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Bohdan Zelinka

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THE DISTANCE BETWEEN A GRAPH AND ITS COMPLEMENT

BOHDAN ZELINKA, Liberec

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In [3] the distance between isomorphism classes of graphs was introduced. Here we shall investigate this distance between a graph and its complement.

An isomorphism class of graphs is the class of all graphs which are isomorphic to a given graph.

Now let n be a positive integer and let $\mathcal{G}_n$ be the set of all isomorphism classes of graphs with n vertices. Let $\mathfrak{G}_1 \in \mathcal{G}_n$, $\mathfrak{G}_2 \in \mathcal{G}_n$. Let p be the maximum number of vertices of a graph which is isomorphic simultaneously to an induced subgraph of a graph $G_1 \in \mathfrak{G}_1$ and to an induced subgraph of a graph $G_2 \in \mathfrak{G}_2$. We put $\delta(\mathfrak{G}_1, \mathfrak{G}_2) = n - p$ and call this number the distance between the isomorphism classes $\mathfrak{G}_1, \mathfrak{G}_2$.

For the sake of brevity we shall (not quite accurately) speak about the distance between graphs instead of the distance between isomorphism classes of graphs. By the distance $\delta(G_1, G_2)$ of the graphs G_1, G_2 (with the same number of vertices) we mean the distance $\delta(\mathfrak{G}_1, \mathfrak{G}_2)$ of the isomorphism classes $\mathfrak{G}_1, \mathfrak{G}_2$ such that $G_1 \in \mathfrak{G}_1$, $G_2 \in \mathfrak{G}_2$. By a common induced subgraph of G_1 and G_2 we shall mean a graph which is isomorphic simultaneously to an induced subgraph of G_1 and to a an induced subgraph of G_2.

In this paper we shall study the distance $\delta(G, \bar{G})$ between a graph G and its complement $\bar{G}$. As the complement $\bar{G}$ is uniquely determined by the graph G, the distance $\delta(G, \bar{G})$ is a numerical invariant of G; we denote it by $\bar{\delta}(G)$.

We shall consider only finite undirected graphs without loops and multiple edges.

Obviously $\bar{\delta}(G) = 0$ if and only if G is a self-complementary graph, i.e. a graph isomorphic to its own complement. These graphs were studied by G. Ringel [1] and H. Sachs [2]; these authors have (mutually independently) proved that a self-complementary graph with n vertices exists if and only if $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.

Theorem 1. *Let n be an integer, $n \geqq 2$. If $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$, then for any graph G with n vertices*

$$0 \leqq \delta(G) \leqq n - 1$$

holds and for any integer d such that $0 \leqq d \leqq n - 1$ there exists a graph G with n vertices such that $\bar{\delta}(G) = d$. If $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$, then for any graph

G with n vertices

$$1 \leqq \bar{\delta}(G) \leqq n - 1$$

holds and for any integer d such that $1 \leqq d \leqq n - 1$ *there exists a graph G with n vertices such that* $\bar{\delta}(G) = d$.

Proof. As it was mentioned above, for $n \equiv 0 \pmod 4$ and for $n \equiv 1 \pmod 4$ there exist self-complementary graphs with n vertices, i.e. graphs G for which $\bar{\delta}(G) = 0$. For $n \equiv 2 \pmod 4$ and for $n \equiv 3 \pmod 4$ such graphs do not exist, but in [4] it was proved that there exist almost self-complementary graphs with n vertices. An almost self-complementary graph is a graph G with the property that it can be transformed into a graph isomorphic to $\bar{G}$ by adding or deleting one edge. Thus consider such an almost self-complementary graph G with n vertices. Let e be the edge by whose adding or deleting from G a graph isomorphic to $\bar{G}$ is obtained, let u be one of its end vertices. Then the graph obtained from G by deleting u is an induced subgraph of a graph isomorphic to $\bar{G}$ and thus $\bar{\delta}(G) = \delta(G, \bar{G}) = 1$. This gives the lower bound. Any non-empty graph contains a subgraph consisting of one isolated vertex, hence $\bar{\delta}(G) \leqq n - 1$.

Now let an integer d be given, $0 \leqq d \leqq n - 1$. The case $d = 0$ was yet considered; thus suppose $1 \leqq d \leqq n - 1$. If $n - d \equiv 0 \pmod 4$ or $n - d \equiv 1 \pmod 4$, we take sets V, V_0 of vertices such that $V_0 \subset V$, $|V_0| = n - d$, $|V| = n$. We construct a self-complementary graph G_0 on V_0. Now the graph G is the graph obtained from G_0 by adding the vertices of $V - V_0$ as isolated vertices. The subgraphs of G and $\bar{G}$ induced by V_0 are both isomorphic to G_0. Any subgraph of G having more than $n - d$ vertices contains at least one isolated vertex, while such a subgraph of $\bar{G}$ has not. Therefore $\delta(G, \bar{G}) = n - (n - d) = d$. If $n - d \equiv 2 \pmod 4$, then we take the vertex sets V_0, V such that $V_0 \subset V$, $|V_0| = n - d + 1$, $|V| = n$, construct an almost self-complementary graph G_0 on V_0 and proceed further as in the preceding case. If $n - d \equiv 3 \pmod 4$, then we take again V_0 and V so that $V_0 \subset V$, $|V_0| = n - d + 1$, $|V| = n$, construct a self-complementary graph on V_0 and add an edge to it to obtain G_0; then we proceed as in the preceding case. ∎

Now we shall investigate graphs with the property that all of their connected components are cliques. Their complements are the so-called complete multipartite graphs.

Theorem 2. *Let G be a graph with n vertices having q connected components, all of which are cliques, let r be the maximum number of vertices of a connected component of G. Then*

$$\bar{\delta}(G) = n - \min\{q, r\}.$$

Proof. Denote $s = \min\{q, r\}$. First suppose $s = q$. Then $s \leqq r$ and both G and $\bar{G}$ contain subgraphs which are complete graphs with s vertices. Now consider a subgraph H of G with more than s vertices. All connected components of H are complete graphs and at least one of them has more than one vertex. If H is a complete graph, then no induced subgraph of $\bar{G}$ is isomorphic to H, because the largest clique in $\bar{G}$ has s vertices. If H contains at least two connected components, then also no induced subgraph of $\bar{G}$ is isomorphic to it, because each disconnected induced subgraph of $\bar{G}$ consists of isolated vertices. Hence $\bar{\delta}(G) = n - s$. Now let $s = r$. Then $s \leqq q$ and both G and $\bar{G}$ contain induced subgraphs consisting of s isolated vertices. Now consider a subgraph H of G with more than s vertices. Then this graph is disconnected. If it contains an edge, it is isomorphic to no induced subgraph of $\bar{G}$ as it was mentioned above. If H consists of isolated vertices, it is also isomorphic to no induced subgraph of $\bar{G}$, because the maximum number of vertices of an independent set in $\bar{G}$ is s. Again $\bar{\delta}(G) = n - s$. ∎

Theorem 3. *For a graph G with n vertices $\bar{\delta}(G) = n - 1$ if and only if G is a complete graph or consists of isolated vertices.*

Proof. The sufficiency follows from Theorem 2, where $q = 1$, $r = n$ or $q = n$, $r = 1$. The necessity follows from the fact that any graph which neither is complete, nor consists of isolated vertices contains both possible types of two-vertex subgraphs. ∎

Theorem 4. *For a graph G with n vertices $\bar{\delta}(G) = n - 2$ if and only if G is a graph of someone of the following types:*
(a) *complete bipartite graph;*
(b) *graph consisting of two connected components being cliques;*
(c) *graph consisting of connected components being cliques at which the maximum number of vertices of a clique is* 2;
(d) *the complement of a graph of the type* (c).

Proof. The graphs of the types (b) and (c) are graphs described in Theorem 2 for $q = 2$ or $r = 2$, the graphs of the types (a) and (d) are their complements. This implies the sufficiency. Now let G be a graph which does not belong to the types (a), (b), (c), (d); then evidently $\bar{G}$ also does not belong to them. Suppose that all connected components of G are cliques. If each of them consists of one vertex or there exists only one connected component, then Theorem 3 holds for G. Otherwise there are at least three connected components and at least one of them has at least three vertices. Then both G and $\bar{G}$ contain triangles and $\bar{\delta}(G) \leqq n - 3$. If all connected components of $\bar{G}$ are cliques, the proof is analogous. Finally, if both G and $\bar{G}$ contain a connected component which is not a complete graph, then they both contain an induced subgraph being a path of the length 2 and again $\bar{\delta}(G) \leqq n - 3$. ∎

At the end we shall study paths and circuits. By P_n we denote the path of the length n, i.e. with n edges and $n + 1$ vertices. By C_n we denote the circuit of the length n.

Theorem 5. *For the paths there is*

$$\begin{aligned}\bar{\delta}(P_1) &= 1\,,\\ \bar{\delta}(P_2) &= 1\,,\\ \bar{\delta}(P_3) &= 0\,,\\ \bar{\delta}(P_n) &= n - 4 \quad \text{for} \quad n \geqq 4\,.\end{aligned}$$

Proof. The assertions for P_1 and P_2 are evident. The path P_3 is a self-complementary graph. If $n \geqq 4$, then P_n contains an induced subgraph isomorphic to P_3; the subgraph induced by the same vertex set in $\bar{P}_n$ is also isomorphic to P_3. The graph P_3 has four vertices and thus $\bar{\delta}(P_n) \leqq n - 4$. On the other hand, each induced subgraph of P_n with at least five vertices contains an independent set with three vertices; hence the subgraph of $\bar{P}_n$ induced by the same set contains a triangle, while P_n contains no triangle. This implies $\bar{\delta}(P_n) = n - 4$. ∎

Theorem 6. *For the circuits there is*

$$\begin{aligned} \bar{\delta}(C_3) &= 1\,, \\ \bar{\delta}(C_4) &= 2\,, \\ \bar{\delta}(C_5) &= 0\,, \\ \bar{\delta}(C_n) &= n - 4 \quad \text{for} \quad n \geqq 6\,. \end{aligned}$$

Proof. The assertions for C_3 and C_4 follow from Theorem 2. The circuit C_5 is a self-complementary graph. The assertion for $n \geqq 6$ can be proved in the same way as the assertion for $n \geqq 4$ in Theorem 5. ∎

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Author's address: 461 17 Liberec 1, Studentská 1292, Czechoslovakia (katedra tváření a plastů VŠST).