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ON A PROBLEM OF B. ZELINKA, II

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In [2], B. Zelinka has posed the following problem viz whether there exists a commutative semi-group such that each tolerance relation is compatible with its element set? In [1] we have given an example of such a semi-group. The purpose of the present paper is to give a complete characterisation of B. Zelinka's problem. For definitions and notation refer [2]. Now we prove the following two theorems which completely characterize the problem of B. Zelinka.

Theorem 1. *Let $\langle S, *\rangle$ be a commutative semi-group with a multiplicatively zero element. Let $|S| \geqq 3$. Then every tolerance relation in $\langle S, *\rangle$ is compatible with its element set if and only if $\langle S, *\rangle$ is a zero semi-group, i.e. the product of any two elements is zero.*

Proof. If part. Proof is exactly the same as in the example of [1]. Only if part. Let a, b be two distinct elements in S different from 0. Suppose $a * b \neq 0$, then $a * b = a$ or $a * b \neq a$. Case i. $a * b = a$. Define a tolerance relation ϱ in S as follows. $\varrho = \{(x, x) \mid x \in S\} \cup \{(a, b), (b, a), (0, b), (b, 0)\}$. We shall show that this tolerance relation ϱ is not compatible with its element set in S. For $b \varrho a$ and $0 \varrho b$ but $(b * 0, a * b) = (0, a) \notin \varrho$, a contradiction. Case ii. $a * b \neq a$. In this case define a tolerance relation T in S as follows. $T = \{(z, z) \mid z \in S\} \cup \{(a, b), (b, a), (0, a), (a, 0)\}$. Now $(a, b), (0, a) \in T$ but $(a * 0, b * a) = (0, a * b) \notin T$. Hence T is not compatible yielding a contradiction. Hence the product of any two distinct elements is zero. Now, it remains to prove that $a * a = 0$ for every $a \in S$. If $a * a \neq 0$, then $a * a = a$ or $a * a \neq a$. Since $|S| \geqq 3$, S contains an element b different from 0 and a. Case iii. $a * a = a$. In this case define a tolerance relation ϱ' in $\langle S, *\rangle$ as follows. $\varrho' = \{(s, s) \mid s \in S\} \cup \{(a, b), (b, a), (0, b), (b, 0)\}$. Now $a \varrho' b$, $a \varrho' 0$. But $(a * a, b * 0) = (a, 0) \notin \varrho'$. Hence ϱ' is not compatible. Case iv. $a * a \neq a$. In this case, define a tolerance relation T' in $\langle S, *\rangle$ as follows. $T' = \{(s, s) \mid s \in S\} \cup \{(0, a), (a, 0)\}$. Now $(a, a) \in T'$ and $(0, a) \in T'$ but $(a * 0, a * a) = (0, a * a) \notin T'$, yielding a contradiction. Hence the theorem is proved.

Remark. There is meaning in taking $|S| \geqq 3$. For now we give an example to show that the theorem is not true when $|S| = 2$. Let $S = \{a, b\}$. The mutiplication table of $\langle S, *\rangle$ is given below.

$*$	a	b
a	a	b
b	b	b

One can easily check that every tolerance relation in $\langle S, *\rangle$ is compatible and obviously $\langle S, *\rangle$ is not a zero semi-group.

Now, we are going to prove a theorem which is more powerful than theorem 1. We need the following definition.

Definition. A commutative semi-group $\langle S, *\rangle$ is called a *Zelinka semi-group* if and only if every tolerance relation in $\langle S, *\rangle$ is compatible with the elements of S.

Theorem 2. *Let $\langle S, *\rangle$ be a commutative semi-group. Let $|S| \geqq 3$. Then S is a Zelinka semi-group if and only if S contains a multiplicatively zero element and the product of any two elements in S is zero.*

Proof. If part. Follows from theorem 1. Only if part. It suffices to prove that $\langle S, *\rangle$ contains a zero element, for then the result follows from theorem 1. Let a be any element in S. Now the set $a * S$ is either a single element set or contains more than one element.

Case i. $|a * S| = 1$. Let $a * S = \{x\}$ where $x \in S$. Now clearly x is a zero element in S for $x * S = (a * S) * S = a * (S * S) \subseteq a * S = \{x\}$. This implies x is a zero element in S and the result that the product of any two elements is zero follows from Theorem 1.

Case ii. $|a * S| > 1$. Then $a * S$ contains at least two distinct elements x, y. Clearly $x \neq y$.

Sub case a. $x \neq y$, $x, y \neq a$. Since $x, y \in a * S$, $a * b = x$, $a * c = y$ for some b, c in S. Now, let A be a tolerance relation defined as follows. $A = \{(s, s) \mid s \in S\} \cup \cup \{(b, c), (c, b)\}$. Now, since A is compatible, $(a, a), (b, c) \in A$ implies $(a * b, a * c) = = (x, y) \in A$. Hence the possibilities are $x = b$, $y = c$ or $x = c$, $y = b$.

Sub case a – i. $x = b$, $y = c$. Now we have $a * b = b$, $a * c = c$. Let B be a tolerance relation in $\langle S, *\rangle$ defined as follows: $B = \{(s, s) \mid s \in S\} \cup \{(a, c), (c, a), (a, b), (b, a)\}$. We shall show that B is not compatible. Now $(a, c) \in B$ and $(b, a) \in B$. But $(a * b, c * a) = (b, c) \notin B$ yielding a contradiction. The next possibility is $x = c$ and $y = b$.

Sub case a – ii. $x = c$, $y = b$. Now we have $a * c = b$ and $a * b = c$. Let C be a tolerance relation in $\langle S, *\rangle$ defined as follows. $C = \{(s, s) \mid s \in S\} \cup \{(a, b), (b, a)\} \cup \{(a, c), (c, a)\}$. Now $(a, b), (c, a) \in C$ but $(a * c, b * a) = (b, c) \notin C$, since $x = c$, $y = b$ and $x \neq y$ and $x, y \neq a$. Hence, C is not compatible, a contradiction.

Sub case b. $x \neq y$ and at least one of x, y equals a. W.l.O.g. assume that $x = a$, $y \neq a$. Now $a * b = a$ and $a * c = y$. Let D be a tolerance relation defined in $\langle S, *\rangle$ as follows, $D = \{(s, s) \mid s \in S\} \cup \{(a, b), (b, a), (a, c), (c, a)\}$. By assumption D is compatible. Hence, $(a, b), (c, a) \in D$ implies $(a * c, b * a) = (y, a) \in D$. Since $y \neq a$, the possibilities are $y = b$ or $y = c$. We shall show that both these possibilities are impossible.

Sub case b – i. $y = b$. Then we have $a * y = a$ and $a * c = y$. Since $y \neq a$, clearly $y \neq c$ for $y = c$ implies $a * y = a * c$ which implies $a = y$ which is a contradiction. Let E be a tolerance relation defined in S as follows. $E = \{(s, s) \mid s \in S\} \cup \cup \{(y, c), (c, y)\}$. Since E is compatible $(a, a) \in E$, $(y, c) \in E$ implies $(a * y, a * c) = = (a, y) \in E$ which implies $y = a$ or $c = a$. Clearly $y \neq a$. Hence, other possibility is $c = a$. We shall show that this is also impossible. Let $c = a$. Now we have $a * y = = a$ and $a * a = y$. Since $|S| \geqq 3$, there exists an element d distinct from a and y. Now by assumption a, y, d are three distinct elements in S. Let F be a tolerance relation defined in $\langle S, * \rangle$ as follows. $F = \{(s, s) \mid s \in S\} \cup \{(y, d), (d, y)\}$. Now, (a, a), $(y, d) \in F$ and since F is compatible $(a * y, a * d) = (a, a * d) \in F$ which implies $a * d = a$. We shall show that $a * d = a$ is also not possible. For let G be a tolerance relation defined in $\langle S, * \rangle$ as follows. $G = \{(s, s) \mid s \in S\} \cup \{(a, d), (d, a)\}$. We shall show that G is not compatible. Now (a, a), $(a, d) \in G$. But $(a * a, a * d) = = (y, a) \notin G$ yielding a contradiction.

Sub case b – ii. $y = c$. Then we have $a * b = a$ and $a * y = y$. Let H be a tolerance relation defined as follows. $H = \{(s, s) \mid s \in S\} \cup \{(b, y), (y, b)\}$. Since H is compatible, $(a, a), (y, b) \in B$ implies $(a * y, a * b) = (y, a) \in H$. The possibilities are $y = a$ or $y = b$ or $b = a$. Since $y \neq a$, $b = y$ or $b = a$. Since $x \neq y$, $x = b$ and $y = c$ we have $y \neq b$. Hence the remaining possibility is $b = a$. In this case, $a * a = a$ and $a * y = y$. Since $|S| \geqq 3$, and $a \neq y$, there exist an element d different from a and y. Now let I be a tolerance relation defined in $\langle S, * \rangle$ as follows. $I = \{(s, s) \mid s \in S\} \cup \{(a, d), (d, a)\}$. Since I is compatible $(a, a), (a, d) \in I$ implies $(a * a, a * d) = (a, a * d) \in I$. Hence $a * d = a$ or d. We shall show that both the possibilities are impossible. Suppose $a * d = a$. Now define a tolerance relation J in $\langle S, * \rangle$ as follows. $\{J = (s, s) \mid s \in S\} \cup \{(d, y), (y, d)\}$. Now, $(a, a), (y, d) \in J$. But $(a * y, a * d) = (y, a) \notin J$ showing that J is not compatible, a contradiction. Next possibility is $a * d = d$. Let K be a tolerance relation defined in $\langle S, * \rangle$ as follows. $K = \{(s, s) \mid s \in S\} \cup \{(a, y), (y, a), (a, d), (d, a)\}$. Now $(a, y), (d, a) \in K$. But $(a * d, y * a) = (d, y) \notin K$, since a, y, d are distinct elements thus yielding a contradiction.

All these contradictions show that $a * S$ contains only one element say x. So by case (i), x is a zero element of $\langle S, * \rangle$. Now by theorem 1, $\langle S, * \rangle$ is a zero semigroup. (Q.E.D.)

Finally, I wish to express my thanks to the refereree.

References

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