r(\sigma_1, \dots, \sigma_r) = \prod_{i=1}^{r} \prod_{j=1}^{r} (1 + x_i + x_j)\,,$$

$$w(\eta \otimes \eta) = \Phi_r(w_1, \dots, w_r)\,,$$

where $w(\eta \otimes \eta)$ denotes the total Stiefel-Whitney class of $\eta \otimes \eta$.

Therefore,

$$(2.7) \qquad \Phi_r(\sigma_1, \dots, \sigma_r) = 1 + \bar{\sigma}_1^2 + \dots + \bar{\sigma}_{\binom{r}{2}}^2\,,$$

where $\bar{\sigma}_k$, $k = 1, \dots, \binom{r}{2}$, is the k-th elementary symmetric function in variables $x_i + x_j$, $i < j$.

Observing that each $\bar{\sigma}_k$ is a homogeneous symmetric polynomial of degree k in $x_1, \dots, x_r$, we conclude that each $\bar{\sigma}_k$ is uniquely expressible as a polynomial in $\sigma_1, \dots, \sigma_k$. Thus, it is immediate that $w_k(\eta \otimes \eta) = 0$ for k odd and that in $w_k(\eta \otimes \eta)$ with k even no term of the form $w_1^{i(1)} \dots w_k^{i(k)}$ with some $i(t)$ odd can occur.

In order to make the proof complete, it remains to verify the following lemma.

(2.8) **Lemma.**

(i) $\bar{\sigma}_k = r_0 \sigma_k + r_0 \sigma_1 \sigma_{k-1} + \dots$, *if $k \geqq 3$ is odd,*

(ii) $\bar{\sigma}_k = r_0 \sigma_k + (1 + r_0)\, \sigma_1 \sigma_{k-1} + \dots$, *if $k \geqq 4$ is even.*

Proof. We shall proceed by induction on k. By [2] we know that

$$\bar{\sigma}_3 = (1 + r_0)(1 + r_1)\sigma_1^3 + r_0\sigma_1\sigma_2 + r_0\sigma_3 ,$$
$$\bar{\sigma}_4 = ((1 + r_0)(1 + r_1) + r_2)\sigma_1^4 + (1 + r_0)\sigma_1\sigma_3 + (1 + r_1)\sigma_2^2 + \\ + r_0(1 + r_1)\sigma_1^2\sigma_2 + r_0\sigma_4 .$$

Supposing (i) and (ii) hold for all $j < k$, $k \geqq 5$, we distinguish three possibilities:
(a) $k = 2s + 1$ for some $s \geqq 2$,
(b) k is even, but not a power of 2,
(c) $k = 2^s$ for some $s \geqq 3$.

Before we shall go any further, let us recall that the well-known primary cohomology operation Sq^i, the i-th Steenrod square, can be applied also to any elementary symmetric function (cf. [6], if necessary).

In fact, in the ring $Z_2[\sigma_1, \ldots, \sigma_r]$ one has the classical formula of *Wu*,

$$Sq^i(\sigma_j) = \sum_{k=0}^{i} \binom{j - i + k - 1}{k} \sigma_{i-k}\sigma_{j+k} . \tag{2.9}$$

This is of course still valid, when σ_j, $j = 1, \ldots, r$, are replaced by the Stiefel-Whitney classes $w_j(\eta)$, $j = 1, \ldots, r$, η being an arbitrary r-plane bundle.

Now we are able to show that the induction works indeed in all the cases (a)–(c).

(2.10) Ad (a). By (2.9), we have

$$\bar{\sigma}_{2s+1} = \bar{\sigma}_1\bar{\sigma}_{2s} + Sq^1(\bar{\sigma}_{2s}) .$$

Since $\bar{\sigma}_1 = (1 + r_0)\sigma_1$, as is easily seen, the induction hypothesis implies

$$\bar{\sigma}_{2s+1} = r_0\sigma_1\sigma_{2s} + r_0\sigma_{2s+1} + \ldots ,$$

which verifies the assertion in the case (a).

(2.11) Ad (b). Now $k = 2^i + \sum_{j \geqq i+1} k_j 2^j$ in the dyadic expansion, where $i > 0$ and $k_j = 1$ for some j. Hence, elementary considerations including (2.3) give that $\binom{u - 1}{v} \equiv 1 \bmod 2$ for $u = \frac{1}{2}k + 2^{i-1}$ and $v = \frac{1}{2}k - 2^{i-1}$. Therefore, by *Wu*'s formula (2.9), we obtain

$$\bar{\sigma}_k = \bar{\sigma}_v\bar{\sigma}_u + \binom{u - v + 1}{2}\bar{\sigma}_{v-2}\bar{\sigma}_{u+2} + \ldots + \binom{u - 3}{v - 2}\bar{\sigma}_2\bar{\sigma}_{k-2} + Sq^v(\bar{\sigma}_u) . \tag{2.12}$$

Obviously, neither σ_k nor $\sigma_1\sigma_{k-1}$ can arise in any term other than $Sq^v(\bar{\sigma}_u)$. Thus, by induction hypothesis (note that $k \geqq 6$ ensures $u \geqq 4$), we need to investigate $Sq^v(r_0\sigma_u + (1 + r_0)\sigma_1\sigma_{u-1} + \ldots)$ and, in fact, just $Sq^v(\sigma_u)$ and $Sq^v(\sigma_1\sigma_{u-1})$.

However, it is clear that

$$Sq^v(\sigma_u) = \sigma_k + 0 \,.\, \sigma_1\sigma_{k-1} + \ldots$$

and

$$Sq^v(\sigma_1\sigma_{u-1}) = \sigma_1\sigma_{k-1} + 0 \,.\, \sigma_k + \ldots .$$

Indeed, since $\binom{u-1}{v} = \binom{u-2}{v} + \binom{u-2}{v-1}$ and $\binom{\text{even}}{\text{odd}} \equiv 0 \bmod 2$, we have also $\binom{u-2}{v} \equiv 1 \bmod 2$.

Altogether, $\sigma_k = r_0\sigma_k + (1 + r_0)\,\sigma_1\sigma_{k-1} + \ldots$, as desired.

(2.13) Ad (c). Now we cannot proceed as we did in the previous situation. As a matter of fact, there is no difficulty in proving that the existence of even numbers $u > v > 0$ such that $u + v = z$ and $\binom{u-1}{v} \equiv 1 \bmod 2$ implies that z is not a power of 2.

Nevertheless, we can employ Steenrod squares again. Namely, using (2.9) we get

$$Sq^2(\bar{\sigma}_k) = \bar{\sigma}_4\bar{\sigma}_{k-2} + Sq^4(\bar{\sigma}_{k-2})\,. \tag{2.14}$$

Further, let us denote the coefficients of σ_k and $\sigma_1\sigma_{k-1}$ in $\bar{\sigma}_k$ by p and q, respectively. We will show that $p = r_0$, $q = 1 + r_0$.

It is easily seen that the left-hand side in (2.14) takes the form $p\sigma_{k+2} + q\sigma_1\sigma_{k+1} + \ldots$. On the right-hand side, σ_{k+2} and $\sigma_1\sigma_{k+1}$ may occur only via $Sq^4(\bar{\sigma}_{k-2})$. This is, by the induction hypothesis, $Sq^4(r_0\sigma_{k-2} + (1 + r_0)\,\sigma_1\sigma_{k-3} + \ldots)$, where three dots are put instead of the monomials that can produce neither σ_{k+2} nor $\sigma_1\sigma_{k+1}$.

Since $Sq^4(\sigma_{k-2}) = \sigma_{k+2} + 0\,.\,\sigma_1\sigma_{k+1} + \ldots$ and $Sq^4(\sigma_1\sigma_{k-3}) = 0\,.\,\sigma_{k+2} + \sigma_1\sigma_{k+1} + \ldots$, we necessarily have $p = r_0$ and $q = 1 + r_0$, as was needed. This concludes the proof of both (2.8) and (2.5).

(2.15) Proof of Theorem (1.1). Clearly, $w_1(G_{n,r}) = n_0w_1$. Since $w_k(G_{n,r}) = w_1(G_{n,r})\,w_{k-1}(G_{n,r}) + Sq^1(w_{k-1}(G_{n,r}))$ for any odd k, just the cases (i) and (iii) need a proof.

We know by [2] that

$$w_6(G_{n,r}) = n_0w_6 + (n_1 + r_0)\,w_3^2 + (n_1 + r_0)\,w_1^2w_2^2 + \ldots,$$
$$w_8(G_{n,r}) = n_0w_8 + (n_1 + r_0)\,w_4^2 + (1 + n_1 + r_0)\,w_1^2w_3^2 + \ldots,$$

indeed hold. Hence, we can continue the induction supposing (i) and (iii) to be true for all even numbers less than k. Let us take the case $k = 4s - 2$ for some $s \geqq 3$.

One easily computes (or cf. [2])

$$w_2(G_{n,r}) = (1 + n_1 + r_0)\,w_1^2 + n_0w_2\,, \tag{2.16}$$

and

$$w_2(\gamma \otimes \gamma) = (1 + r_0)\,w_1^2\,. \tag{2.17}$$

From (2.1), by elementary properties of Stiefel-Whitney classes and by (2.5), we obtain

$$w_k(G_{n,r}) = w_{4s-4}(G_{n,r})\,w_2(\gamma \otimes \gamma) + \ldots + w_2(G_{n,r})\,w_{4s-4}(\gamma \otimes \gamma) + w_{4s-2}(\gamma \otimes \gamma) + w_{4s-2}(n\gamma)\,,$$

where we have left out all the terms that cannot produce any one of w_{4s-2}, w_{2s-1}^2, $w_1^2 w_{2s-2}^2$. Now, by the induction hypothesis, (2.16), (2.17), (2.4) and (2.5), one obtains the desired result for $w_k(G_{n,r})$.

Case $k = 4s$ for some $s \geqq 3$ is completely analogous, and therefore omitted.

(2.18) Proof of Proposition (1.2). As is well-known, the cohomology ring $H^*(G_{n,r}; Z_2)$ can be identified with

$$Z_2[w_1(\gamma_{n,r}), \ldots, w_r(\gamma_{n,r})]/J_{n,r},$$

where $J_{n,r}$ is an ideal with its k-th homogeneous component $J_{n,r}^{(k)} = 0$ for $k \leqq n - r$ (cf. [1, 2, 5], for example).

This, when applied to Theorem (1.1), proves Proposition (1.2).

Acknowledgements. This work was done whilst the author was at the Mathematical Institute of the Czechoslovak Academy of Sciences, Prague. It is a pleasure to thank V. Bartík and Z. Frolík for their support.

References

[1] *Borel, A.:* La cohomologie mod 2 de certains espaces homogènes. Comment. Math. Helvetici *27*, 165—197 (1953).

[2] *Bartík, V., Korbaš, J.:* Stiefel-Whitney characteristic classes and parallelizability of Grassmann manifolds. Rend. Circ. Mat. Palermo (2) (Suppl. *6*), 19—29 (1984).

[3] *Hsiang, W. C., Szcarba, R. H.:* On the tangent bundle of a Grassmann manifold. Amer. J. Math. *86*, 698—704 (1964).

[4] *Korbaš, J.:* On the Stiefel-Whitney classes and the span of Grassmann manifolds (to appear).

[5] *Milnor, J., Stasheff, J.:* Characteristic classes. Annals of Mathematics Studies 76. Princeton: Princeton University Press 1974.

[6] *Mosher, R. E., Tangora, M. C.:* Cohomology operations and applications in homotopy theory. New York, Evanston and London: Harper & Row 1968.

[7] *Oproiu, V.:* Some non-embedding theorems for the Grassmann manifolds $G_{2,n}$ and $G_{3,n}$. Proc. Edinburgh. Math. Soc. *20*, 177—185 (1976—77).

[8] *Oproiu, V.:* Some results concerning the non-embedding codimension of Grassmann manifolds in Euclidean spaces. Rev. Roumaine Math. Pures Appl. *XXVI*, 275—286 (1981).

[9] *Thomas, E.:* On tensor products of n-plane bundles. Arch. Math. (Basel) *X*, 174—179 (1959).

[10] *Thomas, E.:* Vector fields on manifolds. Bull. Amer. Math. Soc. *75*, 643—683 (1969).

Author's address: 010 88 Žilina, Marxa-Engelsa 25, Czechoslovakia (VŠDS).