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ON THE STIEFEL-WHITNEY CLASSES AND THE SPAN
OF REAL GRASSMANNIANS

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1. INTRODUCTION AND STATEMENT OF RESULTS

This paper is a contribution to the study of vector fields on the Grassmann
manifold $G_{n,r}$ of $r$-planes in the real $n$-space. As vector fields we always consider
continuous cross-sections of the tangent bundle.

Recall (Thomas [11]) that the span of a closed connected smooth manifold $M$
is defined as the maximal number of linearly independent vector fields on $M$. Obvi-
ously, if span $M \geq j$, then

$$w_{m-j+1}(M) = w_{m-j+2}(M) = \ldots = w_{m}(M) = 0$$

for Stiefel-Whitney classes of $M$, where $m = \dim M$.

This fact, producing an estimate span $M \leq m - k$ provided $w_{k}(M) \neq 0$ for
some $k$, motivates our interest in the Stiefel-Whitney classes of Grassmann manifolds.

Namely, it is known, see for instance [5], that

$$(1.1) \quad TG_{n,r} \oplus \gamma_{n,r} \otimes \gamma_{n,r} \cong n\gamma_{n,r}$$

where $TG_{n,r}$ denotes the tangent bundle and $\gamma_{n,r}$ the canonical $r$-plane bundle over $G_{n,r}$.

By Borel [4],

$$(1.2) \quad H^{*}(G_{n,r}; \mathbb{Z}_{2}) \cong \mathbb{Z}_{2}[w_{1}(\gamma_{n,r}), \ldots, w_{r}(\gamma_{n,r})]/J_{n,r}$$

for the cohomology algebra, where all the ideal $J_{n,r}$ is determined by the only
equation:

$$(1.3) \quad (1 + w_{1}(\gamma_{n,r}) + \ldots + w_{r}(\gamma_{n,r})) (1 + \bar{w}_{1}(\gamma_{n,r}) + \ldots + \bar{w}_{n-r}(\gamma_{n,r})) = 1$$

Here $\bar{w}_{i}(\gamma_{n,r})$ denotes the $i$-th dual Stiefel-Whitney class of $\gamma_{n,r}$ and, moreover,
$J_{n,r}^{(k)} = 0$ for the $k$-th homogeneous component, if $k \leq n - r$ (cf. [9], [3]).

Hence, if one overcomes difficulties arising when computing the Stiefel-Whitney
classes of the tensor square $\gamma_{n,r} \otimes \gamma_{n,r}$ in terms of $w_{i}(\gamma_{n,r})$, $i = 1, \ldots, r$, then one can
explicitly express $w_{k}(G_{n,r})$ in the same terms (dealing with $n\gamma_{n,r}$ is not hard) and, in
addition, decide whether or not $w_{k}(G_{n,r}) \neq 0$.

This is, essentially, what has been done in [3], for $k \leq 9$. The present paper
provides an improvement of the method used there, making the induction in comput-
ing the dependence of $w_{k}(\gamma_{n,r} \otimes \gamma_{n,r})$ on $w_{i}(\gamma_{n,r})$, $i = 1, \ldots, r$, transparent. So,
one can verify the following extension of [3, 1.1], where, as always in this paper, $p_i (i \geq 0)$ means the $i$-th dyadic coefficient of the positive integer $p$:

(1.4) **Theorem.** Let $w_i$ abbreviate $w_i(y_{n,r}) \in H^i(G_{n,r}; Z_2)$. Let $a = n_1 + r_0$, $b = n_2 + r_1$, $c = n_1 + n_2 + r_1$, $d = n_2 + r_0 + r_1$, $e = n_3 + r_2$ and $f = n_4 + r_3$. Then

$$w_{10}(G_{n,r}) = (1 + a) (b r_1 + c + n_i r_0) (a + e + r_0) + n_0 (a + r) + n_0 (a n_1 + b) w_4 +$$
$$+ n_0 (a + b) w_4 + n_0 (a d w_4 + (1 + a e) w_4^2 + n_0 (a n_1 + b) w_4^3 +$$
$$+ (1 + a + b) w_4^2 + n_0 (a + r) + n_0 a w_4^2 + n_0 (a + b) w_4^3 +$$
$$+ n_0 (a + b) w_4^2 + n_0 (a + e) w_4^2 + n_0 (a d w_4 + (1 + a e) w_4^2 +$$
$$+ (1 + a + b) w_4^2 + n_0 (a + b) w_4^2 + n_0 (a + b) w_4^2 + n_0 (a + b) w_4^2 +$$
$$+ n_0 (a + b) w_4^2 + n_0 (a + b) w_4^2 + n_0 (a + b) w_4^2 + n_0 (a + b) w_4^2 +$$
$$+ n_0 (a + b) w_4^2 + n_0 (a + b) w_4^2 + n_0 (a + b) w_4^2 + n_0 (a + b) w_4^2 +$$

$$w_{12}(G_{n,r}) = (1 + a) r_1 + (c + n_i r_0) (a + e + r_0) + n_0 (a n_1 + b) (b r_1 + c + n_i r_0) +$$
$$+ n_0 (a + e + r_0) + n_0 (a n_1 + b) w_4 + n_0 (a + e + r_0) + n_0 (a n_1 + b) w_4 +$$
$$+ n_0 (a + e + r_0) + n_0 (a n_1 + b) w_4 + n_0 (a + e + r_0) + n_0 (a n_1 + b) w_4 +$$
$$+ n_0 (a + e + r_0) + n_0 (a n_1 + b) w_4 + n_0 (a + e + r_0) + n_0 (a n_1 + b) w_4 +$$
$$+ n_0 (a + e + r_0) + n_0 (a n_1 + b) w_4 + n_0 (a + e + r_0) + n_0 (a n_1 + b) w_4 +$$
$$+ n_0 (a + e + r_0) + n_0 (a n_1 + b) w_4 + n_0 (a + e + r_0) + n_0 (a n_1 + b) w_4 +$$
$$+ n_0 (a + e + r_0) + n_0 (a n_1 + b) w_4 + n_0 (a + e + r_0) + n_0 (a n_1 + b) w_4 +$$

$$w_{14}(G_{n,r}) = (1 + a) (b + r_1) + (c + n_i r_0) (a + e + r_0) + n_0 (a n_1 + b) + n_0 (a n_1 + b) + (b + e + r_0) + n_0 (a n_1 + b) +$$
$$+ n_0 (a n_1 + b) + n_0 (a n_1 + b) + n_0 (a n_1 + b) + n_0 (a n_1 + b) + n_0 (a n_1 + b) +$$
$$+ n_0 (a n_1 + b) + n_0 (a n_1 + b) + n_0 (a n_1 + b) + n_0 (a n_1 + b) + n_0 (a n_1 + b) +$$

$$w_{16}(G_{n,r}) = (1 + a) (b + r_1) + (c + n_i r_0) (a + e + r_0) + n_0 (a n_1 + b) + n_0 (a n_1 + b) + (b + e + r_0) + n_0 (a n_1 + b) +$$
$$+ n_0 (a n_1 + b) + n_0 (a n_1 + b) + n_0 (a n_1 + b) + n_0 (a n_1 + b) + n_0 (a n_1 + b) +$$

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+ b(c + n_3 r_0 + n_1 r_0 r_1) + (1 + b) e w_2^4 + n_0 (1 + a(e + n_1) + 
+ ab(n_1 + r_1) + b) w_2^2 w_3^2 + (1 + a(e + n_1 r_1) + b + 
+ (e r_0 + n_1 r_1) w_2^3 + n_0 (a n_1 + r_1) b + e) w_2^2 w_8 + 
+ n_0 (1 + a) (1 + b) w_1^2 w_2^5 + n_0 (1 + a) (1 + b) w_1^2 w_4 + 
+ n_0 (1 + a) (1 + b) w_1^2 w_3^3 w_4 + (1 + a) b w_1^2 w_5^2 + n_0 (1 + a) b w_1^2 w_{10} + 
+ (1 + (1 + a) (1 + b)) w_1^2 w_3^3 + n_0 (a n_1 + b) w_1^2 w_{12} + (a(e d + r_0) + 
+ b(n_1 n_2 r_0 + n_1 r_1 + r_0)) w_1^2 w_2^3 + n_0 (a(e + r_0) + 
+ b(n_1 r_0 + n_2 r_0 + n_1 r_1)) w_1^2 w_2^3 + n_0 a c w_1^2 w_2 w_4^2 + a c w_1^2 w_6 + 
+ n_0 a c w_1^2 w_2^2 + n_0 a c w_1^2 w_2 w_5^2 + n_0 b (1 + a) w_1^2 w_2 w_3 + n_0 a w_1^2 w_2 w_6 + 
+ n_0 (a(e + r_0) + b(n_1 r_1 + n_1 r_0 + n_2 r_0)) w_1^2 w_2 w_4 + n_0 a d w_1^2 w_2 w_6 + 
+ (1 + a) b w_1^2 w_5 w_3^2 + n_0 (1 + a) (1 + b) w_1^2 w_4 w_6 + n_0 a d w_1^2 w_2 w_6 + 
+ n_0 (1 + a) w_1^2 w_2 w_8 + n_0 a w_1^2 w_3 w_10 + n_0 (1 + a c) w_1^2 w_2^3 w_6 + 
+ n_0 (1 + a d) w_1^2 w_2^3 + n_0 (1 + a) w_1^2 w_4 w_5 + n_0 a w_1^2 w_3 w_6 + a d w_1^2 w_6 + 
+ (1 + a) w_1^2 w_7 + n_0 (1 + a) w_1^2 w_4 w_5 + (1 + a) (1 + b) w_1^2 w_2 w_3 + 
+ a w_1^2 w_2 w_4 + ((a r_0 + n_2) b + e) w_1^2 + n_0 a c w_1^2 w_4 + 
+ n_0 (1 + a r_0 + b) w_1^2 w_5 + n_0 (1 + a) w_1^2 w_3^3 w_4 + a c w_1^2 w_2^2 + 
+ n_0 a w_2^2 w_12 + n_0 a c w_2^3 w_3 + n_0 (1 + a r_0 + b) w_2^3 + a w_1^2 w_3 w_4^2 + w_2^3 + w_2^3 + 
+ n_0 a w_3 w_10 + a c w_4 w_2 + (a r_0 + b) w_4 + n_0 a w_2 w_6 + n_0 a w_3 w_6 + 
+ n_0 a w_4 w_6 + n_0 a w_2 w_7 + a w_3 + n_0 w_{16}.

Since the formulae are very long, we have omitted those for \( w_k(G_{n,r}) \) with odd values of \( k \). Nevertheless, one can close the gaps easily, if required.

Namely, by the classical formula of Wu (e.g. Borel [4, 7.1]), if \( i \leq j \), then

\[
(1.5) \quad Sq^j(w_j(\xi)) = \sum_{k=0}^{i} \binom{j - i + k - 1}{k} w_{i-k}(\xi) w_{j+k}(\xi)
\]

for an arbitrary vector bundle \( \xi \) (\( Sq^i \) being the \( i \)-th Steenrod square,

\[
\left( \frac{u}{v} \right) = u!(u-v)!v!.
\]

Therefore we have

\[
w_k(G_{n,r}) = w_1(G_{n,r}) w_{k-1}(G_{n,r}) + Sq^1(w_{k-1}(G_{n,r}))
\]

whenever \( k \) is odd. Hence, keeping in mind that ([3])

\[
w_1(G_{n,r}) = n_0 w_1,
\]

one can compute the omitted formulae without difficulties.

Moreover, using elementary facts about the binomial coefficients, including

\[
(1.6) \quad \left( \frac{u}{v} \right) \equiv \prod_{i} \left( \frac{u_i}{v_i} \right) \mod 2,
\]

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it is easy to verify the following assertion:

(1.7) For a positive even integer $x$ there exist even integers $m > n > 0$ such that $m + n = x$ and $\binom{m-1}{n} \equiv 1 \mod 2$ if $x$ is not a power of 2.

This implies (cf. (1.5)) that for an arbitrary $w_k(G_n,r)$ with $2^i < k < 2^{i+1}$, the formula can be derived from those for $w_j(G_n,r)$ with $j \leq 2^i$.

In particular, the formulae for $w_k(G_n,r)$, $k = 10, 12, 14$ could be computed using only Steenrod squares and the knowledge already contained in [3]. Nevertheless, we have preferred to give them here not only for reader's convenience but also for future references.

More specifically speaking, one can observe, for instance, that $w_7$ has the coefficient $n_1 + r_0 \mod 2$ in $w_{2i}(G_n,r)$ $i = 5, 6, 7, 8$. One can suspect that this will be the case for $i = 9, 10$ etc., as well.

As a matter of fact, this property can be verified. A research of phenomena of this kind was initiated in [7], and we hope that the formulae for $w_k(G_n,r)$, $k = 10, 12, 14$, will be useful in its development.

Finally, supposing $n \geq 2r$ (which is not restrictive because $G_n,r$ and $G_n,n-r$ are diffeomorphic) we obtain from Theorem 1.4 the following

(1.8) Corollary. If $n \geq 10$ is even and $r \geq 3$ is odd then

\[ \text{span } G_{n,r} \leq r(n-r) - 16 \]

with the exception of

\[ \text{span } G_{10,3} \leq 7. \]

Recall that $G_{n,1}$ is the projective space (therefore $\text{span } G_{n,1} = \text{span } S^{n-1}$; cf. [1] for its values) and that $\text{span } G_{n,r} = 0$ if $\dim G_{n,r} = r(n-r)$ is even. Moreover, $\text{span } G_{6,3} \leq 7$ and $\text{span } G_{8,3} = 7$ (cf. [3]).

We also remark that the minimal from the two upper bounds should be always taken: one given by 1.8, the other by [7], when estimating the span of some grassmannian. For lower bounds we refer to [8] or [6].

Concluding this section we observe that although our method has produced the best known upper bounds for the span of grassmannians, its disadvantage is considerable. Namely, briefly speaking, the higher we go, the longer and more complicated all the computations involved become.

Hence, in this way we can enrich our knowledge to some extent, but must remember that we stay still very far from achieving the final, general solution of the vector fields problem on $G_{n,r}$'s (if such is possible at all), unless some new, intensive approach appears.
2. PROOFS OF RESULTS

We postpone the proof of Theorem 1.4, proving first its corollary.

(2.1) Proof of (1.8). As we have mentioned already, \( J^{(k)}_{n,r} = 0 \) for \( k \leq n - r \), in (1.2). From 1.4 we read that the coefficients of \( w_i^4 w_3^4 \) and \( w_i^0 w_2^2 w_3^2 \) in \( w_{16}(G_{n,r}) \) are always different mod 2. This yields that \( w_{16}(G_{n,r}) = 0 \) and therefore \( \text{span } G_{n,r} \leq r(n - r) - 16 \), when \( 16 < n - r \).

To make the proof complete, we are left with 26 cases where \( n - r < 16 \).

Thanks to (1.3), we can find all generators of \( J^{(16)}_{n,r} \) and also decide whether \( w_{16}(G_{n,r}) \in J^{(16)}_{n,r} \) or not.

To facilitate this task, it is useful to recall that if

\[
i_1: G_{n,r} \subseteq G_{n+1,r}
i_2: G_{n+1,r} \subseteq G_{n+2,r+1}
\]

are the usual inclusions, then

\[
i^*(w_i(G_{n+2,r+1})) = w_k(G_{n,r})
\]

for \( i = i_2 \circ i_1 \). Hence, for example,

\[
w_k(G_{n,r}) = 0 \text{ implies } w_k(G_{n+4,r+2}) = 0.
\]

It turns out that from all the cases, only for \( G_{10,3} \) the 16-th class vanishes. Fortunately, in a similar way one checks that \( w_{14}(G_{10,3}) = 0 \).

Besides some other facts, we shall need the following two lemmas for the proof of Theorem 1.4.

(2.2) Lemma. Let \( \eta \) be an \( r \)-plane bundle over a paracompact space \( M \). Let us abbreviate the Stiefel-Whitney class \( w_i(\eta) \in H^i(M; \mathbb{Z}_2) \) to \( w_i \). Then:

\[
w_{10}(\eta \otimes \eta) = (1 + r_0)(1 + r_1 + r_2) w_1^1 + (1 + r_0 r_1) w_1^4 w_3^2 + \]
\[
+ (1 + r_0)(1 + r_1) w_1^4 w_2^4 + r_0(1 + r_1) w_1^2 w_2^2 + r_0 w_1^2 w_2^4 + r_0 w_3^2;
\]

\[
w_{12}(\eta \otimes \eta) = ((1 + r_0)(1 + r_1 + r_2) w_1^1 + r_0(1 + r_0 + r_1) r_2) w_1^4 + r_0(1 + r_0) w_1^2 w_3^2 + \]
\[
+ r_0(1 + r_1) w_1^4 w_2^4 + r_0(1 + r_1 + r_2) w_1^2 w_2^2 + \]
\[
+ (1 + r_0)(1 + r_1) w_1^2 w_2^4 + r_0(1 + r_1) w_2^2 w_3^2 + (1 + r_0) w_1^2 w_2^2 w_3^2 + \]
\[
+ (1 + r_0) w_1^2 w_2^2 w_3^2 + (1 + r_1) w_2^4 + r_0 w_3^2;
\]

\[
w_{14}(\eta \otimes \eta) = (1 + r_0)(1 + r_1)(1 + r_2) w_1^4 + (r_0(1 + r_2) + r_1) w_1^8 w_3^2 + \]
\[
+ r_0 r_1 w_1^6 w_2^2 + r_0 r_1 w_1^4 w_3^2 + r_0(1 + r_0) w_1^2 w_3^4 + \]
\[
+ r_0(1 + r_1 + r_2) w_1^4 w_3^4 + r_0(1 + r_1) w_1^6 w_2^2 + r_0(1 + r_1) w_1^2 w_3^4 + \]
\[
+ r_0 w_1^2 w_2^2 w_3^2 + r_0 w_1^2 w_2^2 + r_0 w_2^4;
\]

\[
w_{16}(\eta \otimes \eta) = ((1 + r_0)(1 + r_1)(1 + r_2) + r_3) w_1^4 + \]
\[
+ r_0(1 + r_1)(1 + r_2) w_1^4 w_2^2 + (1 + r_1)(1 + r_2) w_1^4 w_2^2 + \]
\[
+ (1 + r_0)(1 + r_2) w_1^4 w_3^2 + r_1(1 + r_0) w_1^2 w_3^4 + (r_0(1 + r_2) + \]

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\[ + r_1(1 + r_0)w^8_1w^8_2 + (1 + r_1 + r_2)w^8_1w^8_3 + r_1(1 + r_0)w^8_1w^8_2w^8_3 + + r_0(1 + r_1)w^8_1w^8_3 + r_0w^8_1w^8_2w^8_3 + + r_0(1 + r_1)w^8_1w^8_3 + r_0(1 + r_1)w^8_1w^8_2w^8_3 + + r_0(1 + r_1)w^8_1w^8_2 + r_1w^8_3 + w^8_5 + (1 + r_0)w^8_1w^8_2 + r_0w^8_3. \]

(2.3) **Lemma.** With the notation of (2.2) we have for the n-fold Whitney sum \(n\eta\):

\[ w_{16}(n\eta) = n_0w_{16} + n_0n_1w_1^2w_{14} + n_0n_1w_1^2w_{12} + n_0n_1w_1^4w_{12} + n_0n_1w_1^3w_{10} + + n_0n_1n_2w_1^6w_{10} + n_1w_1^2 + n_0n_1w_1^2w_8 + n_0n_1w_1^2w_8 + n_0n_1n_2w_1^3w_8 + + n_0n_3w_1^8w_8 + n_1n_2w_1^2w_6 + n_0n_1w_1^2w_6 + n_0n_1n_2w_1^3w_6 + + n_0n_1n_2w_1^3w_6 + n_0n_1n_3w_1^10w_6 + n_0n_1w_4w_8 + n_2w_4 + + n_0n_1n_2w_1^4w_4 + n_1n_2w_1^2w_4 + n_1n_3w_1^8w_4 + n_0n_2w_3w_4 + + n_0n_1n_2w_2w_4 + n_0n_1n_3w_1^4w_2 + n_0n_2n_3w_1^{12}w_4 + n_0n_1w_2w_7 + + n_0n_1n_2w_1^6w_5 + n_0n_2n_2w_1^2w_5 + n_0n_1n_3w_1^8w_5 + n_1n_2w_2^4w_5 + + n_3w_1^8 + n_2n_3w_1^4w_2 + n_1n_2n_3w_1^{12}w_2 + n_0n_1n_2w_1^2w_3 + + n_0n_1n_2w_1^4w_2 + n_4w_1^6. \]

Assuming (2.2) and (2.3) we are able to prove (1.4).

(2.4) **Proof of Theorem 1.4.** By (1.5), we have the following Wu's formulae for an arbitrary vector bundle \(\xi\):

(2.5) \[ w_{10}(\xi) = w_2(\xi)w_8(\xi) + Sq^2(w_8(\xi)), \]

(2.6) \[ w_{12}(\xi) = w_4(\xi)w_8(\xi) + Sq^4(w_8(\xi)), \]

(2.7) \[ w_{14}(\xi) = w_2(\xi)w_{12}(\xi) + Sq^2(w_{12}(\xi)). \]

By \([3, 1.1]\), we know \(w_4(G_{n,r})\) for \(k = 2, 4, 8\). Hence, we can compute the formulae for \(w_k(G_{n,r})\), \(k = 10, 12, 14\), putting \(\xi = TG_{n,r}\) and using only elementary properties of Steenrod squares.

Having done this, we can compute \(w_{16}(G_{n,r})\) as well (of course, in another way: cf. (1.7)).

Namely, recalling that odd-dimensional Stiefel-Whitney classes of \(\gamma_{n,r} \otimes \gamma_{n,r}\) vanish (cf. \([3, 2.1]\)), we obtain from Hsiang and Szczarba's relation (1.1):

\[ w_{16}(G_{n,r}) = \sum_{i=1}^{8} w_{16-2i}(G_{n,r})w_{2i}(\gamma_{n,r} \otimes \gamma_{n,r}) + w_{16}(n\gamma_{n,r}). \]

Clearly \([3, 11.1, 2.1]\), Lemma 2.2 and Lemma 2.3 now provide all the information needed.

As the proof of Lemma 2.3 is very easy (therefore omitted), all that now remains is to prove (2.2).

(2.8) **Proof of Lemma 2.2.** Let \(w(\eta \otimes \eta)\) denote the total Stiefel-Whitney class of the tensor square \(\eta \otimes \eta\), and \(\sigma_1, \ldots, \sigma_r\) the elementary symmetric functions in
variables \( x_1, \ldots, x_r \). Then (cf. [10])

\[
(2.9) \quad w(\eta \otimes \eta) = \Phi_r(w_1, \ldots, w_r),
\]

where \( \Phi_r \) is the only element in \( Z_2[x_1, \ldots, x_r] \) such that

\[
(2.10) \quad \Phi_r(\sigma_1, \ldots, \sigma_r) = \prod_{1 \leq i < j \leq r} (1 + x_i + x_j).
\]

This makes it clear that our ability to express Stiefel-Whitney classes of \( \eta \otimes \eta \) in terms of \( w_1, \ldots, w_r \) is determined by our knowledge of the polynomial \( \Phi_r \).

However, (2.10) obviously yields

\[
(2.11) \quad \Phi_r(\sigma_1, \ldots, \sigma_r) = 1 + \bar{\sigma}_1^2 + \ldots + \bar{\sigma}_r^2,
\]

where \( \bar{\sigma}_k \) denotes the \( k \)-th elementary symmetric function in variables \( x_i + x_j \), \( i < j \).

Since each \( \bar{\sigma}_k \) can be expressed in a unique way as a polynomial in \( \sigma_1, \ldots, \sigma_k \), our strategy is straightforward.

As a matter of fact, an improved (as compared with [3]) algorithm for computing \( \bar{\sigma}_k \) in terms of \( \sigma_1, \ldots, \sigma_k \) mod 2 now follows.

(2.12) Algorithm for computing \( \bar{\sigma}_p \) provided that we have already computed \( \bar{\sigma}_k \) for \( k < p \).

The procedure will have three steps.

(2.13) Step 1. Let

\[
\tilde{\mathcal{M}}_p = \{ \sigma^{(1)}_i \ldots \sigma^{(p)}_i; \ i(1) + 2i(2) + \ldots + pi(p) = p \}.
\]

We define the leading monomial of \( \sigma^{(1)}_i \ldots \sigma^{(p)}_i \in \tilde{\mathcal{M}}_p \) to be

\[
\text{Im} (\sigma^{(1)}_1 \ldots \sigma^{(p)}_p) := x^{(1)}_1 \ldots x^{(1)}_i \ldots x^{(i)}_2 \ldots x^{(i)}_p \ldots x^{(p)}_p.
\]

Further, we order the set

\[
\{ \text{Im} (\sigma^{(1)}_1 \ldots \sigma^{(p)}_i); \sigma^{(1)}_1 \ldots \sigma^{(p)}_i \in \tilde{\mathcal{M}}_p \}
\]

by the rule

\[
(R_p) \quad x^{(1)}_1 \ldots x^{(p)}_i < x^{(1)}_1 \ldots x^{(p)}_i \quad \text{if} \quad (s(1), \ldots, s(p)) < (i(1), \ldots, i(p)) \quad \text{in NLO},
\]

where NLO is an abbreviation for the natural lexicographical order.

Finally, we order \( \tilde{\mathcal{M}}_p \) as follows:

\[
\sigma^{(1)}_1 \ldots \sigma^{(p)}_i < \sigma^{(1)}_1 \ldots \sigma^{(i)}_p \quad \text{if} \quad \text{Im} (\sigma^{(1)}_1 \ldots \sigma^{(p)}_i) < \text{Im} (\sigma^{(1)}_1 \ldots \sigma^{(i)}_p).
\]

This completes the first step of the procedure. The set \( (\tilde{\mathcal{M}}_p, <) \) has the following very important property:

if we interpret \( \sigma^{(1)}_1 \ldots \sigma^{(p)}_i \in \tilde{\mathcal{M}}_p \) as an element of \( Z_2[x_1, \ldots, x_r] \), then the coef-
ficient of
\[
\text{Im} \left( \sigma_1^{(1)} \ldots \sigma_p^{(p)} \right) \quad \text{in} \quad \sigma_1^{(1)} \ldots \sigma_p^{(p)},
\]
\[
\text{Im} \left( \sigma_1^{(1)} \ldots \sigma_p^{(p)} \right) + \sigma_1^{(1)} \ldots \sigma_p^{(p)}
\]
for short, can be 1 only when
\[
(2.14) \quad \sigma_1^{(1)} \ldots \sigma_p^{(p)} \leq \sigma_1^{(1)} \ldots \sigma_p^{(p)}
\]
in the set ($p,<$). This is easily implied by the definition of the order in $M_p$ and by
the simple fact that $\text{Im} \left( \sigma_1^{(1)} \ldots \sigma_p^{(p)} \right)$ is the greatest element in the set
\[
\{x_1^{s(1)} \ldots x_r^{s(r)}, (x_1^{s(1)} \ldots x_r^{s(r)} + \sigma_1^{(1)} \ldots \sigma_p^{(p)}) = 1\}
\]
ordered by the rule ($R_r$).

\[
(2.15) \quad \text{Examples.}
\]
$p = 1$: $M_1 = \{s_1\}$.
$p = 2$: $M_2 = \{s_1, s_2\}$.
Since $\text{Im} (s_1^2) = x_1^2$ and $\text{Im} (s_2) = x_1x_2$, we have $\text{Im} (s_2) < \text{Im} (s_1^2)$ and
therefore
\[
(M_2, <) = \{s_1^2 < s_2\}.
\]
$p = 3$: $M_3 = \{s_1^3, s_1s_2, s_3\}$.
Now, $\text{Im} (s_1^3) = x_1^3$, $\text{Im} (s_1s_2) = x_1^2x_2$ and $\text{Im} (s_3) = x_1x_2x_3$. Hence
\[
(M_3, <) = \{s_1^3 < s_1s_2 < s_3\}.
\]
We observe that the number of all elements in $M_p$ is part $(p)$, the number of partitions
of $p$. For instance (cf. [2]):
\[
\begin{array}{cccccccccccc}
 p & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\hline
\text{part } (p) & 1 & 2 & 3 & 5 & 7 & 11 & 15 & 22 & 30 & 42 & 55 \\
\end{array}
\]

(2.16) Step 2. Let us search for the number
\[
(2.17) \quad x_1^{s(1)} \ldots x_h^{s(h)} + \sigma_p
\]
with $s(1) \geq \ldots \geq s(h) > 0$, $\sum_{i=1}^{h} s(i) = p$.

To begin with, let us denote
\[
\mathcal{S}_r = \{x_i + x_j; 1 \leq i < j \leq r\},
\]
\[
\mathcal{L}_h = \{x_1 + x_j; 2 \leq j \leq h\},
\]
both considered as naturally lexicographically ordered, and
\[
\mathcal{G}_h = \{x_1^{s(1)} \ldots x_h^{s(h)}; (x_1^{s(1)} \ldots x_h^{s(h)}) + \prod_{j=2}^{h} (1 + x_j + x_j) \equiv 1 \mod 2,
\]
and $s(i) - g(i) \geq 0$ for $i = 1, \ldots, h$.  

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By definition, \( \bar{\sigma}_p \) is the sum of all products of the form

\[
\prod_{k=1}^{p} (x_{i(k)} + x_{j(k)}),
\]

where \( \{(x_{i(k)} + x_{j(k)})\}_{k=1}^{p} \) is an increasing sequence in \( \mathcal{P}_r \).

This fact, a little thinking of the “list”

\[
x_1 + x_2, \ldots, x_1 + x_h, x_1 + x_{h+1}, \ldots, x_1 + x_r \mid x_2 + x_3, \ldots, x_2 + x_r, \ldots, x_{r-1} + x_r
\]

with realizing how the elements of the set \( \mathcal{G}_h \) are related with \( \mathcal{P}_h \), clarify the following result:

\[
(x_1^r + \bar{\sigma}_p) = \binom{r-1}{p} \mod 2, \quad \text{and}
\]

\[
(x_1^{g(1)} \ldots x_h^{g(h)} + \bar{\sigma}_p) = \sum \binom{r-h}{g(1)-g(1)} N_{g(2)-g(2),\ldots,g(h)-g(h)} \mod 2
\]

if \( h \geq 2 \).

Here the sum is taken over all \( h \)-tuples \( (g(1), \ldots, g(h)) \) such that \( x_1^{g(1)} \ldots x_h^{g(h)} \in \mathcal{G}_h \), and \( N_{g(2)-g(2),\ldots,g(h)-g(h)} \in Z_2 \) is

\[
x_1^{g(2)-g(2)} \ldots x_h^{g(h)-g(h)} \bar{\sigma}_t(x_2 + x_3, \ldots, x_2 + x_r; \ldots; x_{r-1} + x_r)
\]

with \( t = \sum_{i=2}^{h} (s(i) - g(i)) \).

It is clear, however, that \( N_{g(2)-g(2),\ldots,g(h)-g(h)} \) coincides with

\[
x_1^{g(2)-g(2)} \ldots x_h^{g(h)-g(h)} \bar{\sigma}_t(x_1 + x_2, \ldots, x_1 + x_{r-1}; \ldots; x_{r-2} + x_{r-1})
\]

So an induction can come in, finally. Indeed, since \( t < p \) and therefore we have

\[
\bar{\sigma}_t(x_1 + x_2, \ldots, x_1 + x_{r-1}; \ldots; x_{r-2} + x_{r-1})
\]

in terms of \( \sigma_i(x_1, \ldots, x_{r-1}), i = 1, \ldots, t \), already computed (cf. 2.12), we are able to find all the numbers \( N_{g(2)-g(2),\ldots,g(h)-g(h)} \).

Taking successively all the leading monomials \( \text{L.m.}(\sigma_1^{g(1)} \ldots \sigma_p^{g(p)}), \sigma_1^{g(1)} \ldots \sigma_p^{g(p)} \in \bar{\mathcal{M}}_p \) for \( x_1^{g(1)} \ldots x_h^{g(h)} \) in (2.17), we accomplish the second step of our algorithm.

We just note that the binomial coefficients in (2.18) are easily expressible in terms of dyadic coefficients, using (1.6), and that it is very useful to remember that

\[
\prod_{j=2}^{h}(1 + x_1 + x_j) = \sum_{i=1}^{h} (1 + x_1)^{h-i} \sigma_{i-1}(x_2, \ldots, x_h),
\]

when forming the set \( \mathcal{G}_h \).

(2.19) Step 3. For a while let us denote by \( A_k \) the \( k \)-th element of the ordered set \( (\bar{\mathcal{M}}_p, <) \). Then, of course,

\[
\bar{\sigma}_p = a(1) A_1 + a(2) A_2 + \ldots + a(\text{part } (p)) A_{\text{part}(p)}
\]

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for some \( a(k) \in \mathbb{Z}_2 \), and our final aim is to find all these \( a(k) \). We are ready to do this.

Namely, for any fixed \( A_k \) it is easy to find all \( A_j \) with the property

\[
(\text{Im}(A_k) \downarrow A_j) = 1.
\]

Recall that \( (\text{Im}(A_k) \downarrow A_k) = 1 \), and that all candidates for (2.21) have to satisfy \( A_j \leq A_k \) (cf. (2.14)). Hence, from (2.20) we get a linear equation over \( \mathbb{Z}_2 \), where the left-hand side is the sum of \( a(k) \) and some \( a(i) \)'s, \( i < k \), while the right-hand side is the number \( \text{Im} (A_k) \downarrow \tilde{\sigma}_p \), computed in Step 2.

So, finding such an equation for every \( k = 1, 2, \ldots, \text{part } (p) \), we obtain a very simple system of linear equations over \( \mathbb{Z}_2 \) giving us all \( a(k) \) as desired.

This completes the last step of our algorithm.

(2.22) Example. Say, we have

\[
\tilde{\sigma}_1 = (1 + r_0) \sigma_1,
\]

\[
\tilde{\sigma}_2 = (1 + r_0 + r_1) \sigma_1^2 + r_0 \sigma_2,
\]

and we wish to compute \( \sigma_3 \).

The first step was made in 2.15. Recall its result:

\[
(\mathcal{M}_3, <) = \{ \sigma_1^3 < \sigma_1 \sigma_2 < \sigma_3 \}.
\]

Step 2.

a) \((x_1^3 \uparrow \tilde{\sigma}_3) = \left( \frac{r - 1}{3} \right) = 1 + r_0 + r_1 + r_0 r_1 \mod 2\).

b) For the leading monomial \( x_1^2 x_2 \) of \( \sigma_1 \sigma_2 \) we have

\[
\mathcal{G}_2 = \{ 1, x_1, x_2 \}.
\]

So (2.18) gives

\[
(x_1^2 x_2 \uparrow \tilde{\sigma}_3) = \left( \frac{r - 2}{2} \right) N_1 + \left( \frac{r - 2}{1} \right) N_0.
\]

We find

\[
N_0 = (1 + \tilde{\sigma}_0 (x_1 + x_2, \ldots, x_{r-2} + x_{r-1})) = 1, \quad \text{and}
\]

(c.f. (2.23))

\[
N_1 = (x_1 + \tilde{\sigma}_1 (x_1 + x_2, \ldots, x_{r-2} + x_{r-1}; \ldots; x_{r-2} + x_{r-1})) = r_0.
\]

Therefore

\[
(x_1^2 x_2 \uparrow \tilde{\sigma}_3) = 1 + r_1 + r_0 r_1 \mod 2.
\]

c) For the leading monomial \( x_1 x_2 x_3 \) of \( \sigma_3 \) we get

\[
\mathcal{G}_3 = \{ 1, x_2, x_3, x_1 x_2, x_1 x_3, x_2 x_3 \}.
\]

Now (2.18) reads

\[
(x_1 x_2 x_3 \uparrow \tilde{\sigma}_3) = \left( \frac{r - 3}{1} \right) N_{1,1} + \left( \frac{r - 3}{0} \right) N_{0,1} + \left( \frac{r - 3}{1} \right) N_{1,0} + \left( \frac{r - 3}{0} \right) N_{0,1} +
\]

\[
\left( \frac{r - 3}{0} \right) N_{1,0} + \left( \frac{r - 3}{1} \right) N_{0,0}.
\]

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We find
\[ N_{0,0} = (1 + \bar{\sigma}_0(x_1 + x_2, \ldots, x_{r-2} + x_{r-1})) = 1 , \]
\[ N_{0,1} = N_{1,0} = (x_1 + \bar{\sigma}_1(x_1 + x_2, \ldots, x_1 + x_{r-1}; \ldots; x_{r-2} + x_{r-1}) = r_0 , \]
and (cf. (2.24))
\[ N_{1,1} = (x_1 x_2 + \bar{\sigma}_2(x_1 + x_2, \ldots, x_1 + x_{r-1}; \ldots; x_{r-2} + x_{r-1}) = 1 + r_0 . \]

Hence we obtain
\[(x_1 x_2 x_3 + \bar{\sigma}_3) = 0 \mod 2 . \]

Step 3. Writing
\[ \bar{\sigma}_3 = a(1) \sigma_1^3 + a(2) \sigma_1 \sigma_2 + a(3) \sigma_3 \]
we get the system
\[ a(1) = 1 + r_0 + r_1 + r_0 r_1 , \]
\[ a(1) + a(2) = 1 + r_1 + r_0 r_1 , \]
\[ a(2) + a(3) = 0 . \]

Clearly, \( \bar{\sigma}_3 = (1 + r_0)(1 + r_1) \sigma_1^3 + r_0 \sigma_1 \sigma_2 + r_0 \sigma_3 . \)

Continuing these computations for \( \bar{\sigma}_k , k = 4, 5, 6, 7, 8 , \) one is able to check Lemma 2.2.

We observe that Wu’s formula (1.5) is also true for elementary symmetric functions (in arbitrary variables). Hence, when \( k \) is not a power of 2, also the Steenrod squares techniques can be used in order to compute \( \bar{\sigma}_k . \)

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