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ON CENTRAL RELATIONS OF COMPLETE LATTICES

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Several important properties of a lattice L can be described by the reflexive, symmetric, compatible binary relations of L which are called *tolerances*. The tolerances can be also considered as sublattices of L^2 which contain the diagonal relation $\Delta = \{(a, a)\ a \in L\}$ (identity relation) and are symmetric. A lattice L is called *simple* if besides Δ and $L \times L$ there exist no transitive tolerances i.e. congruence relations of L. Of course congruence relations have been studied to a great extent in order to develope the structure theory of lattices. But already the theorem of Baker-Pixley points out that the other binary compatible relations of L may play an important role. In this paper we study central relations which are tolerances having a center Z, $\emptyset \subsetneq Z \subsetneq L$, such that $(a, z) \in \varrho$ for every $a \in L$ if and only if $z \in Z$. In [5] it was proved that a maximal tolerance of a lattice of finite height is either a central relation or a congruence relation. In this paper we characterize the existence of central relations by filters and ideals under the hypothesis that the sublattices of L^2 are complete and L is distributive. We give some illustrations to this result and show that a modular lattice L of finite height is a projective geometry if and only if L is simple and has no central relation. We like to thank the referee for his suggestions.

Proposition 1. *Let ϱ be a central relation of the complete lattice L. Furthermore let ϱ be a complete sublattice of L^2 and $a = \sup\{x \mid (0, x) \in \varrho\}$, $b = \inf\{x \mid (1, x) \in \varrho, x \in L\}$. Then the following holds:*

1) *If Z is the center of ϱ then $Z = \{x \mid b \leqq x \leqq a,\ x \in L\}$ where $0 < b \leqq a < 1$.*
2. *If $\{a_i \mid i \in I\}$ is the set of atoms of L then $a \geqq \sup\{a_i \mid i \in I\}$.*
3) *If $\{b_i \mid i \in I\}$ is the set of coatoms of L then $b \leqq \inf\{b_i \mid i \in I\}$.*

Proof. As ϱ is a central relation with the center Z we have for $z \in Z$ that $(1, z) \in \varrho$ and $(0, z) \in \varrho$. Hence we have $b \leqq z$ and $z \leqq a$ and hence $Z \subset \{x \mid b \leqq x \leqq a, x \in L\} = [b, a]$. If $u \in [b, a]$ then $(1, u) \in \varrho$ because $b \leqq u$ and $(0, u) \in \varrho$ because $u \leqq a$. We conclude that $(x, u) \in \varrho$ for all $x \in L$ and hence $Z = [b, a]$. Because of $\emptyset \subsetneq Z \subsetneq L$ we have $0 < b \leqq a < 1$. If a_i is an atom of L and $a_i \nleqq a$ then we have $a \wedge a_i = 0$. Considering $(a_i, a_i) \in \varrho$ and $(a, a_i) \in \varrho$ we have $(0, a_i) \in \varrho$ and hence $a_i \leqq a$, a contradiction.

3) is proved in a similar way. □

Proposition 2. *Let ϱ be a tolerance of the complete lattice L and let ϱ be a complete sublattice of L^2 such that*

$$a = \sup \{x \mid (0, x) \in \varrho,\ x \in L\} \quad \textit{and} \quad b = \inf \{x \mid (1, x) \in \varrho,\ x \in L\}\,.$$

ϱ is a central relation if and only if $0 < b \leqq a < 1$.

Proof. We have only to show that $Z = [b, a]$ is a center of ϱ. If $z \in [b, a]$ then $(1, z) \in \varrho$ because of $b \leqq z$ and $(0, z) \in \varrho$ because $z \leqq a$. We have $(w, z) = [(w, w) \wedge \wedge (1, z)] \vee (0, z) \in \varrho$ for every $w \in L$. Obviously we have $\emptyset \subsetneqq Z \subsetneqq L$. □

Proposition 3. *Let L be a lattice with* 0, 1. *Assume that there are elements $a, b \in \in L \setminus \{0, 1\}$, $b \leqq a$, such that from $b \nleqq x$ it follows $x \leqq a$. Then L has a central relation.*

Proof. We consider the sublattice ϱ of L^2 which is generated by $\{(c, c);\ c \in L\}$, $(b, 0), (0, b), (b, 1), (1, b)$. ϱ is a reflexive and symmetric relation because of its generators. Furthermore ϱ is compatible with the lattice operations and b is an element of the center of ϱ. ϱ is a central relation if $\varrho \neq L^2$. We show that the condition $(*)$ "If $b \nleqq k$ then $l \leqq a$" holds for every pair $(k, l) \in \varrho$. At first we show that $(*)$ holds for the generators of ϱ and then for all elements of ϱ using induction for $\vee$ and $\wedge$. Obviously $(*)$ holds for (c, c) because of the hypothesis that from $b \nleqq c$ it follows $c \leqq a$. Similarly we have for $(0, b)$ that $b \nleqq 0$ but $b \leqq a$.

Consider $(e, g) \vee (s, t) = (e \vee s, g \vee t)$ and assume $b \nleqq e \vee s$. It follows $b \nleqq e$ and $b \nleqq s$ and hence $g \vee t \leqq a$. Consider $(e, g) \wedge (s, t) = (e \wedge s, g \wedge t)$ and assume $b \nleqq e \wedge s$. Then there is $b \nleqq e$ or $b \nleqq s$. For $b \nleqq e$ we have $g \leqq a$ and hence $g \wedge t \leqq a$. Now by the condition $(*)$ it follows that $\varrho \neq L^2$. □

In [2] Chajda, Niederle and Zelinka showed that the existence of certain ideals and filters is connected to the existence of intransitive tolerances. Following this line we prove

Lemma 4. *Let L be a complete lattice with complete ideals and filters. If I is a non-trivial ideal and F a non-trivial filter, such that*
1) $I \cap F \neq \emptyset$,
2) $I \cup F = L$,
then L has a central relation.

Proof. We consider the elements $a = \sup \{x \mid x \in I\}$ and $b = \inf \{x \mid x \in F\}$. As $I \cap F \neq \emptyset$ we have $b \leqq a$. If $c \in L = I \cup F$ with $b \nleqq c$ it follows $c \in I$ and $c \leqq a$. By proposition 3 follows that L has a central relation. □

A function $d\colon L \to L$ is called a $\vee$*-preserving subjection* if $d(x) \leqq x$ and $d(x \vee y) = d(x) \vee d(y)$. We use this concept which was introduced by Wille [7] to show the reverse direction of lemma 4 for distributive lattices. For the convenience of the reader we prove

Theorem 5. *Let L be a lattice such that every sublattice of L^2 is complete. Then* *there is a Galois connection between the lattice $T(L)$ of the tolerances of L and the lattice $D(L)$ of the $\vee$-preserving subjections of L.*

Proof. For every tolerance ϱ we define the map $d(x) = \inf\{y \mid (y, x) \in \varrho\}$. The map d has the property $d(x) \leqq x$ and is order preserving. Hence we have $d(x) \vee d(y) \leqq d(x \vee y)$. If we put $u = \inf\{z \mid (z, x) \in \varrho\}$ and $v = \inf\{z \mid (z, y) \in \varrho\}$ then we have $(u, x) \in \varrho$ and $(v, y) \in \varrho$ and hence $(u \vee v, x \vee y) \in \varrho$. Therefore we have $d(x \vee y) \leqq u \vee v = d(x) \vee d(y)$. We conclude that d is a $\vee$-preserving subjection.

On the other hand for every $\vee$-preserving subjection d we define the reflexive and symmetric relation θ by $(u, v) \in \theta$ if and only if $d(u \vee v) \leqq u \wedge v$. Considering $(u, v) \in \theta$ and $(r, s) \in \theta$ we have $d(u \vee r \vee v \vee s) = d(u \vee v) \vee d(r \vee s) \leqq (u \wedge v) \vee (r \wedge s) \leqq (u \vee r) \wedge (v \vee s)$. Hence $(u \vee v, r \vee s) \in \theta$. Considering $(u \wedge r, v \wedge s)$ we have $d((u \wedge r) \vee (v \wedge s)) \leqq d(u \vee v) \wedge d(r \vee s) \leqq (u \wedge v) \wedge (r \wedge s)$ and hence $(u \wedge r, v \wedge s) \in \theta$. We conclude that θ is a tolerance.

If we have $(u, v) \in \varrho$ then we have $(u \wedge v, v \vee u) \in \varrho$ and hence $d(u \vee v) = \inf\{y \mid (y, u \vee v) \in \theta, y \in L\} \leqq u \wedge v$. Therefore we have $\varrho \subseteq \theta$. Now let $(u, v) \in \theta$. We have $(u, u \vee v) \in \theta$ and $(d(u \vee v), u \vee v) \in \varrho$ by the definition of d. As $d(u \vee v) \leqq u \wedge v$ we have $(u \wedge v, u \vee v) \in \varrho$. It follows $(u \wedge v, u) \in \varrho, (u \wedge v) \in \varrho$ and hence $(u, v) \in \varrho$. Therefore we have $\theta \subseteq \varrho$. We have shown $\theta = \varrho$ and conclude there is a bijective function from $T(L)$ to $D(L)$. If $\varrho_1 \subseteq \varrho_2$ then $d_1(x) = \inf\{y \mid (y, x) \in \varrho_1, y \in L\} \geqq \inf\{y \mid (y, x) \in \varrho_2, y \in L\} = d_2(x)$. □

Theorem 6. *Let L be a distributive lattice such that every sublattice of L^2 is complete. L has a central relation if and only if there exists a non-trivial ideal I and a non-trivial filter F on L such that*

1) $I \cap F \neq \emptyset$,

2) $L = I \cup F$.

Proof. Let θ be a central relation of L. If ϱ is a (non-trivial) maximal tolerance with $\theta \subseteq \varrho$ then ϱ is a central relation. We consider ϱ with the center $Z = [b, a] = \{z \mid b \leqq z \leqq a\}$ and put $I = [0, a]$ and $F = [b, 1]$. Obviously we have $I \cap F \neq \emptyset$. It remains to show $L = I \cup F$. If $c \in L, c \notin I \cup F$ then $b \nleqq c$ and $c \nleqq a$. Furthermore we have from $(0, a) \in \varrho$, $(c, c) \in \varrho$ that $(c, c \vee a) \in \varrho$ and from $(b, 1) \in \varrho$ that $(c \wedge b, c \vee a) \in \varrho$. If $c \wedge b = 0$ then $c \vee a \leqq a$ because $a = \sup\{x \mid (0, x) \in \varrho, x \in L\}$. Hence $b > c \wedge b > 0$. By theorem 5 a $\vee$-preserving subjection d corresponds to the tolerance ϱ. We consider $\bar{d}(x) = d(x) \wedge c$. $\bar{d}$ has the properties $\bar{d}(x) \leqq d(x) \leqq x$ and $\bar{d}(x \vee y) = d(x \vee y) \wedge c = [d(x) \vee d(y)] \wedge c = [d(x) \wedge c] \vee [d(y) \wedge c] = \bar{d}(x) \vee \bar{d}(y)$. Hence $\bar{d}$ is a $\vee$-preserving subjection and by theorem 5 we have a tolerance $\bar{\varrho}$ corresponding to $\bar{d}$. $\bar{\varrho}$ is not trivial because $\bar{d}(1) = d(1) \wedge c = b \wedge c > 0$. We have $\bar{d} < d$ and by theorem 5 $\varrho \subsetneqq \bar{\varrho}$ which contradicts the maximality of ϱ. □

We conclude the paper with examples demonstrating the role of central relations.

Theorem 7. *Let L be a simple modular lattice of finite length. L is a projective geometry if and only if L has no central relation.*

This result is implied by theorem 5 in [4] and theorem 4 in [5]. As Fig. 1 shows,

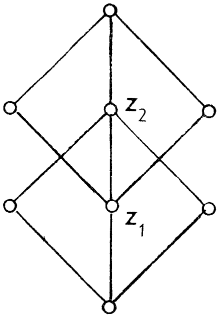

Fig. 1.

one can separate finite simple modular lattices in those without non-trivial tolerances and those having a central relation.

Theorem 8. *Let L be a lattice such that every sublattice of L^2 is complete.*

8.1. *If the greatest element* 1 *of L is the join of atoms then L has no central relations* (see also Wille [7] Satz 7).

8.2. *If L is orthocomplemented then L has no central relation.*

Proof. 8.1 follows from Proposition 1 property 2).

8.2. If ϱ is a central relation of L with the center Z and $z \in Z$ then we have $(z, 0) \in \varrho$ and $(1, z) \in \varrho$. It is $(z, 0) \vee (z', z') = (1, z') \in \varrho$ for the orthocomplement z' of z and hence $(1, z) \wedge (1, z') = (1, 0) \in \varrho$, a contradiction. □

8.1 and 8.2 does not imply that there are no intransitive tolerances on L as Fig. 2 shows.

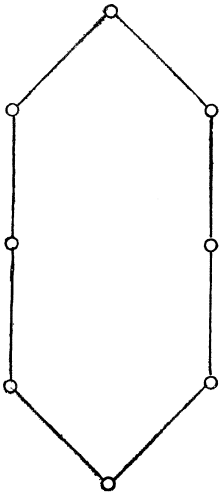

Fig. 2.

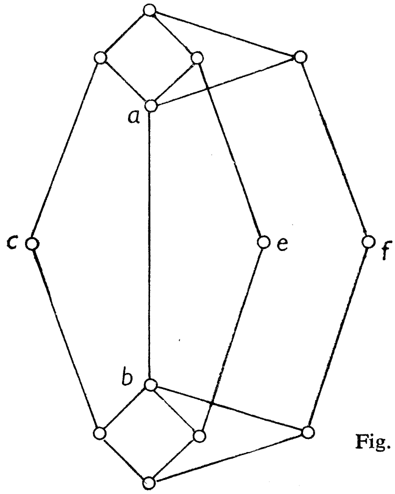

Fig. 3.

Remark. Theorem 6 does not hold for arbitrary lattices. Consider the lattice L of Fig. 3 for which every non-trivial ideal I and non-trivial filter F have the property $I \cup F \subsetneq L$ if $I \cap F \neq \emptyset$. On the other hand L has a central relation ϱ with the center $[b, a]$. To show that ϱ is not the allrelation we use the technique of Proposition 3. We verify that the condition "If $x \leqq c$ then $y \leqq c \vee a$" holds for every pair $(x, y) \in \varrho$. As in Proposition 3 we show that this condition holds for the generators $(a, 0)$, $(0, a)$, $(a, 1)$, $(1, a)$, $(b, 0)$, $(0, b)$, $(b, 1)$, $(1, b)$ of ϱ and then by induction for $\vee$ and $\wedge$.

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