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OSCILLATIONS OF ALL SOLUTIONS
OF FUNCTIONAL DIFFERENTIAL INEQUALITIES

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1. INTRODUCTION

In this paper we are concerned with the oscillatory behavior of the functional
differential inequality of the form

\[(E_2) \quad (-1)^{\nu} x^{(\nu)}(t) \operatorname{sgn} x(t) \geq p(t) \prod_{i=1}^{m} |x(g_i(t))|^{\alpha_i}, \quad n \geq 2,\]

where $\nu$ is a natural number, $\alpha_i \in \mathbb{R}_+ = [0, \infty)$ with $\alpha_1 + \ldots + \alpha_m = 1$, the function
$p: \mathbb{R}_+ \to \mathbb{R}_+$ is integrable on each finite segment and is not identically zero in every
neighbourhood of infinity, the functions $g_i: \mathbb{R}_+ \to \mathbb{R}_+ \ (i = 1, \ldots, m)$ are continuous
and \( \lim_{t \to \infty} g_i(t) = \infty. \)

By a solution of $(E_2)$ we mean a function $x: [t_0, \infty) \to \mathbb{R}$, $t_0 \in \mathbb{R}_+$, such that
(a) $x^{(k)}$ ($k = 0, 1, \ldots, n - 1$) is absolutely continuous on $[t_0, \infty)$,
(b) $\sup \{|x(s)| : t \leq s < \infty\} > 0$ for any $t \geq t_0$,
(c) there exists a $t_1 \in (t_0, \infty)$ such that $x$ satisfies $(E_2)$ almost everywhere on $[t_1, \infty)$.

A solution of $(E_2)$ is called oscillatory, if it has an infinite sequence of zeros
tending to infinity. Otherwise it is called nonoscillatory.

The purpose of this paper is to study oscillations of solutions of $(E_2)$ generated
by general deviating arguments $g_i$ (not necessarily delay or advanced arguments).
Some results on oscillation of functional differential equations and inequalities
with general deviating arguments have been obtained in the papers [2, 5, 7, 9, 15, 19, 22, 24, 26–28]. The main results of this paper are new and are independent of
the analogous ones known for delay and advanced differential equations. Some
specific comparisons to known results will be made in the text of the paper.

The following notation will be used throughout this paper:

\[D = \{ t \in \mathbb{R}_+: g_i(t) \leq \delta (i = 1, \ldots, m) \}, \quad A = \{ t \in \mathbb{R}_+: g_i(t) \geq \delta (i = 1, \ldots, m) \}.\]

Let $g_i^*, d_i, a_i: \mathbb{R}_+ \to \mathbb{R}_+ \ (i = 1, \ldots, m)$ be nondecreasing functions such that

\[g_i^*(t) \leq \min \{ t, g_i(t) \} \quad \text{and} \quad d_i(t) \leq t \leq a_i(t) \quad \text{for} \quad t \in \mathbb{R}_+,
\]

\[g_i(t) \leq d_i(t) \quad \text{for} \quad t \in D \quad \text{and} \quad a_i(t) \leq g_i(t) \quad \text{for} \quad t \in A.
\]
Let $D_1(t) = D \cap [d_i(t), t]$ and $A_1(t) = A \cap [t, a_i(t)]$ for $t \in R_+$. To obtain our results we need the following two lemmas.

**Lemma 1** [6]. Let $x$ be a nonoscillatory solution of (E$_z$). Then there exist an integer $l \in \{0, 1, \ldots, n\}$ with $n + 1 + z$ even and a number $t_1 \in [t_0, \infty)$ such that

\[
\begin{align*}
(N_1) & \quad x(t) x^{(k)}(t) > 0 \quad (k = 0, 1, \ldots, l - 1), \\
(-1)^{k+1} x(t) x^{(k)}(t) & > 0 \quad (k = l, \ldots, n - 1)
\end{align*}
\]

for $t \geq t_1$.

**Lemma 2** [7]. Let $x$ be a nonoscillatory solution of (E$_z$) satisfying the inequality $(N_1)$ with $l \in \{1, \ldots, n - 1\}$ and $n + l + z$ even. In addition, let

\[
(1) \quad \int_{t_1}^\infty t^{-n-l} |x^{(l)}(t)| \, dt = \infty.
\]

Then the following inequalities hold for sufficiently large $t \geq t_1$:

\[
(2) \quad |x^{(l-1)}(t)| \geq \frac{t}{(n-l)!} \int_{t}^{\infty} s^{n-l-1} |x^{(l)}(s)| \, ds
\]

and

\[
(3) \quad k |x^{(l-k)}(t)| \geq t |x^{(l-k+1)}(t)|, \quad (k = 1, \ldots, l).
\]

2. **MAIN RESULTS**

**Theorem 1.** Consider the differential inequality (E$_z$) subject to the condition

\[
(4) \quad \lim_{t \to \infty} \sup \prod_{j=1}^{m} \left[ \int_{g_{j}(t)}^{t} p(s) \prod_{i=1}^{m} [g_{i}\ast(s)]^{(n-1)\ast_i} \, ds + \right.
\]

\[
\left. + \prod_{i=1}^{m} [g_{i}\ast(s)]^{\ast_i} \int_{t}^{\infty} p(s) \prod_{i=1}^{m} [g_{i}\ast(s)]^{(n-2)\ast_i} \, ds \right]^{\ast_j} > (n-1)!.
\]

Then

(i) for $n$ even, every solution of (E$_1$) is oscillatory,

(ii) for $n$ odd, every solution of (E$_1$) is either oscillatory or $\lim_{t \to \infty} x^{(k)}(t) = 0$ $(k = 0, 1, \ldots, n - 1)$ monotonically,

(iii) for $n$ even, every solution of (E$_2$) is either oscillatory or $\lim_{t \to \infty} x^{(k)}(t) = \infty$ $(k = 0, 1, \ldots, n - 1)$ monotonically,

(iv) for $n$ even, every solution of (E$_2$) is either oscillatory or $\lim_{t \to \infty} x^{(k)}(t) = 0$ or $\lim_{t \to \infty} |x^{(k)}(t)| = \infty$ $(k = 0, 1, \ldots, n - 1)$ monotonically.

**Proof.** Suppose that the inequality (E$_z$) has a nonoscillatory solution $x(t) \equiv 0$ for $t \geq t_0$. Therefore for sufficiently large $t \geq t_1$, by Lemma 1, there exists an integer $l \in \{0, 1, \ldots, n\}$ with $n + l + z$ even, such that $x(t)$ satisfies the inequalities $(N_1)$. Case (i). Then we have $n$ even, $z = 1$ and an odd $l \in \{1, 3, \ldots, n - 1\}$. We observe
that (4) and \((N_i)\) \((1 \leq l \leq n - 1)\) imply that the condition (1) of Lemma 2 is satisfied. Therefore (2) and \((E_i)\) yield
\[
|x^{(l-1)}(t)| \geq \frac{t}{(n-l)!} \int_t^\infty s^{n-l-1} p(s) \prod_{i=1}^m |x(g_i(s))|^{x_i} \, ds \geq \\
\geq \frac{t}{(n-l)!} \int_t^\infty s^{n-l-1} p(s) \prod_{i=1}^m |x(g_i^*(s))|^{x_i} \, ds ,
\]
which yields, by (3),
\[
|x^{(l-1)}(t)| \leq \frac{t}{l! (n-l)!} \int_t^\infty s^{n-l-1} p(s) \prod_{i=1}^m [g_i^*(s)]^{(l-1)x_i} |x^{(l-1)}(g_i^*(s))|^{x_i} \, ds .
\]
From the above inequality for \(j \in \{1, \ldots, m\}\) and \(t \geq t_1\) we get
\[
l! (n-l)! \left| \frac{x^{(l-1)}(g_j^*(t))}{g_j^*(t)} \right| \geq \int_t^\infty s^{n-l-1} p(s) \prod_{i=1}^m [g_i^*(s)]^{(l-1)x_i} |x^{(l-1)}(g_i^*(s))|^{x_i} \, ds + \\
+ \int_t^\infty s^{n-l-1} p(s) \prod_{i=1}^m [g_i^*(s)]^{(l-1)x_i} |x^{(l-1)}(g_i^*(s))|^{x_i} \, ds .
\]
Since \(|x^{(l-1)}(t)| t^{-1}\) is nonincreasing for \(1 \leq l \leq n - 1\) and \(t \geq t_1\) by (3), we obtain for \(s \in [g_j^*(t), t]\) and \(i \in \{1, \ldots, m\}\)
\[
\left| x^{(l-1)}(g_i^*(s)) \right| \leq \frac{g_i^*(s)}{g_i^*(t)} x^{(l-1)}(g_i^*(t)) .
\]
Therefore from (5) and (6), in view of the increasing character of \(|x^{(l-1)}(t)|\), we derive for \(t \geq t_1\)
\[
l! (n-l)! \left| \frac{x^{(l-1)}(g_j^*(t))}{g_j^*(t)} \right| \geq \prod_{i=1}^m \left[ \left| x^{(l-1)}(g_i^*(t)) \right| \right]^{x_i} \int_t^\infty s^{n-l-1} p(s) \prod_{i=1}^m [g_i^*(s)]^{x_i} \, ds + \\
+ \prod_{i=1}^m \left| x^{(l-1)}(g_i^*(t)) \right|^{x_i} \int_t^\infty s^{n-l-1} p(s) \prod_{i=1}^m [g_i^*(s)]^{(l-1)x_i} \, ds .
\]
For \(l \in \{1, \ldots, n - 1\}\) the following inequalities hold for \(s \geq t_1\):
\[
s^{n-l-1} \prod_{i=1}^m [g_i^*(s)]^{x_i} = s^{n-1} \prod_{i=1}^m \left[ \frac{g_i^*(s)}{s} \right]^{x_i} \geq \\
\geq s^{n-1} \prod_{i=1}^m \left[ \frac{g_i^*(s)}{s} \right]^{(n-1)x_i} = \prod_{i=1}^m [g_i^*(s)]^{(n-1)x_i}
\]
and
\[
s^{n-l-1} \prod_{i=1}^m [g_i^*(s)]^{(l-1)x_i} \geq \prod_{i=1}^m [g_i^*(s)]^{(n-2)x_i} .
\]
Using now (8) and (9) in (7) we get

\[
(n - 1)! \frac{x^{(l-1)}(g_i^*(t))}{g_j^*(t)} \geq \prod_{i=1}^{m} \left[ \frac{x^{(l-1)}(g_i^*(t))}{g_i^*(t)} \right]^{x_i} \int_{t}^{t} p(s) \prod_{i=1}^{m} \left[ g_i^*(s) \right]^{(n-1)x_i} ds + \\
+ \prod_{i=1}^{m} \left[ g_i^*(t) \right]^{x_i} \int_{t}^{\infty} p(s) \prod_{i=1}^{m} \left[ g_i^*(s) \right]^{(n-2)x_i} ds.
\]

Raising both sides of the above inequality to \( x_j \) and then multiplying the resulting inequalities we obtain

\[
(n - 1)! \prod_{i=1}^{j} \left[ \frac{x^{(l-1)}(g_i^*(t))}{g_i^*(t)} \right]^{x_i} \geq \\
\geq \prod_{i=1}^{m} \left[ \frac{x^{(l-1)}(g_i^*(t))}{g_i^*(t)} \right]^{x_i} \prod_{i=1}^{m} \int_{t}^{t} p(s) \prod_{i=1}^{m} \left[ g_i^*(s) \right]^{(n-1)x_i} ds + \\
+ \prod_{i=1}^{m} \left[ g_i^*(t) \right]^{x_i} \int_{t}^{\infty} p(s) \prod_{i=1}^{m} \left[ g_i^*(s) \right]^{(n-2)x_i} ds \right]^{x_i},
\]

which contradicts (4).

Case (ii). Then \( x(t) \) satisfies the inequalities (N\( l \)) for \( l \in \{0, 2, 4, \ldots, n - 1\} \). By arguments similar to those in the proof of (i), we prove that the case \( l \in \{2, 4, \ldots, n - 1\} \) is impossible. Therefore \( x(t) \) satisfies (N\( l \)) for \( l = 0 \), i.e.

\[
(-1)^{k} x(t) x^{(k)}(t) > 0 \quad (k = 0, 1, \ldots, n - 1) \quad \text{for} \quad t \geq t_1.
\]

We shall prove that \( \lim_{t \to \infty} x(t) = 0 \). Suppose to the contrary that \( \lim_{t \to \infty} x(t) = C > 0 \).

Then \( |x(g_i(t))| \geq C \) for \( t \geq t_2 \). From (10) it follows that (see [7])

\[
\int_{t_2}^{\infty} t^{n-1} |x^{(n)}(t)| dt < \infty,
\]

which implies, by (E\( l \)),

\[
\infty > \int_{t_2}^{\infty} t^{n-1} |x^{(n)}(t)| dt \geq \int_{t_2}^{\infty} t^{n-1} p(t) \prod_{i=1}^{m} |x(g_i(t))|^{x_i} dt \geq C \int_{t_2}^{\infty} t^{n-1} p(t) dt.
\]

But this gives a contradiction, since (4) implies that

\[
\int_{t_2}^{\infty} p(t) \prod_{i=1}^{m} \left[ g_i^*(t) \right]^{(n-1)x_i} dt = \infty.
\]

Case (iii) and (iv). Then \( x(t) \) satisfies (N\( l \)) for \( l \in \{0, 2, 4, \ldots, n\} \). The case \( l = 0 \) holds only when \( n \) is even. Then, by arguments analogous to those in the proof of (ii), we have \( \lim_{t \to \infty} x^{(k)}(t) = 0 \) for \( k = 0, 1, \ldots, n - 1 \). Similarly as in the proof of (i), we prove that the case \( l \in \{2, 4, \ldots, n - 2\} \) is impossible. In the case \( l = n \) we have

\[
x(t) x^{(n)}(t) \geq 0 \quad \text{and} \quad x(t) x^{(k)}(t) > 0 \quad (k = 0, 1, \ldots, n - 1)
\]

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for \( t \geq t_2 \). We shall prove that \( \lim_{t \to \infty} |x^{(k)}(t)| = \infty \) \((k = 0, 1, \ldots, n - 1)\). From (12) it follows that there exist a point \( t_3 \geq t_2 \) and a positive constant \( \gamma \) such that
\[
|x(g_i(t))| \geq \gamma g_i^{n-1}(t) \quad (i = 1, \ldots, m) \quad \text{for} \quad t \geq t_3.
\]
Integrating now \((E_2)\) from \( t_3 \) to \( t \), by (13) we obtain
\[
|x^{(n-1)}(t)| \geq |x^{(n-1)}(t_3)| + \int_{t_3}^{t} p(s) \prod_{i=1}^{m} |x(g_i(s))|^{\eta_i} \, ds \geq \gamma \int_{t_3}^{t} p(s) \prod_{i=1}^{m} [g_i^*(s)]^{(n-1)\eta_i} \, ds.
\]
From the above inequality and (11) we get \( \lim_{t \to \infty} |x^{(k)}(t)| = \infty \) \((k = 0, 1, \ldots, n - 1)\). Thus the proof is complete.

**Corollary 1.** Consider the linear differential equation with general deviating argument
\[
(14) \quad x^{(n)}(t) + p(t) x(g(t)) = 0, \quad n \geq 2,
\]
where \( p \) is the same as in \((E_2)\), \( g: \mathbb{R}_+ \to \mathbb{R}_+ \) is continuous and \( \lim g(t) = \infty \). Let the function \( g^*(t) \leq \min(t, g(t)) \) be nondecreasing on \( \mathbb{R}_+ \). If
\[
(15) \quad \lim_{t \to \infty} \sup \left\{ \int_{g^*(t)}^{t} \left[ g^*(s) \right]^{n-1} p(s) \, ds + g^*(t) \int_{t}^{\infty} \left[ g^*(s) \right]^{n-2} p(s) \, ds \right\} > (n - 1)!,
\]
then every solution of (14) is oscillatory, if \( n \) is even, and every solution of (14) is either oscillatory or \( \lim_{t \to \infty} x^{(k)}(t) = 0 \) \((k = 0, 1, \ldots, n - 1)\) monotonically if \( n \) is odd.

**Remark 1.** From Corollary 1, in the case of ordinary linear differential equations \((g(t) = t)\), we obtain the result of Chanturia [1, Th. 2.3]. In the case of advanced differential equations \((g(t) \geq t)\), Corollary 1 gives the result of Kusano [8, Th. 3].

**Remark 2.** Recently, conditions of nature similar to that in Corollary 1 have been obtained by Oláh [15, Th. 1], [16, Th. 2] and [17, Th. 1] (cf. also [2, C. 2.17] and [14, Th. 8]). These conditions for the equation (14) have the following forms:
\[
(16) \quad \lim_{t \to \infty} \sup \left[ g^*(t) \right]^{n-1} \int_{t}^{\infty} p(s) \, ds > (n - 1)!,
\]
for the general deviating argument \( g(t) \) and
\[
(17) \quad \lim_{t \to \infty} \sup g(t) \int_{t}^{\infty} g^{n-2}(s) p(s) \, ds > (n - 1)!,
\]
\[
(18) \quad \lim_{t \to \infty} \left[ \int_{t}^{\infty} sg^{n-1}(s) p(s) \, ds + t \int_{t}^{\infty} \frac{g^{n-1}(s)}{s} p(s) \, ds \right] > (n - 1)!
\]
in the case \( g(t) \leq t \).

We note that the condition (15) of Corollary 1 is independent of the above con-
ditions (16)–(18) for oscillations. For example, if we put \( p(t) = (n - 1)!e^{3n-4t-n} \) and \( g(t) = e^{-3t} \) in the equation (14), then the condition (15) of Corollary 1 is satisfied. In this case none of the conditions (16)–(18) is satisfied.

**Corollary 2.** Suppose that \( \prod_{i=1}^{m} [g_{i}^*(t)]^{s_{i}} \) has a nonnegative derivative on \( R_+ \). If there exists a positive nonincreasing function \( \psi \) on \( R_+ \), such that

\[
\int_{t}^{\infty} t^{-1} \psi(t) \, dt < \infty \quad \text{and} \quad \int_{t}^{\infty} p(t) \prod_{i=1}^{m} [g_{i}^*(t)]^{(n-1)s_{i}} \psi(\prod_{i=1}^{m} [g_{i}^*(t)]^{s_{i}}) \, dt = \infty,
\]

then the conclusion of Theorem 1 holds.

**Proof.** Proceeding identically as in the proof of Corollary of [15], we prove that

\[
\lim_{t \to \infty} \prod_{i=1}^{m} [g_{i}^*(t)]^{s_{i}} \int_{t}^{\infty} p(s) \prod_{i=1}^{m} [g_{i}^*(s)]^{(n-2)s_{i}} \, ds = \infty.
\]

Therefore the condition (4) is satisfied and the conclusion of Theorem 1 holds.

**Remark 3.** Similar conditions as in Corollary 2 can be found in the papers [7, 15, 17, 21, 23].

**Theorem 2.** If

\[
\lim_{t \to \infty} \prod_{j=1}^{m} \left[ \int_{D_{j}(t)} [s - d_{j}(t)]^{n-v-1} \prod_{i=1}^{m} [d_{i}(t) - g_{i}(s)]^{s_{i}} p(s) \, ds \right]^{s_{j}} > v! (n - v - 1)!
\]

for some \( v \in \{0, 1, \ldots, n - 1\} \) then every bounded solution of \((E_{n})\) is oscillatory.

**Proof.** Assume that \((E_{n})\) has a bounded nonoscillatory solution \( x(t) \neq 0 \) for \( t \geq t_{0} \). Then for sufficiently large \( t \geq t_{1} \geq t_{0} \) we have, by \((E_{n})\),

\[
(-1)^{n} x(t) x^{(v)}(t) \geq 0 \quad \text{and} \quad (-1)^{k} x(t) x^{(k)}(t) > 0 \quad (k = 0, 1, \ldots, n - 1).
\]

From the equality

\[
x^{(v)}(t) = \sum_{k=v}^{n-1} \frac{(t - u)^{k-v}}{(k - v)!} x^{(k)}(u) + \int_{u}^{t} \frac{(t - s)^{n-v-1}}{(n - v - 1)!} x^{(n)}(s) \, ds
\]

we obtain, in view of (20), for \( v \in \{0, 1, \ldots, n - 1\} \) and \( u \geq t \geq t_{1} \)

\[
|x^{(v)}(t)| \geq \int_{t}^{u} \frac{(s - t)^{n-v-1}}{(n - v - 1)!} |x^{(n)}(s)| \, ds
\]

and

\[
|x(t)| \geq \frac{(u - t)^{v}}{v!} |x^{(v)}(u)|.
\]

From (23) we have for \( s \in D_{j}(t) \) (\( j = 1, \ldots, m \)) and \( t \geq t_{1} \)

\[
|x(d_{j}(s))| \geq \frac{[d_{j}(t) - g_{j}(s)]^{v}}{v!} |x^{(v)}(d_{j}(t))| \quad (i = 1, \ldots, m).
\]
Therefore from (22), (Eₐ) and (24) we get for \( t \geq t_1 \) and \( v \in \{0, 1, \ldots, n - 1\}, \ j \in \{1, \ldots, m\} \)
\[
|x^{(\nu)}(d_j(t))| \geq \int_{d_j(t)}^{t} \frac{[s - d_j(t)]^{n - \nu - 1}}{(n - \nu - 1)!} |x^{(\nu)}(s)| \, ds \geq \\
\geq \prod_{i=1}^{m} |x^{(\nu)}(d_i(t))|^{\nu_i} \int_{d_i(t)}^{t} \frac{[s - d_i(t)]^{n - \nu - 1}}{v! (n - \nu - 1)!} p(s) \prod_{i=1}^{m} [d_i(t) - g_i(s)]^{v_{i2}} \, ds .
\]
Raising both sides of the above inequality to \( x_j \) and then multiplying, we obtain
\[
\prod_{j=1}^{m} |x^{(\nu)}(d_j(t))|^{x_j} \geq \\
\geq \prod_{i=1}^{m} |x^{(\nu)}(d_i(t))|^{x_i} \prod_{j=1}^{m} \left[ \int_{d_i(t)}^{t} \frac{[s - d_i(t)]^{n - \nu - 1}}{v! (n - \nu - 1)!} p(s) \prod_{i=1}^{m} [d_i(t) - g_i(s)]^{v_{i2}} \, ds \right]^{x_j},
\]
which contradicts (19). Thus the proof is complete.

**Corollary 3.** Suppose that in the equation \((E_\alpha)\), \(g_i(t) \leq t (i = 1, \ldots, m)\) on \(R_+\). Then every bounded solution of \((E_\alpha)\) is oscillatory, if
\[
\limsup_{t \to \infty} \prod_{i=1}^{m} \left[ \int_{g_i(t)}^{t} \frac{[s - g_i(t)]^{n - 1}}{p(s)} \, ds \right]^{x_i} > (n - 1)! .
\]

**Remark 4.** The oscillatory character of the bounded solutions of delay differential equations and inequalities has been considered in many papers, see for example [3, 4, 7, 10−13, 18, 20, 25, 28]. A conditions similar to that in Corollary 3 for delay differential equations and inequalities can be found in the papers [13, Th. 2], [18, Th. 1] and [25, Th. 2]. From these results it follows that every bounded solution of \((E_\alpha)\) is oscillatory, if
\[
\limsup_{t \to \infty} \int_{g(t)}^{t} \frac{[s - g(t)]^{n - 1}}{p(s)} \, ds > (n - 1)! ,
\]
where \(g(t) = \max(g_1(t), \ldots, g_m(t))\) and \(g_j(t)\) are nondecreasing. We note that the condition (25) of Corollary 3 is better than (26).

**Theorem 3.** Let \( n \geq 3 \) be odd. Consider the differential inequality \((E_1)\) subject to the conditions (4) and (19). Then every solution of \((E_1)\) is oscillatory.

**Proof.** The above theorem follows from Theorem 1 and 2.

**Theorem 4.** Let \( n \geq 3 \) be odd. Consider the differential inequality \((E_2)\) subject to the conditions
\[
\limsup_{t \to \infty} \prod_{j=1}^{m} \left[ \int_{g_j(t)}^{t} s p(s) \prod_{i=1}^{m} [g_i(s)]^{(n-2)x_i} \, ds + \right.
\]

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\[ + \prod_{j=1}^{m} [g_j^*(t)]^{p_j} \int_{t}^{\infty} sp(s) \prod_{i=1}^{m} [g_i^*(s)]^{(n-3)p_j} \, ds \right]^{p_j} > (n - 1)! \]

and

\[ \limsup_{t \to \infty} \prod_{j=1}^{m} \left[ \int_{A_j(t)} \left[ a_j(t) - s \right]^{n-1} \prod_{i=1}^{m} [g_i^*(s) - a_i(t)]^{p_j} \, ds \right]^{p_j} > v! (n - v - 1)! \]

for some \( v \in \{0, 1, \ldots, n - 1\} \). Then every solution of (E₂) is oscillatory.

**Proof.** Suppose that the inequality (E₂) has a nonoscillatory solution \( x(t) \neq 0 \) for \( t \geq t_0 \). Then Lemma 1 implies that either

\[ x(t) x^{(k)}(t) \geq 0, \quad x(t) x^{(k)}(t) > 0 \quad (k = 0, 1, \ldots, n - 1) \]

or there exists an odd \( l \in \{1, 3, \ldots, n - 2\} \) such that

\[ x(t) x^{(k)}(t) > 0 \quad (k = 0, 1, \ldots, l - 1), \]

\[ (-1)^{k+1} x(t) x^{(k)}(t) > 0 \quad (k = l, \ldots, n - 1) \quad \text{for} \quad t \geq t_1 \geq t_0. \]

Let (29) hold. Then from (21) we have for \( u \geq t \geq t_1 \) and \( v \in \{0, 1, \ldots, n - 1\} \)

\[ |x^{(v)}(u)| \geq \int_{t}^{u} \frac{(u - s)^{n-v-1}}{(n - v - 1)!} |x^{(n)}(s)| \, ds \]

and

\[ |x(u)| \geq \frac{1}{v!} (u - t)^v |x^{(v)}(t)|. \]

From (32) for \( s \in A_j(t) \) \( (j = 1, \ldots, m) \) and \( t \geq t_1 \) we obtain

\[ |x(g_j(s))| \geq \frac{1}{v!} [g_j(s) - a_j(t)]^v |x^{(v)}(a_j(t))| \quad i = 1, \ldots, m. \]

Then from (31), (E₂) and (33) we derive for \( j \in \{1, \ldots, m\} \), \( v \in \{1, \ldots, n - 1\} \) and \( t \geq t_1 \)

\[ |x^{(v)}(a_j(t))| \geq \int_{t}^{a_j(t)} \frac{[a_j(t) - s]^{n-v-1}}{(n - v - 1)!} |x^{(n)}(s)| \, ds \geq \]

\[ \geq \int_{A_j(t)} \frac{[a_j(t) - s]^{n-v-1}}{(n - v - 1)!} p(s) \prod_{i=1}^{m} |x(g_i(s))|^{p_i} \, ds \geq \]

\[ \geq \prod_{i=1}^{m} |x^{(v)}(a_i(t))|^{p_i} \int_{A_j(t)} \frac{[a_j(t) - s]^{n-v-1}}{(n - v - 1)!} p(s) \prod_{i=1}^{m} [g_i(s) - a_i(t)]^{p_i} \, ds. \]

Raising both sides of the above inequality to \( \alpha_j \) and then multiplying the resulting inequalities we get

\[ \prod_{j=1}^{m} |x^{(v)}(a_j(t))|^{\alpha_j} \geq \]

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\[ \geq \prod_{i=1}^{m} |x^{(v)}(a_i(t))|^{x_i} \prod_{j=1}^{m} \left[ \int_{a_j(t)}^{s} \frac{[a_j(t) - s]^{n-1}}{v!(n-v-1)!} \frac{p(s)}{s} \prod_{i=1}^{m} [g_i(s) - a_i(t)]^{x_i} \, ds \right]^{y_i}, \]

which contradicts (28). Thus, the case (29) is impossible.

Suppose now that (30) holds. Then, in view of (27) and (30), the assumptions of Lemma 2 are satisfied. Therefore, by arguments similar to those in the proof of Theorem 1, we obtain the inequality (7). Since \( l \in \{1, 3, \ldots, n-2\} \), we get for \( s \geq t_1 \)

\[
(34) \quad s^{n-2} \prod_{i=1}^{m} \left[ g_i(s) \right]^{(n-2)x_i} = s^{n-2} \prod_{i=1}^{m} \left[ g_i(s) \right]^{(n-2)x_i} \geq
\]

and

\[
(35) \quad s^{n-2} \prod_{i=1}^{m} \left[ g_i(s) \right]^{(n-3)x_i} = s^{n-2} \prod_{i=1}^{m} \left[ g_i(s) \right]^{(n-3)x_i} \geq
\]

Using now (34) and (35) in (7) we obtain

\[
(n-1)! \frac{|x^{(l-1)}(g_j^*(t))|}{g_j^*(t)} \geq \prod_{i=1}^{m} \left[ \frac{|x^{(l-1)}(g_i^*(t))|}{g_i^*(t)} \right]^{x_i} \int_{a_j^*(t)}^{s} \frac{s}{p(s)} \prod_{i=1}^{m} \left[ g_i(s) \right]^{(n-2)x_i} \, ds +
\]

\[
+ \prod_{i=1}^{m} \left[ g_i^*(t) \right]^{x_i} \int_{a_j^*(t)}^{\infty} s \prod_{i=1}^{m} \left[ g_i(s) \right]^{(n-3)x_i} \, ds. \]

Proceeding as in the corresponding part of the proof of Theorem 1, we get a contradiction with the assumption (27). Thus, the inequalities (30) cannot hold. This completes the proof.

**Theorem 5.** Let \( n \) be even. Consider the differential inequality (E\(_2\)) subject to the conditions (19) and (28). In addition, let for \( n \geq 4 \)

\[
(36) \quad \lim_{t \to \infty} \sup \prod_{j=1}^{m} \left[ \int_{a_j^*(t)}^{s} \frac{s}{p(s)} \prod_{i=1}^{m} \left[ g_i(s) \right]^{(n-2)x_i} \, ds +
\]

\[
+ \prod_{i=1}^{m} \left[ g_i^*(t) \right]^{x_i} \int_{a_j^*(t)}^{\infty} s \prod_{i=1}^{m} \left[ g_i(s) \right]^{(n-3)x_i} \, ds \right]^{y_j} > 2(n-2)!.
\]

Then every solution of (E\(_2\)) is oscillatory.

**Proof.** Let \( x \) be a nonoscillatory solution of (E\(_2\)) on an interval \([t_0, \infty)\). Thus, by Lemma 1, \( x \) satisfies the inequalities (N\(_l\)) with \( l \in \{0, 2, \ldots, n-2, n\} \). The cases \( l = 0 \) and \( l = n \) are impossible, by the assumptions (19), (28) and Theorem 2 and 4, respectively.

Suppose now that \( l \in \{2, \ldots, n-2\} \), which is possible only if \( n \geq 4 \). Therefore,
by arguments similar to those in the corresponding part of the proof of Theorems 1 and 4, we obtain the inequalities (7), (34) and (35). Combining (34) and (35) with (7) and using the fact that \(2 \leq l \leq n - 2\), we have
\[
2(n - 2)! \left| \frac{x^{(l-1)}(g_i^*(t))}{g_i^*(t)} \right| \geq \prod_{i=1}^{m} \left[ \frac{x^{(l-1)}(g_i^*(t))}{g_i^*(t)} \right]_{t}^{\infty} \int_{g_i^*(t)}^{\infty} s p(s) \prod_{i=1}^{m} \left[ g_i^*(s) \right]^{(\alpha-2)x_t} ds + \]
\[
+ \prod_{i=1}^{m} \left| x^{(l-1)}(g_i^*(t)) \right|_{t}^{\infty} \int_{t}^{\infty} s p(s) \prod_{i=1}^{m} \left[ g_i^*(s) \right]^{(\alpha-3)x_t} ds.
\]
From this inequality, similarly as in the proof of Theorem 4, we obtain a contradiction with the assumption (36). Thus \(l \notin \{2, \ldots, n - 2\}\) and the proof is complete.

3. FINAL REMARKS

For simplicity, we consider the linear differential equation with a deviating argument
\[
(x^{(n)}(t) = p(t) x(g(t)) ,
\]
where \(n\) is even, \(p: R_+ \to (0, \infty)\) and \(g: R_+ \to R_+\) are continuous, \(g(t)\) is non-decreasing and \(\lim_{t \to \infty} g(t) = \infty\). Let \(g_0(t) = \min\{t, g(t)\}, g_0^*(t) = \max\{t, g(t)\}, D = \{t \in R_+: g(t) < t\}\) and \(A = \{t \in R_+: g(t) > t\}\).

It is known that in the case of ordinary differential equation, i.e. \(g(t) = t\), the equation (L) always has nonoscillatory solutions satisfying the inequalities \((N_0)\) and \((N_n)\). The situation is different when \(g(t) \neq t\). For example, in view of Theorems 1 and 2, every solution \(x\) of (L) is either oscillatory or \(\lim_{t \to \infty} |x^{(k)}(t)| = \infty (k = 0, 1, \ldots, n - 1)\) monotonically, if the following conditions hold:

\[
\lim_{t \to \infty} \sup \left\{ \int_{g_0^*(t)}^{\infty} g_0^{n-1}(s) p(s) ds + g_0(t) \int_{t}^{\infty} g_0^{n-2}(s) p(s) ds \right\} > (n - 1)! ,
\]
and for some \(v \in \{0, 1, \ldots, n - 1\},

\[
\lim_{t \to \infty} \sup \int_{D_0}[g_0(t), t] \left[ s - g_0(t) \right]^{n-v-1} \left[ g_0(t) - g(s) \right]^{v} p(s) ds > v!(n - v - 1)!. 
\]

On the basis of Theorems 1 and 4 we can prove that every solution \(x\) of (L) is either oscillatory or \(\lim_{t \to \infty} x^{(k)}(t) = 0 (k = 0, 1, \ldots, n - 1)\) monotonically, if (37) holds and

\[
\lim_{t \to \infty} \sup \int_{A_0}[t, g_0^*(t)] \left[ g_0^*(t) - s \right]^{n-v-1} \left[ g(s) - g_0^*(t) \right]^{v} p(s) ds > v!(n - v - 1)! 
\]
for some \(v \in \{0, 1, \ldots, n - 1\}.

From Theorem 5 it follows that every solution of (L) is oscillatory if (38) and (39)
hold. In addition, when \( n \geq 4 \), the following inequality is satisfied:

\[
\limsup_{t \to \infty} \left\{ \int_{g^\ast(t)}^{\tau(t)} sg_\ast^{n-2}(s) p(s) \, ds + g_\ast(t) \int_{t}^{\infty} sg_\ast^{n-3}(s) p(s) \, ds \right\} > 2(n - 2)!.
\]

Recently, Kusano [9, Th. 1] has proved that every solution of (L) is oscillatory if there exist two sequences \( \{t_k\}, \{\tau_k\} \) such that \( t_k \in A \), \( t_k \to \infty \) as \( k \to \infty \), \( \tau_k \in D \), \( \tau_k \to \infty \) as \( k \to \infty \),

\[
\min \left\{ \int_{g(t_k)}^{\tau_k} [g(\tau_k) - g(s)]^{n-1} p(s) \, ds, \int_{t_k}^{\tau_k} [g(s) - g(t_k)]^{n-1} p(s) \, ds \right\} \geq (n - 1)!
\]

for all \( k = 1, 2, \ldots \) and

\[
\int_{0}^{\infty} [g_\ast(t)]^{n-1-\varepsilon} p(t) \, dt = \infty \quad \text{for some} \quad \varepsilon > 0.
\]

In some cases the conditions (38)–(40) are better than the conditions (41) and (42). For example, consider the differential equation with general deviating argument

\[
x^{(\sigma)}(t) = p(x(t + \sin t)), \quad p \text{ a positive constant}.
\]

Then

\[
D = \bigcup_{k=0}^{\infty} ((2k + 1)\pi, (2k + 2)\pi), \quad A = \bigcup_{k=0}^{\infty} (2k\pi, (2k + 1)\pi)
\]

and

\[
g_\ast(t) = \begin{cases} t + \sin t & \text{for} \quad t \in D, \\ t & \text{for} \quad t \notin D \end{cases}, \quad g^\ast(t) = \begin{cases} t + \sin t & \text{for} \quad t \in A, \\ t & \text{for} \quad t \notin A \end{cases}.
\]

If we choose \( t_k = (2k + 1)\pi + \frac{1}{2}\pi \) (\( k = 1, 2, \ldots \)), then \( D \cap [g_\ast(t_k), t_k] = [g_\ast(t_k), t_k] \) and

\[
\int_{D \cap [g_\ast(t_k), t_k]} [s - g_\ast(t_k)]^{n-1} p(s) \, ds = p \int_{g_\ast(t_k)}^{t_k} [s - g_\ast(t_k)]^{n-1} \, ds = \frac{p}{n} [t_k - g_\ast(t_k)]^n = \frac{p}{n}.
\]

Thus, the condition (38) is satisfied for \( p > n! \). If we choose \( t_k = 2k\pi + \frac{1}{2}\pi \) (\( k = 1, 2, \ldots \)), then we can prove by a similar argument as above that also the condition (39) is satisfied for \( p > n! \). Therefore for \( p > n! \) all solutions of (43) are oscillatory. The conditions (41) and (42) (cf. [9]) imply that every solution of (43) is oscillatory for \( p \geq (n - 1)! \left( \sin 1 - \frac{1}{2} \right)^{1-n} \). Finally, we remark that \((n - 1)! \left( \sin 1 - \frac{1}{2} \right)^{1-n} > n! \) for \( n \geq 2 \).
References


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