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ON A PROBLEM OF B. ZELINKA, II

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In [2], B. Zelinka has posed the following problem viz whether there exists a commutative semi-group such that each tolerance relation is compatible with its element set? In [1] we have given an example of such a semi-group. The purpose of the present paper is to give a complete characterisation of B. Zelinka's problem. For definitions and notation refer [2]. Now we prove the following two theorems which completely characterize the problem of B. Zelinka.

**Theorem 1.** Let \( \langle S, * \rangle \) be a commutative semi-group with a multiplicatively zero element. Let \(|S| \geq 3\). Then every tolerance relation in \( \langle S, * \rangle \) is compatible with its element set if and only if \( \langle S, * \rangle \) is a zero semi-group, i.e. the product of any two elements is zero.

**Proof.** If part. Proof is exactly the same as in the example of [1]. Only if part. Let \( a, b \) be two distinct elements in \( S \) different from 0. Suppose \( a * b \neq 0 \), then \( a * b = a \) or \( a * b \neq a \). Case i. \( a * b = a \). Define a tolerance relation \( \varrho \) in \( S \) as follows. \( \varrho = \{(x, x) \mid x \in S\} \cup \{(a, b), (b, a), (0, b), (b, 0)\} \). We shall show that this tolerance relation \( \varrho \) is not compatible with its element set in \( S \). For \( b \neq a \) and \( 0 \notin b \) but \( (b * 0, a * b) = (0, a) \notin \varrho \), a contradiction. Case ii. \( a * b \neq a \). In this case define a tolerance relation \( T \) in \( S \) as follows. \( T = \{(z, z) \mid z \in S\} \cup \{(a, b), (b, a), (0, a), (a, 0)\} \). Now \( (a, b), (0, a) \notin T \) but \( (a * 0, b * a) = (0, a * b) \notin T \). Hence \( T \) is not compatible yielding a contradiction. Hence the product of any two distinct elements is zero. Now, it remains to prove that \( a * a = 0 \) for every \( a \in S \). If \( a * a \neq 0 \), then \( a * a = a \) or \( a * a \neq a \). Since \(|S| \geq 3\), \( S \) contains an element \( b \) different from 0 and \( a \). Case iii. \( a * a = a \). In this case define a tolerance relation \( \varrho' \) in \( \langle S, * \rangle \) as follows. \( \varrho' = \{(s, s) \mid s \in S\} \cup \{(a, b), (b, a), (0, b), (b, 0)\} \). Now \( a \notin \varrho' \), \( b \notin \varrho' \). But \( (a * a, b * 0) = (a, 0) \notin \varrho' \). Hence \( \varrho' \) is not compatible. Case iv. \( a * a \neq a \). In this case, define a tolerance relation \( T' \) in \( \langle S, * \rangle \) as follows. \( T' = \{(s, s) \mid s \in S\} \cup \{(0, a), (a, 0)\} \). Now \( (a, a) \notin T' \) and \( (0, a) \notin T' \) but \( (a * 0, a * a) = (0, a * a) \notin T' \), yielding a contradiction. Hence the theorem is proved.

**Remark.** There is meaning in taking \(|S| \geq 3\). For now we give an example to show that the theorem is not true when \(|S| = 2\). Let \( S = \{a, b\} \). The mutiplication...
A table of \((S, *)\) is given below.

<table>
<thead>
<tr>
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<th>a</th>
<th>b</th>
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<tr>
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One can easily check that every tolerance relation in \((S, *)\) is compatible and obviously \((S, *)\) is not a zero semi-group.

Now, we are going to prove a theorem which is more powerful than theorem 1. We need the following definition.

**Definition.** A commutative semi-group \((S, *)\) is called a Zelinka semi-group if and only if every tolerance relation in \((S, *)\) is compatible with the elements of \(S\).

**Theorem 2.** Let \((S, *)\) be a commutative semi-group. Let \(|S| \geq 3\). Then \(S\) is a Zelinka semi-group if and only if \(S\) contains a multiplicatively zero element and the product of any two elements in \(S\) is zero.

**Proof.** If part. Follows from theorem 1. Only if part. It suffices to prove that \((S, *)\) contains a zero element, for then the result follows from theorem 1. Let \(a\) be any element in \(S\). Now the set \(a * S\) is either a single element set or contains more than one element.

Case i. \(|a * S| = 1\). Let \(a * S = \{x\}\) where \(x \in S\). Now clearly \(x\) is a zero element in \(S\) for \(x \in S\). This implies \(x\) is a zero element in \(S\) and the result that the product of any two elements is zero follows from Theorem 1.

Case ii. \(|a * S| > 1\). Then \(a \in S\) contains at least two distinct elements \(x, y\). Clearly \(x \neq y\).

Sub case a. \(x \neq y\), \(x, y \neq a\). Since \(x, y \in a \in S\), \(a \neq b = x, a \neq c = y\) for some \(b, c\) in \(S\). Now, let \(A\) be a tolerance relation defined as follows. \(A = \{(s, s) | s \in S\} \cup \{(b, c), (c, b)\}\). Now, since \(A\) is compatible, \((a, a), (b, c) \in A\) implies \((a \neq b, a \neq c) = (x, y) \in A\). Hence the possibilities are \(x = b, y = c\) or \(x = c, y = b\).

Sub case a. \(x = b, y = c\). Now we have \(a \neq b = a \neq c = c\). Let \(B\) be a tolerance relation in \((S, *)\) defined as follows: \(B = \{(s, s) | s \in S\} \cup \{(a, c), (c, a), (a, b), (b, a)\}\). We shall show that \(B\) is not compatible. Now \((a, c) \in B\) and \((b, a) \in B\). But \((a \neq b, c \neq a) = (b, c) \neq B\) yielding a contradiction. The next possibility is \(x = c\) and \(y = b\).

Sub case b. \(x = c, y = b\). Now we have \(a \neq c = b\) and \(a \neq b = c\). Let \(C\) be a tolerance relation in \((S, *)\) defined as follows: \(C = \{(s, s) | s \in S\} \cup \{(a, b), (a, c), (c, a)\}\). Now \((a, b), (c, a) \in C\) but \((a \neq c, b \neq a) = (b, c) \neq C\), since \(x = c, y = b, x \neq y\) and \(x, y \neq a\). Hence, \(C\) is not compatible, a contradiction.

Sub case c. \(x \neq y\) and at least one of \(x, y\) equals \(a\). W.l.o.g. assume that \(x = a, y \neq a\). Now \(a \neq b = a \neq c = y\). Let \(D\) be a tolerance relation defined in \((S, *)\) as follows, \(D = \{(s, s) | s \in S\} \cup \{(a, b), (b, a), (a, c), (c, a)\}\). By assumption \(D\) is compatible. Hence, \((a, b), (c, a) \in D\) implies \((a \neq c, b \neq a) = (y, a) \in D\). Since \(y \neq a\),
the possibilities are $y = b$ or $y = c$. We shall show that both these possibilities are impossible.

Sub case $b - i$. $y = b$. Then we have $a * y = a$ and $a * c = y$. Since $y \neq a$, clearly $y \neq c$ for $y = c$ implies $a * y = a * c$ which implies $a = y$ which is a contradiction. Let $E$ be a tolerance relation defined in $S$ as follows. $E = \{(s, s) \mid s \in S\} \cup \{(y, c), (c, y)\}$. Since $E$ is compatible $(a, a) \in E$, $(y, c) \in E$ implies $(a * y, a * c) = (a, y) \in E$ which implies $y = a$ or $c = a$. Clearly $y \neq a$. Hence, other possibility is $c = a$. We shall show that this is also impossible. Let $e = a$. Now we have $a * y = a$ and $a * a = y$. Since $|S| \geq 3$, there exists an element $d$ distinct from $a$ and $y$.

Now by assumption $a, y, d$ are three distinct elements in $S$. Let $F$ be a tolerance relation defined in $\langle S, * \rangle$ as follows. $F = \{(s, s) \mid s \in S\} \cup \{(y, d), (d, y)\}$. Now, $(a, a), (y, d) \in F$ and since $F$ is compatible $(a * y, a * d) = (a, a * d) \in F$ which implies $a * d = a$. We shall show that $a * d = a$ is also not possible. For let $G$ be a tolerance relation defined in $\langle S, * \rangle$ as follows. $G = \{(s, s) \mid s \in S\} \cup \{(a, d), (d, a)\}$. We shall show that $G$ is not compatible. Now $(a, a), (a, d) \in G$. But $(a * a, a * d) = (a, a \neq d)$ yielding a contradiction.

Sub case $b - ii$. $y = c$. Then we have $a * b = a$ and $a * y = y$. Let $H$ be a tolerance relation defined as follows. $H = \{(s, s) \mid s \in S\} \cup \{(b, y), (y, b)\}$. Since $H$ is compatible, $(a, a), (y, b) \in H$ implies $(a * y, a * b) = (y, a) \in H$. The possibilities are $y = a$ or $y = b$ or $b = a$. Since $x \neq a, b = y$ or $b = a$. Since $x \neq a, x = b$ and $y = c$ we have $y \neq b$. Hence the remaining possibility is $b = a$. In this case, $a * a = a$ and $a * y = y$. Since $|S| \geq 3$, and $a \neq y$, there exist an element $d$ different from $a$ and $y$. Now let $I$ be a tolerance relation defined in $\langle S, * \rangle$ as follows. $I = \{(s, s) \mid s \in S\} \cup \{(a, d), (d, a)\}$. Since $I$ is compatible $(a, a), (a, d) \in I$ implies $(a * a, a * d) = (a, a * d) \in I$. Hence $a * d = a$ or $d$. We shall show that both the possibilities are impossible. Suppose $a * d = a$. Now define a tolerance relation $J$ in $\langle S, * \rangle$ as follows. $J = \{(s, s) \mid s \in S\} \cup \{(d, y), (y, d)\}$. Now, $(a, a), (y, d) \in J$. But $(a * y, a * d) = (y, a) \notin J$ showing that $J$ is not compatible, a contradiction. Next possibility is $a * d = d$. Let $K$ be a tolerance relation defined in $\langle S, * \rangle$ as follows. $K = \{(s, s) \mid s \in S\} \cup \{(a, y), (y, a), (a, d), (d, a)\}$. Now $(a, y), (d, a) \in K$. But $(a * d, y * a) = (d, y) \notin K$, since $a, y, d$ are distinct elements thus yielding a contradiction.

All these contradictions show that $a * S$ contains only one element say $x$. So by case (i), $x$ is a zero element of $\langle S, * \rangle$. Now by theorem 1, $\langle S, * \rangle$ is a zero semigroup. (Q.E.D.).

Finally, I wish to express my thanks to the refereree.

References


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