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AN ORDERING OF SOME METRICS DEFINED ON THE SPACE OF GRAPHS

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1. INTRODUCTION

Recently, a number of metrics have been defined on the space Γ of graphs or various subsets of Γ. Zelinka [3] defined a metric d_i over all graphs $\Gamma(p)$ having order p where $d_i(A, B)$ was based upon the largest graph which is an induced subgraph of both A and B; we shall call d_i the *induced-subgraph metric*. Chartrand, Saba and Zou [1]) defined a metric d_{er} over all graphs $\Gamma(p, q)$ having order p and size q where $d_{er}(A, B)$ was based upon the minimum number of edge rotations required to transform A into B; we shall call d_{er} the *edge-rotation metric*. Johnson [2] defined a metric d_s over all graphs Γ where $d_s(A, B)$ was based upon the largest graph which is a subgraph of both A and B; we shall call d_s the *subgraph metric*.

Johnson also showed that metrics defined on graphs may be applied to problems in medicinal chemistry. Such applications of metrics raise problems of selecting the appropriate metric. This selection cannot be based upon topological properties because each of these metrics induces the discrete topology on its respective domain. However, these metrics are differentiated by their graphs. Since the graph of the discrete metric defined on any finite set is the complete graph, and since the graph of any other metric defined on that same set is always a subgraph of the complete graph, it makes sense to partially order the metrics based upon the partial ordering arising from the subgraph relation.

This paper explores this partial ordering. The preceding metrics are defined in section 2 and another metric d_{es} is defined on the space $\Gamma_c(p, q)$ of connected graphs of order p and size q where $d_{es}(A, B)$ is based upon a more restricted notion of an edge rotation which will be called an *edge shift*. Some terminology for comparing these metrics is developed in section 3. In section 4, we establish that $d_s \mid \Gamma(p) \geqq d_i$ and that $d_{er} \geqq d_s \mid \Gamma(p, q)$ where $\geqq$ denotes the expansion relation. We also show that $d_{es} \geqq d_{er} \geqq d_s \geqq d_i \geqq d_d$ when all metrics are restricted to $\Gamma_c(p, q)$ and that there exists (p, q) such that strict inequality holds in each case. In the last section, we show that the preceding metrics are graphable and that $d_i \mid \Gamma_c(p, q)$ and $d_{er} \mid \Gamma_c(p, q)$

are connected for all p and q, but not graphable for all p and q. We also show that $d_{er}(A, B) \geqq d_s(A, B)/2 = q - s$ where s is the size of any maximum common subgraph. (Chartrand, Saba and Zou [1] have already shown $2(q - s) \geqq d_{er}(A, B)$.)

2. SOME METRICS DEFINED ON GRAPHS

We start by reviewing the definitions of the preceding metrics beginning with those defined on the largest domains.

Let Γ denote the space of finite graphs. The *discrete* metric $d_d\colon \Gamma \times \Gamma \to Z^+$ is defined by $d_d(A, B) = 0$ if $A \cong B$ and $d_d(A, B) = 1$, otherwise.

Define the *cardinality* $|G|$ of a graph G to be $|V(G)| + |E(G)|$ where $V(G)$ and $E(G)$ denote the vertex set and edge of G. Johnson [1] defined the *subgraph* metric d_s: $\Gamma \times \Gamma \to Z^+$ such that $d_s(A, B)$ is the minimum of $|A| + |B| - 2|C|$ taken over all graphs C which are isomorphic to subgraphs of both A and B. Note that there always exist graphs A', B' and C' such that

$$(1) \qquad d(A', B') = |V(A' \setminus C')| + |V(B' \setminus C')| + |E(A' \setminus C')| + E(B' \setminus C')|$$

where $V(A' \setminus C')$ and $E(A' \setminus C')$ denote $V(A') \setminus V(C')$ and $E(A') \setminus E(C')$, respectively, and where $A' \cong A$, $B' \cong B$ and $C' \cong C$.

Zelinka defined the *induced subgraph* metric $d_i\colon \Gamma(p) \times \Gamma(p) \to Z^+$ such that $d(A, B) = n$ where $p - n$ is the order of a largest graph that is an induced subgraph of both A and B.

We shall say A can be transformed into a graph B by an edge rotation if A contains distinct vertices u, v and w such that $uv \in E(A)$, $uw \notin E(A)$ and $B \cong A - uv + uw$. Denote this *edge rotation* by (u, v, w) and the graph $A - uv + uw$ by tA where $t = (u, v, w)$. Chartrand, Saba and Zou [1] defined the *edge rotation* metric d_{er}: $\Gamma(p, q) \times \Gamma(p, q) \to Z^+$ by $d_{er}(A, B) = 0$ if $A \cong B$ and by $d_{er}(A, B) = n$, otherwise where n is the smallest positive integer for which there exists a sequence $t_1, \ldots, t_n$ of edge rotations such that $t_n \ldots t_1 A \cong B$.

An *edge shift* on a graph A is an edge rotation $t = (u, v, w)$ such that vw is an edge of A. As with an edge rotation, tA will denote the newly formed graph $A - uv + uw$. The *edge shift* metric $d_{es}\colon \Gamma_c(p, q) \times \Gamma_c(p, q) \to Z^+$ is defined by $d_{es}(A, B) = 0$ if $A \cong B$ and by $d_{es}(A, B) = n$ otherwise, where n is the smallest integer for which there exist edge shifts, $t_1, \ldots, t_n$, such that $t_n \ldots t_1 A \cong B$. That d_{es} is a metric follows immediately from propositions 1 and 2.

Proposition 1. *Let $A \in \Gamma_c(p, q)$. Let t be any edge shift. Then $tA \in \Gamma_c(p, q)$.*

Proof. Clearly, $tA \in \Gamma(p, q)$. Thus, we need only show that tA is connected. Write $t = (a, b, c)$ and let x and y be any two vertices of tA. Since A is connected, there exists a path $P = x_1, \ldots, x_n$ in A connecting x and y which we shall assume is a shortest such path. If P does not pass through the edge ab, then P is also a path of tA. Thus, we need only consider the case in which ab occurs once in P.

If ab or ba is a subpath of P and bc or cb is not, construct the walk P' from P by replacing ab or ba by acb or bca. If ab or ba is a subpath of P and bc or cb is also, then P contains a subpath of the form abc or cba since P is a shortest path. Form the path P' by replacing abc or cba with ac or ca, respectively. In either case, P' is a path connecting x and y. ∎

Proposition 2. *For nonisomorphic graphs $A, B \in \Gamma_c(p, q)$, there exists a sequence $t_1, \ldots, t_n$ of edge shifts such that $t_n \ldots t_1 A \cong B$.*

Proof. Let $A \in \Gamma_c(p, q)$. We shall say $G = (\{1, \ldots, p\}, \{e_1, \ldots, e_q\})$ is a *standard form* of A if (1) $G \cong A$, (2) $e_i = ab$ implies $a < b$ and (3) $i < j$ implies $e_i < e_j$ based upon the lexicographical ordering of the edges, i.e. $ab < cd$ if $a < c$, or if $a = c$ and $b < d$. Call $s(G) = e_1, \ldots, e_q$ the *edge sequence* of G. If $e_k(G) = ij$, we shall say $s(G)$ *increases minimally at* k if $e_{k+1}(G) = i(j + 1)$ for $j \neq p$ and $e_{k+1}(G) = (i + 1)(i + 2)$ for $j = p$. We shall call G *k-minimal* if $s(G)$ increases minimally at j for $j \leqq k$. Clearly, if G and H are both in $\Gamma_c(p, q)$ and are both q-minimal, then $G \cong H$. A q-minimal sequence has the following form:

$$12, \ldots, 1p, \ldots, (i - 1)\, i, \ldots, (i - 1)\, p, i(i + 1), \ldots, i(i + m)\,.$$

We will show that there exists a standard form G of A and a sequence $t_1, \ldots, t_n$ of edge shifts such that $t_n \ldots t_1 G \cong B$. Following the approach of Chartrand, Saba and Zou [1], we first prove that there exists a sequence $t_1, \ldots, t_n$ of edge shifts such that $t_n \ldots t_1 G$ is q-minimal.

Assume otherwise. Then there is a largest k, $k < q$, such that $t'_m \ldots t'_1 G$ is k-minimal for some edge shift sequence $t'_1, \ldots, t'_m$ and some standard form G of A. Let $H = t'_m \ldots t'_1 G$. We shall obtain a contradiction by showing there exists a standard form of H and a sequence $t_1, \ldots, t_n$ of edge shifts such that $t_n \ldots t_1 H$ is $(k + 1)$-minimal.

Case $k < (p - 1)$: Then $e_k(H) = 1(k + 1)$, but $e_{k+1}(H) = ij$ where either $i = 1$ and $j > k + 2$, or $i > 1$.

There must exist an edge uv where $u \leqq k + 1 < v$ for otherwise the subgraph induced by the vertex set $\{1, \ldots, k + 1\}$ would form a component of H. Form the graph H' by interchanging the labels v and $k + 2$. Clearly H' is k-minimal. If $u = 1$, H' is also $(k + 1)$-minimal. If $u \neq 1$, then the edge shift $t = (k + 2, u, 1)$ exists, and tH' is $(k + 1)$-minimal.

Case $p - 1 \leqq k < q$: Let $e_k(H) = ij$ and $e_{k+1}(H) = uv$ where (1) $j < p$ implies $e_{k+1}(H) \neq i(j + 1)$ and (2) $j = p$, implies $e_{k+1}(H) \neq (i + 1)(i + 2)$. It follows that $i > 1$ and that 1 is a star vertex, i.e. the edges $1m$ exist for $m = 2, \ldots, p$.

Assume $j < p$ and $e_{k+1}(H) = (j + 1)\, v$. Then $t_1((j + 1), 1, i)$ exists, deletes $1(j + 1)$ and creates $i(j + 1)$. It follows that $t_2((j + 1), v, 1)$ exists for $t_1 H$, deletes $(j + 1)\, v$ and recreates $1(j + 1)$. Clearly, $t_2 t_1 H = H - (j + 1)\, v + i(j + 1)$. It follows that $t_2 t_1 H$ is $(k + 1)$-minimal.

By the same argument, one can show that if $j < p$ and $e_{k+1}(H) = u(j+1)$, then there exist t_1 and t_2 such that $t_2 t_1 H$ is $(k+1)$-minimal.

Assume $j < p$ and $e_{k+1}(H) = uv$ where $u, v \neq (j+1)$. Then $t_1(u, 1, j+1)$ exists for H and $t_2(u, v, 1)$ exists for $t_1 H$. It follows that $t_2 t_1 H = H - uv + u(j+1)$ is k-minimal and has an edge of the form $(j+1)u$ or $u(j+1)$, and, by the preceding argument, H can be transformed into a $(k+1)$-minimal graph.

The case $j = p$ is proved similarly with i replaced by $i+1$ and with $j+1$ replaced by $i+2$ in the preceding argument. By contradiction then, there exists a sequence $t_1, \ldots, t_n$ of edge shifts such that $t_n \ldots t_1 H$ is q-minimal where H is a standard form of A.

Now let $A, B \in \Gamma_c(p, q)$. We have shown that for some standard forms G of A and H of B, there exist edge shift sequences, $t_1, \ldots, t_n$ and $u_1, \ldots, u_m$, such that $t_n \ldots t_1 G$ and $u_m \ldots u_1 H$ are both q-minimal. If $t = (u, v, w)$, let $t^{-1} = (u, w, v)$ and call t^{-1} the *inverse* of t. If t is well-defined on G, then t^{-1} is well-defined on tG and $t^{-1}tG = G$. It follows that $u_1^{-1} \ldots u_m^{-1} t_n \ldots t_1 G \cong H$. ■

3. COMPARING THE GRAPHS OF INTEGER METRICS

We shall now develop some terminology for relating integer metrics and the path metrics of their associated graphs.

A metric $d: W \times W \to Z^+$ taking values on the positive integers will be called an *integer* metric with *unit* λ where $\lambda = \min\{d(w, w') \mid w, w' \in W \text{ and } w \neq w'\}$. It will be convenient to say an integer metric defined on a singleton set has unit λ for any λ. One can associate the graph $G(d) = (W, E)$ with integer metric d by putting $ww' \in E$ if and only if $d(w, w') = \lambda$. Let d' be another integer metric. If $G(d)$ is a subgraph of $G(d')$, then d will be said to *expand* d' and we shall write $d \geqq d'$. Since the subgraph relation is a partial order, this relation of expansion is also a partial order. We shall say d strictly expands d' if d expands d', but not vice versa; and we shall write $d > d'$. The following propositions will be needed to establish the expansion relationship between restrictions of integer metrics. The proof of the first proposition is trivial.

Proposition 3. *Let d and d_d be integer metrics defined on W where d has unit λ and d_d denotes the discrete metric. Then $d \geqq d_d$, and if $d(w, w') > 1$ for any $w, w' \in W$, then $d > d_d$.*

Proposition 4. *Let d and d' be integer metrics defined on W with units λ and λ'. If $\lambda' \geqq \lambda^*$ and if $d(w, w') = \lambda$ implies $d'(w, w') \leqq \lambda^*$, then $\lambda' = \lambda^*$ and $d \geqq d'$.*

Proof. If W is a singleton set, the proposition is true by setting $\lambda^* = \lambda'$.

If W is not a singleton set, there exists $w, w' \in W$ such that $d(w, w') = \lambda$. This implies $d'(w, w') \leqq \lambda^*$, and consequently, $\lambda' \leqq \lambda^*$. It follows that $d(w, w') = \lambda$ implies $d'(w, w') = \lambda'$. Consequently, $d \geqq d'$. ■

A similar proof can be used to establish the following proposition.

Proposition 5. *Let d and d' be integer metrics defined on W with units λ and λ'. Let $W' \subset W$. If $d \geqq d'$ and if $d \mid W'$ has unit λ, then $d' \mid W'$ has unit λ' and $d \mid W' \geqq d' \mid W'$.*

Let d be any integer metric on W having graph $G(d)$. If $G(d)$ is connected, then d will be called *connected.* Since $G(d)$ is connected, there exists a shortest path connecting $w, w' \in G(d)$ whose length we denote by $\delta(w, w')$. The associated function $\delta\colon W \times W \to Z^+$ is a metric which will be called the *path metric associated with d.* Before proceeding further, note that the metric d defined on $\{1, 2, 3\}$ by $d(1, 2) = 1$, $d(1, 3) = 2$ and $d(2, 3) = 2$ is not connected since 3 is an isolated vertex of $G(d)$. The following proposition follows immediately from the triangle inequality.

Proposition 6. *Let d be a connected integer metric with unit λ defined on W, and let δ be the associated path metric. Then for every $w, w' \in W$,*

$$d(A, B) \leqq \lambda\delta(A, B)\,.$$

A number of metrics were defined in the preceding section. The following proposition will be useful in proving various restrictions of them are connected.

Proposition 7. *Let d and d' be any two integer metrics defined on W. If $d \geqq d'$ and d is connected, then d' is connected.*

Proof. The proof follows immediately from the fact $G(d)$ is a connected subgraph of $G(d')$ having the same vertex set as $G(d')$. ■

Let d be any connected integer metric with graph $G(d)$ and path metric δ. If $d(w, w') = \lambda\delta(w, w')$ for all $w, w' \in W$ then d will be said to be *graphable.* Note that δ is always graphable with unit 1. By defining $d'\colon W \times W \to Z^+$ by $d'(w, w') = d(w, w')/\lambda$, we see that any graphable metric with unit λ is equivalent to a graphable metric with unit 1. However, if d is not graphable, d' may not be an integer metric. Unless specifically stated otherwise, we shall assume all metrics have unit 1.

A metric can be connected, but not graphable. The metric d defined on $\{1, 2, 3, 4\}$ by $d(i, i + 1) = 1$ for $i = 1, 2, 3$, $d(i, i + 2) = 2$ for $i = 1, 2$ and $d(1, 4) = 2$ is connected, but not graphable. The graph of this metric is given by

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but misrepresents d since $\delta(1, 4) = 3 \neq 2 = d(1, 4)$. Clearly, this last metric is changed into a graphable metric by redefining $d(1, 4) = 3$.

Our use of the symbol $\geqq$ for the expansion relation is a consequence of the following proposition.

Proposition 8. *Let d and d' be integer metrics defined on W with units λ and λ', respectively. Let d be graphable. If $d \geqq d'$, then $d(w, w') \geqq (\lambda/\lambda')\, d'(w, w')$ for all $w, w' \in W$.*

Proof. Let $w, w' \in W$. By assumption, $d(w, w')/\lambda = \delta(w, w')$. Proposition 7 and $d \geqq d'$ imply that d' is connected. Consequently, δ' is defined. Since $G(d)$ is a subgraph of $G(d')$, $\delta(w, w') \geqq \delta'(w, w')$. By proposition 6, $\delta'(w, w') \geqq d'(w, w')/\lambda'$. ∎

4. SOME COMPARISONS BETWEEN THE DISCRETE METRICS DEFINED ON GRAPHS

The expansion relation is developed in this section for the metrics defined in section 2. Proposition 9 is an immediate consequence of proposition 3.

Proposition 9. *The integer metrics d_i, d_s, d_{er}, and d_{es} strictly expand d_d an their respective domains.*

Proposition 10. *d_s restricted to $\Gamma(p)$ expands d_i, and strictly expands d_i for some p.*

Proof. Let e denote any edge of K_p. Then $d_s(K_p, K_p - e) = 1$. Thus the unit of $d_s \mid \Gamma(p)$ is 1. Let $G(d_s \mid \Gamma(p))$ denote the graph of d_s restricted to $\Gamma(p)$.

To show $d_s \mid \Gamma(p) \geqq d_i$, let AB be any edge in $G(d_s \mid \Gamma(p))$. Then $d_s(A, B) = 1$. Thus, one of the sets on the right hand side of equation 1 has one member and the others are null. It follows that either A is a proper subgraph of B or vice versa.

Without loss of generality, we can assume that A is a proper subgraph of B and either $V(A) = V(B)$ and $|E(B) \setminus E(A)| = 1$, or $E(A) = E(B)$ and $|V(B) \setminus V(A)| = 1$. In the first case, assume $uv \in E(B) \setminus E(A)$. Then $A - u$ is an induced subgraph of both A and B, in which case $d_i(A, B) = 1$. In the second case, A is an induced, subgraph of both A and B, in which case $d_i(A, B) = 1$. It follows that $d_i(A, B) \leqq 1$. Since A is not isomorphic to B, $d_i(A, B) = 1$. Thus $AB \in G(d_i)$, i.e. $d_s \geqq d_i$.

To show $d_s > d_i$ for $p \geqq 3$, simply note that if $p \geqq 3$, there exist $u, v, w \in V(K_p)$. Let $A = K_p$ and $B = K_p - uv - uw$. It follows that $d_s(A, B) = 2 > 1 = d_i(A, B)$, i.e. $AB \in E(G(d_i))$, but $AB \notin E(G(d_s))$. ∎

Proposition 11. *d_{er} expands d_s restricted to $\Gamma(p, q)$, and there exists p and q such that d_{er} strictly expands d_s.*

Proof. We will establish the conditions of proposition 4. Let λ' denote the unit of $d_s \mid \Gamma(p, q)$. Note that $d_s(A, B) = 1$ implies either A or B is a proper subgraph of the other, i.e. $\{A, B\}$ is not a subset of $\Gamma(p, q)$ for any p and q. Thus, $\lambda' \geqq 2$.

Let AB be any edge of $G(d_{er})$ i.e. $d_{er}(A, B) = 1$. By definition, we can write $B \cong A - uv + uw$ where u, v and w are vertices of A, $uv \in E(A)$ and $uw \notin E(A)$, and where A is not isomorphic to B. It follows that $A - uv$ is a subgraph of both A and B, and, consequently, $d_s(A, B) \leqq 2$. Thus, by proposition 4, $d_{er} \geqq d_s \mid \Gamma(p, q)$ where $\lambda^* = 2$.

To show there exists p and q such that $d_{er} > d_s$, let A and B be defined by Figure 1.

Clearly, a single edge rotation of any edge lying on the 6-cycle of A will not suffice to transform A into B, for such a rotation results in a graph without a 6-cycle. Likewise, a single rotation of the 35 edge of A eliminates any 4-cycle of A unless a vertex of degree 4 is formed, of which B has none. Thus, $d_{er}(A, B) \geqq 2$, i.e. AB is not an edge of $G(d_{er})$. Since, A-35 is a subgraph of both A and B, $d_s(A, B) \leqq 2$, and since A

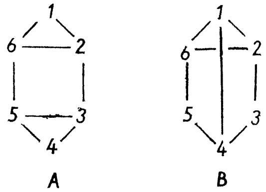

Fig. 1.

is not isomorphic to B, $d_s(A, B) = \lambda'$. Thus AB is an edge of $G(d_s \mid \Gamma(6, 8)$. It follows that $d_{er} < d_s$ on $\Gamma(6, 8)$. ■

Proposition 12. *If d_d, d_i, d_s, d_{er} and d_{es} are restricted to $\Gamma_c(p, q)$, then $d_d \leqq d_i \leqq \leqq d_s \leqq d_{er} \leqq d_{es}$. Moreover, there exist p and q such that strict inequality holds in each case.*

Proof. From proposition 5 and the transitivity of the extension relation, we need only show that $d_{es} \geqq d_{er}$ to establish the first set of inequalities. But this is obvious, because any edge shift is a special case of an edge rotation.

To establish the strict extension relation for some (p, q), consider the graphs in the Figure 2, and let $\lambda\ (=2)$ denote the unit of $d_s \mid \Gamma_c(p, q)$.

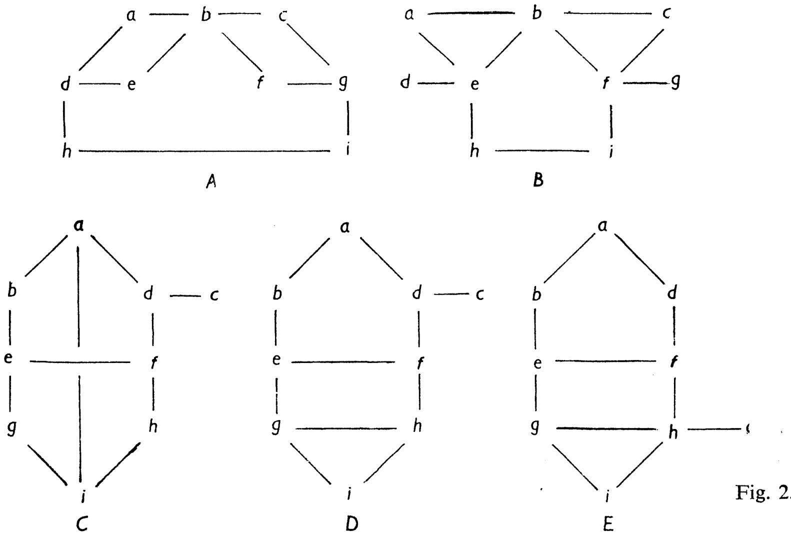

Fig. 2.

First note that $C - v$ contains a 5-cycle for all v. Since A does not contain a 5-cycle, $d_i(A, C) > 1$. It follows that $d_i \mid \Gamma_c(9, 11) > d_d\Gamma_c(9, 11)$.

Clearly, $A - b \cong B - b$. Thus $d_i(A, B) = 1$. However, $B - e'$ contains a 3-cycle for all edges e'. Since A contains no 3 cycles, $d_s(A, B) > \lambda$. Thus $d_s \mid \Gamma_c(9, 11) > > d_i \mid \Gamma_c(9, 11)$.

Since $C - ai \cong D - gh$, $d_s(C, D) = \lambda$. However, no single edge rotation will convert C into D. For rotating an edge on an 8-cycle of C would destroy C's only 8-cycle, and D has an 8-cycle. Rotating any of the other edges, ef, ai, or dc, to create the desired 3-cycle in D either creates a vertex of degree 4 or eliminates the only terminal vertex of D. Since C has a terminal vertex, but no vertex of degree 4, $d_{er}(C, D) > 1$. Thus, $d_{er} \mid \Gamma_c(9, 11) > d_s \mid \Gamma_c(9, 11)$.

Finally, $t(D) \cong E$ where t is the edge rotation (c, d, h). Thus, $d_{er}(D, E) = 1$. To show that $d_{es}(D, E) > 1$, note that the edges gi and hi cannot be shifted. Any edge shift of any other edges lying on the 8-cycle of D, eliminates either the only 8-cycle or the only terminal vertex of D, and E has both an 8-cycle and a terminal vertex. Finally, the edges gh, ef and dc of C cannot be shifted to form a 3-cycle with a vertex adjacent to a terminal vertex, and D has such a vertex. It follows $d_{es}(C, D) > 1$, and consequently, $d_{es} \mid \Gamma_c(9, 11) > d_{er} \mid \Gamma_c(9, 11)$. ∎

5. CONNECTEDNESS AND GRAPHABILITY OF d_s, d_i, d_{er} AND d_{es}

In this section, the subgraph, induced subgraph, edge-rotation and edge-shift metrics will be shown to be graphable on Γ, $\Gamma(p)$, $\Gamma(p, q)$ and $\Gamma_c(p, q)$, respectively. The restrictions of the subgraph metric to $\Gamma(p)$, $\Gamma(p, q)$ and $\Gamma_c(p, q)$ will be shown to be graphable for all p and q. The restrictions to $\Gamma_c(p, q)$ of the induced subgraph metric and the edge-rotation metric will be shown to be connected, but not graphable for all p and q.

Proposition 13. *The subgraph, induced subgraph, edge-rotation and edge-shift metrics are graphable.*

Proof. The edge-rotation and edge-shift metrics are trivially graphable because they are defined to be the path metric of their associated graphs.

To show d_i is graphable, let A and B be any two graphs of order p where $d_i(A, B) = n$. Let C have the largest vertex set of any graph that is isomorphic to an induced subgraph of both A and B. Without loss of generality, we can assume that A and B are defined on the same vertex set, V, and that C is a subgraph of both A and B. Let v be any vertex in $V \setminus V(C)$. By definition, $|V \setminus V(C)| = n$.

Let $v \in V \setminus V(C)$ and let $E_A(v)$ and $E_B(v)$ denote the edges of A and of B, respectively, that are adjacent to v. Clearly, $E_A(v) \neq E_B(v)$, for otherwise $C + v + E_A(v)$, which is larger that C, would be an induced subgraph of A and B – a contradiction. Let $v_1, \ldots, v_n$ be any ordering of the vertices in $V \setminus V(C)$. Define $H_0, \ldots, H_n$ by $H_0 = A$ and $H_{i+1} = H_i - E_A(v_i) + E_B(v_i)$ for $i = 1, \ldots, n - 1$. Since $H_i \in \Gamma(p)$, $i = 0, \ldots, n$, $d_i(H_i, H_{i+1}) \leqq 1$ for $i = 0, \ldots, n$, and $H_n \cong B$, it follows that there exists a subsequence of $H_0, \ldots, H_n$ which is a path of $G(d_i)$ connecting A and B. Thus, $\delta(A, B) \leqq n = d_i(A, B)$, and hence, d_i is graphable by proposition 6.

To show d_s is graphable, let C be any maximum common subgraph of A and B. Without loss of generality, we can assume C is a subgraph of both A and B.

We will define a sequence of transformations converting A to B that begins by deleting vertices in $V(A \setminus C)$ and then edges in $E(A \setminus C)$ and ends by adding vertices in $V(B \setminus C)$ and then edges in $E(B \setminus C)$. Specifically, let $x_1, \ldots, x_n$ be any sequencing of the elements in the union of the sets on the right hand side of equation 1 such that each element occurs once and only once in the sequence and such that $x_i \in E(A \setminus C)$ and $x_j \in V(A \setminus C)$ implies $i < j$, $x_i \in V(A \setminus C)$ and $x_j \in V(B \setminus C)$ implies $i < j$, and $x_i \in V(B \setminus C)$ and $x_j \in E(B \setminus C)$ implies $i < j$. Define the sequence $H_0, \ldots, H_n$ by $H_0 = A$, and $H_{i-1} = H_i \Delta x_i$ for $i = 1, \ldots, n$ where Δ is $-$ if $x_i \in V(A \setminus C) \cup E(A \setminus C)$ and Δ is $+$ if $x_i \in V(B \setminus C) \cup E(B \setminus C)$. Clearly, $d_s(H_i, H_{i+1}) = 1$ for $i = 0, \ldots, n-1$ and $H_n = B$. Thus $H_0, \ldots, H_n$ is a path of length $d(A, B)$ connecting A and B. It follows that the graph of d_s is connected and that $\delta_s(A, B) \leqq d_s(A, B)$. ∎

Propositions 8, 11, and 13 together with equation 1 imply that $d_{er}(A, B) \geqq d_s(A, B)/2 = q - s$ where s is the size of the maximum common subgraph of A and B. Moreover, the lower bound is achieved for A and B in Figure 1. An achieved upper bound $2(q - s) \geqq d_{er}$ was established in [1].

Proposition 14. *The restrictions of the subgraph metric to $\Gamma(p)$, $\Gamma(p, q)$ and $\Gamma_c(p, q)$ are graphable.*

Proof. The proof that $d_s \mid \Gamma(p)$ is graphable is a special case, where $V(A) = V(B)$, of the preceding proof that d_s is graphable.

Turning to $d_s \mid \Gamma(p, q)$ and $d_s \mid \Gamma_c(p, q)$, recall from the proof of proposition 11 that both metrics have unit 2. The connectedness and graphability of these metrics will both follow from the construction of paths of length $d_s(A, B)/2$ for any A, B in $\Gamma(p, q)$ and $\Gamma_c(A, B)$, respectively.

Let $A, B \in \Gamma(p, q)$, and let $(e_i, e_1'), \ldots, (e_n, e_n')$ be any one-to-one correspondence between the elements of $E(A \setminus C)$ and $E(B \setminus C)$. Define the sequence $H_0, \ldots, H_n$ by $H_0 = A$ and $H_i = H_{i-1} - e_i + e_i'$, for $i = 1, \ldots, n$. Clearly, $H_n = B$. Since, $H_i \in \Gamma(p, q)$ and $d(H_i, H_{i+1}) = 2$ for $i = 0, \ldots, n-1$, we have $2\delta'(A, B) \leqq d_s(A, B)$ where δ' is the path metric of $d_s \mid \Gamma(p, q)$. Thus, $d_s \mid \Gamma(p, q)$ is graphable.

Let A, B and C and $H_0, \ldots, H_n$ be as defined in the preceding proof except that $A, B \in \Gamma_c(p, q)$. We shall show that there exists a sequential pairing $(e_i, e_1'), \ldots, (e_n, e_n')$ of the elements of $E(A \setminus C)$ and $E(B \setminus C)$ such that the $H_i \in \Gamma_c(p, q)$ for $i = 0, \ldots, n$.

By definition, $H_0 \in \Gamma_c(p, q)$. Assume $H_i \in \Gamma_c(p, q)$ for $i \leqq k$. To show $H_{i+1} \in \Gamma_c(p, q)$, note that this is certainly the case if $H_i - e_{i+1}$ is connected. If $H_i - e_{i+1}$ is not connected, then it must have exactly two components, H and H'. Since B is connected and $H_n = B$ for any sequential pairing $(e_i, e_1'), \ldots, (e_n, e_n')$ of the elements of $E(A \setminus C)$ and $E(B \setminus C)$, there exists an edge $e' \in E(B \setminus C)$ such that $u \in V(H)$ and $v \in V(H')$. Clearly, $e_j = e'$ implies $j > i$, for otherwise H and H' could not be components of $H_i - e_{i+1}$. Put $e'_{i+1} = e'$. Then, H_{i+1} is connected. It follows that $2\delta'(A, B) \leqq d_s(A, B)$ where δ' is the path metric of $d_s \mid \Gamma_c(p, q)$. ■

Proposition 15. *The restrictions of the induced subgraph and edge-rotation metrics to $\Gamma_c(p, q)$ are connected, but not graphable for all p and q.*

Proof. The connectedness property for both $d_i \mid \Gamma_c(p, q)$ and $d_{er} \mid \Gamma_c(p, q)$ follows directly from propositions 7, 12 and 13.

To show $d_i \mid \Gamma_c(8, 7)$ is not graphable, let $A = K(1, 7)$ and $B = C \cup C'$ where C and C' are $K(1, 4)$ graphs with one edge in common. It is easy to show that $d_i \mid \Gamma_c(8, 7)$ has unit 1 and that $d_i(A, B) = 2$. We shall show that there does not exist a path joining A and B of length 2, i.e. there does not exist a connected graph G such that $d(A, G) = d(G, B) = 1$.

Assume G exist. Then there exist vertices u of G and v of A such that $G - u \cong$ $\cong A - v$. Assume v has degree 7, then $G - u \cong \overline{K}_7$. Since G is connected, it follows that $G \cong A$, i.e. $d(G, A) = 0$. Thus v must have degree 1, i.e. $G - u \cong K(1, 6)$. Since G is connected with 7 edges, it must have exactly one vertex u^* of degree 6 or more, and all other vertices must have degree 2 or less. Since $G - u^*$ has less than 2 edges and $B - w$ has more than 2 edges for any w, $G - u^*$ is not isomorphic to $B - w$ for any w. If $u \neq u^*$, then $G - u$ has a vertex of degree four or more. Thus $G - u$ is not isomorphic to $B - w$ for any w. It follows that $d(B, G) > 1$ – a contradiction.

To show $d_{er} \mid \Gamma_c(63, 62)$ is not graphable, let graphs A and B be the trees defined by Figure 3 where all terminal vertices are suppressed in the diagram and where a, b, c, d, e and f denote vertices of degrees 4, 8, 11, 11, 15 and 18 in A and degrees 5, 7, 11, 11, 14, and 19 in B.

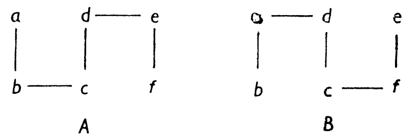

Fig. 3. Two trees where only non terminal vertices are depicted.

Let $t_1 = (d, e, a)$ and $t_2 = (c, b, f)$. Clearly, $t_2(t_1(A)) = B$. Thus $d_{er}(A, B) \leqq 2$. Since an edge rotation changes the degree of 2 vertices and since A has 4 vertices with degrees not found in B, $d_{er}(A, B) \geqq 2$. Thus $d(A, B) = 2$.

We will show that t_1, t_2 and t_2, t_1 are the only edge rotation sequences of length 2 which transform A into B. The proof will then follow from the fact that neither t_1A nor t_2A is connected. Consider any sequence u, v of 2 edge rotations leading to a graph isomorphic to B. Then vuA must have a vertex in A corresponding to vertex b in B. If uA is formed by adding an edge to a terminal vertex of A or by deleting an edge from any non terminal vertex of A other than b or e of A, then $d_{er}(uA, B) \geqq 2$ since uA would contain at least 3 vertices with degrees not found in B. Thus, u must contain an edge-rotation which deletes an edge adjacent to b or e and adds it to a non terminal vertex such that uA has only 2 vertices with degrees not present in B. But t_1 and t_2 are the only edge rotations satisfying these constraints. Since the unit of $d_{er} \mid \Gamma_c(56, 55)$ is 1, the proof is complete. ■

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