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LOCALLY FINE UNIFORMITIES AND NORMAL COVERS

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Introduction. All topological spaces are assumed to be completely regular Hausdorff.

When investigating uniform spaces one finds out very quickly that fine uniform spaces (i.e. uniform spaces endowed with the finest uniformity which induces the given topology) allow to adopt many topological arguments. (Recall the well-known categorical fact that the category of all fine uniform spaces and uniformly continuous mappings and the category of all topological spaces and continuous mappings are naturally isomorphic.) Two coreflective classes of uniform spaces were introduced in [GI], [I], namely the class of subfine uniform spaces (= uniform subspaces of fine uniform spaces) and the class of locally fine uniform spaces (to be defined below). It is easy to see that each subfine uniform space is locally fine; it was exactly this rather technical internal property defining local fineness which made an application of local (topological-like) arguments possible. Nevertheless, the question of whether the class of subfine uniform spaces (the description of which is external) and the class of locally fine uniform spaces (the description of which is internal) coincide, was left open. We will give an affirmative answer. As Z. Frolík [F1] pointed out, this affirmative answer implies that the class of all fine uniform spaces is the smallest coreflective class $\mathscr{C}$ such that the subspaces of objects of $\mathscr{C}$ are exactly the locally fine uniform spaces. (Z. Frolík described this situation saying that topological spaces are defined purely "algebraically" over uniform spaces.)

Z. Frolík suggested another approach to localization in a uniform space: a uniform space $(X, \mathscr{U})$ is said to be *functionally locally fine* (FLF) if each uniformly locally uniformly continuous (ULUC) mapping from $(X, \mathscr{U})$ into a metric space is uniformly continuous (a mapping $f: (X, \mathscr{U}) \to (Y, \mathscr{V})$ is ULUC if there is a uniform cover $\mathscr{P} \in \mathscr{U}$ such that for each $P \in \mathscr{P}$, $f \dashv P$ is uniformly continuous with respect to $\mathscr{U} \dashv P$ and $\mathscr{V}$). Each locally fine uniform space is obviously FLF and it is proved in [PPV] that each FLF space is locally fine. So we are going to obtain three different descriptions of subspaces of fine uniform spaces.

Finally, let us remark that the locally fine uniform spaces play an important role not only in a transformation of topological methods into a uniform setting but also in the investigation proper of the structure of uniform spaces (see e.g. [I], [F] etc.).

Preliminaries. Let $(X, \mathscr{U})$ be a uniform space. The Ginsburg-Isbell derivative $\mathscr{U}^{(1)}$ of the uniformity $\mathscr{U}$ (see [GI], [I]) is defined as the collection of all uniformly locally uniform covers with respect to $\mathscr{U}$, i.e. covers of the form $\{U_\alpha \cap V_\beta^\alpha\}$ where $\{U_\alpha\} \in \mathscr{U}$ and for each α, $\{V_\beta^\alpha\}$ is again a member of $\mathscr{U}$. A uniform space $(X, \mathscr{U})$ is called *locally fine* if $\mathscr{U}^{(1)} = \mathscr{U}$. It is proved in [GI], [I] that locally fine spaces form a coreflective subcategory of UNIF and the corresponding coreflection is denoted by λ. The functor λ is constructed in [GI], [I] by a transfinite iterative process using the Ginsburg-Isbell derivative (this construction involved one formal complication as the Ginsburg-Isbell derivative of a uniformity need not form a uniformity, see [P]: so quasiuniformities had to be used in this construction). The following theorem [GI], [I] is crucial for our procedure:

Theorem 1. *Let M be a complete metric space. Denote its metric uniformity by $\mathscr{U}$. Then $\lambda(M, \mathscr{U})$ is a fine uniform space.*

Using the Gleason factorization theorem, J. R. Isbell [I] further proved that each locally fine uniform space containing no uncountable uniformly discrete subspace is subfine.

Let $\mathscr{T} = (T, \leqq)$ be a partially ordered set. $\mathscr{T}$ will be called a *special tree* if:

1) $\mathscr{T}$ is a tree (i.e. $\mathscr{T}$ has the least element and for each $x \in T$ the set $\{y \in T \mid y < x\}$ is well-ordered by $\leqq$),

2) each chain (= linearly ordered subset of $\mathscr{T}$) is finite. For $x \in T$, we denote the set of all immediate successors of x in $\mathscr{T}$ by $S(x)$. We put $\text{End}(\mathscr{T}) = \{x \in T \mid S(x) = \emptyset\}$.

Let $(X, \mathscr{U})$ be a uniform space. Let $\varphi\colon T \to \exp X$ (the set of subsets of X) be a mapping such that a collection $\varphi[S(x)]$ is a $\mathscr{U}$-uniform cover for each $x \in T - \text{End}(\mathscr{T})$ and $\varphi(\min \mathscr{T}) = X$. Such a mapping φ will be called a *$(\mathscr{T}, \mathscr{U})$-mapping*. Put $[\mathscr{T}, \varphi] = \{\bigcap \varphi(C) \mid C \subset T \text{ and } C \text{ is a maximal chain in } \mathscr{T}\}$.

Proposition 1. *For each special tree $\mathscr{T}$, each uniform space $(X, \mathscr{U})$ and each $(\mathscr{T}, \mathscr{U})$-mapping φ, $[\mathscr{T}, \varphi] \subset \lambda\mathscr{U}$.*

Proof is a straightforward application of Lemma VII.8 [I] (= Lemma 4.1 [GI]).

Remark 1. 1) Let us observe the obvious fact that each endpoint $x \in \text{End}(\mathscr{T})$ is contained in a single maximal chain $C(x)$ of $\mathscr{T}$ and each maximal chain C contains a single point $x_C \in \text{End}\,\mathscr{T}$. Hence the members of $[\mathscr{T}, \varphi]$ can be indexed by points of $\text{End}(\mathscr{T})$ as well.

2) One can prove (using Ginsburg-Isbell's construction of λ) that each cover belonging to $\lambda\mathscr{U}$ can be refined by the cover of the form $[\mathscr{T}, \varphi]$. Hence

$$\lambda\mathscr{U} = \{[\mathscr{T}, \varphi] \mid \mathscr{T} \text{ is a spacial tree and } \varphi \text{ is a } (\mathscr{T}, \mathscr{U})\text{-mapping}\}. \tag{1}$$

So the transfinite procedure of constructing the locally fine coreflection from [GI], [I] can be replaced by a one-step construction using "transfinite" objects.

We shall employ the following notation and terminology. Let $\mathscr{T} = (T, \leqq)$ be a special tree. Let $M \subset \mathrm{End}(\mathscr{T})$. For each $x \in M$ take a special tree $\mathscr{T}_x = (T_x, \leqq_x)$. We construct a new special tree $\mathscr{T}' = (T', \leqq')$ by setting $T' = T \vee \bigvee_{x \in M} T_x$ and defining $\leqq'$ as the transitive closure of R:

$$(a, b) \in R \quad \text{iff} \quad (a, b \in T,\ a \leqq b) \quad \text{or} \quad (a, b \in T_x,\ x \in M,\ a \leqq_x b) \quad \text{or}$$
$$(a = x,\ x \in M,\ b = \min \mathscr{T}_x)\,.$$

The special tree $\mathscr{T}'$ will be denoted by $\langle \mathscr{T} \to \{\mathscr{T}_x \mid x \in M\}\rangle$. Let $(X, \mathscr{U})$ be a uniform space. If a $(\mathscr{T}, \mathscr{U})$-mapping φ and $(\mathscr{T}, \mathscr{U})$-mappings φ_x, $x \in M$, are given then the mapping $\Phi\colon T' \to \exp X$ such that $\Phi \restriction T = \varphi$, $\Phi \restriction T_x = \varphi_x$ for each $x \in M$, will be denoted by $\langle \varphi \to \{\varphi_x \mid x \in M\}\rangle$. For a metric space (M, ϱ) the collection of all ϱ-balls of diameter ε will be denoted by $\mathscr{K}^{\varrho}(\varepsilon)$.

An open set G in a topological space X is called *regular open* if $\mathrm{Int}(\bar{G}) = G$.

Let $\{X_\iota \mid \iota \in A\}$ be a collection of topological spaces. An open set $G \subset \prod_{\iota \in A} X_\iota$ is said to be a *basic open set* if $G = \prod_{\iota \in A} Z_\iota$ where all but finitely many Z_ι are equal to X_ι. We put $P(G) = \{\iota \in A \mid Z_\iota \neq X_\iota\}$. For an open set H in $\prod_{\iota \in A} X_\iota$ put $\mathscr{B}(H) = \{G \subset \prod_{\iota \in A} X_\iota \mid G \subset H,\ G$ is a basic open set$\}$. For $K \subset A$, denote by π_K the projection from $\prod_{\iota \in A} X_\iota$ onto a partial product $\prod_{\iota \in K} X_\iota$.

Main results:

Proposition 2. *Let $\{(M_\iota, \varrho_\iota) \mid \iota \in A\}$ be a collection of complete metric spaces. For each $\iota \in A$ denote by $\mathscr{U}_\iota$ the metric uniformity induced by ϱ_ι. Put $\mathscr{U} = \prod_{\iota \in A} \mathscr{U}_\iota$. Let $\mathscr{R}$ be a locally finite cover of $\prod_{\iota \in A} (M_\iota, \varrho_\iota)$ consisting of regular open sets. Then there exist a special tree $\mathscr{T}$ and a $(\mathscr{T}, \mathscr{U})$-mapping φ such that*

$$[\mathscr{T}, \varphi] \prec \mathscr{R}\,.$$

Corollary. *If $(X, \mathscr{U}) = \prod_{\iota \in A} (M_\iota, \varrho_\iota)$ where each (M_ι, ϱ_ι) is a complete metric space then $\lambda(X, \mathscr{U})$ is a fine uniform space.*

Remark 2. a) An analogous statement was proved by J. R. Isbell [I] for separable complete metric spaces.

b) By the results proved by E. V. Shchepin ([S1], [S2]) any product of κ-metrizable spaces is κ-metrizable (in particular, κ-normal); hence it follows that the base of the fine uniformity of the topological product of metric spaces is formed by all locally finite covers consisting of regular open sets.

Theorem 2. *Each locally fine uniform space is subfine.*

Proof. Let $X = \prod_{\iota \in A} X_\iota$, each X_ι being a topological space. We need the following simple facts:

Observation 1. *Let* G_1, G_2 *be basic open sets in* X. *If there is a set* $K \subset A$ *such that* $\pi_K(G_1) \cap \pi_K(G_2) \neq \emptyset$ *and* $(P(G_1) \cap P(G_2)) - K = \emptyset$ *then* $G_1 \cap G_2 \neq \emptyset$.

Proof is trivial.

Observation 2'. *If* $\{G_n \mid n \in \omega\}$ *is a collection of basic open sets such that* $P(G_n) \cap \cap P(G_m) = \emptyset$ *whenever* $n \neq m$ *then* $\overline{\bigcup\{G_n \mid n \in \omega\}} = X$ (ω *is the set of all finite cardinals*).

Proof is again trivial.

Observation 2. *Let* H *be an open subset of* X. *Let* D *be a regular open subset of* X. *If there exists* $K \subset A$ *and* $\{G_n \mid n \in \omega\} \subset \mathscr{B}(D)$ *such that* $\pi_K(G_n) \supset \pi_K(H)$ *for each* $n \in \omega$ *and* $(P(G_n) \cap P(G_m)) - K = \emptyset$ *whenever* $n \neq m$, *then* $H \subset D$.

Proof. Using Observation 2' we obtain that $\pi_K(H) \prod_{\iota \in A} \{X_\iota \mid \iota \in A - K\} \subset \subset \overline{\bigcup\{G_n \mid n \in \omega\}}$. As D is regular open, $D \supset \operatorname{Int}(\overline{\bigcup\{G_n \mid n \in \omega\}}) \supset H$.

Proof of Proposition 2: Put $X = \prod_{\iota \in A} M_\iota$. For each point $x \in X$ there is a basic open set H such that $x \in H$ and H intersects only finitely many members of $\mathscr{R}$. Choose a cover of X by such basic open sets and denote it by $\mathscr{D}$.

Take any point $y \in X$ and $G \in \mathscr{D}$ such that $y \in G$. By Observation 1, there is a finite set $\mathscr{R}' \subset \mathscr{R}$ such that for all $R \in \mathscr{R} - \mathscr{R}'$, $P(G) \cap P(H) \neq \emptyset$ for each $H \in \mathscr{B}(R)$. If there is $R \in \mathscr{R}'$ such that the collection $\{P(H) \mid H \in \mathscr{B}(R)\}$ contains an infinite disjoint subfamily, then by Observation 2 $R = X$ and the proof is finished. So we shall suppose that for each $R \in \mathscr{R}'$ there is $I(R) \subset A$ such that $I(R)$ is finite and for each $H \in \mathscr{B}(R)$, $P(H) \cap I(R) \neq \emptyset$.

Put $\mathscr{F} = \bigcup\{\mathscr{B}(R) \mid R \in \mathscr{R}\}$.

Put $U(0) = P(G) \cup \bigcup\{I(R) \mid R \in \mathscr{R}'\}$.

Define $\mathscr{W}(0) = \pi_{U(0)}(\mathscr{D}) \wedge \pi_{U(0)}(\mathscr{F}) \wedge \prod_{\iota \in U(0)} \mathscr{K}^{\varrho_\iota}(1)$. $\mathscr{W}(0)$ is an open cover of a complete metric space $\prod_{\iota \in U(0)} (M_\iota, \varrho_\iota)$. So by Theorem 1 and the equality (1) there is a special tree $\mathscr{T}(0)$ and a $(\mathscr{T}(0), \prod_{\iota \in U(0)} \{\mathscr{U}_\iota \mid \iota \in U(0)\})$-mapping $\tilde{\varphi}_0$ such that $[\mathscr{T}(0), \tilde{\varphi}_0] \prec \mathscr{W}(0)$.

Define $\varphi_0 = \pi_{U(0)}^{-1} \circ \tilde{\varphi}_0$. Clearly, φ_0 is a $(\mathscr{T}(0)\,\mathscr{U})$-mapping.

Put $\mathscr{V}(0) = [\mathscr{T}(0), \varphi_0]$. We may suppose that $\mathscr{V}(0)$ consists of basic open sets. For $x \in \operatorname{End}(\mathscr{T}(0))$, put $V_x^0 = \bigcap \varphi_0(C(x))$ (see Remark 1.1). Denote by $E(\mathscr{T}(0))$ the set of all points $x \in \operatorname{End}(\mathscr{T}(0))$ such that V_x^0 is not contained in any member of $\mathscr{R}$. If $E(\mathscr{T}(0)) = \emptyset$, the proof is finished. Consider V_x^0 for $x \in E(\mathscr{T}(0))$. Put $\mathscr{Z}_x^0 = = \{R \in \mathscr{R} \mid$ there exists $G \in \mathscr{B}(R)$ such that $\pi_{U(0)}(G) \supset \pi_{U(0)}(V_x^0)\}$. By the definition of $\mathscr{V}(0)$, there is a basic open set $D \in \mathscr{D}$ such that $\pi_{U(0)}(D) \supset \pi_{U(0)}(V_x^0)$. Hence there is a finite $\mathscr{Z}' \subset \mathscr{Z}_x^0$ such that for all $R \in \mathscr{Z}_x^0 - \mathscr{Z}'$, $(P(D) \cap P(G)) - U(0) \neq \emptyset$ for all $G \in \mathscr{A}(R)$. Because of Observation 2, for each $R \in \mathscr{Z}_x^0$ there is a finite $I(R) \subset A$ such that $(I(R) \cap P(G)) - U(0) \neq \emptyset$ for each $G \in \mathscr{B}(R)$ with $\pi_{U(0)} \supset \pi_{U(0)}(V_x^0)$; denote such G's by $\mathscr{A}_x^0$ or $\mathscr{A}(R)$, where $\mathscr{A}(R) = \mathscr{A}(R, 0) = \{G \in \mathscr{B}(R) \mid \pi_{U(0)}(G) \supset \supset \pi_{U(0)}(V_x^0)\}$ and $\mathscr{A}_x^0 = \bigcup\{\mathscr{A}(R) \mid R \in \mathscr{Z}_x^0\}$.

Put $U(0, x) = U(0) \cup P(D) \cup \bigcup\{I(R) \mid R \in \mathscr{L}'\}$.

Observe that $U(0, x)$ has the following very important property: if $G \in \mathscr{A}_x^0$ then $(P(G) \cap U(0, x)) - U(0) \neq \emptyset$.

Put $\mathscr{W}(0, x) = \pi_{U(0,x)}(\mathscr{D}) \wedge \pi_{U(0,x)}(\mathscr{F}) \wedge \prod_{\iota \in U(0,x)} \mathscr{K}^{\varrho_\iota}(2^{-1})$. $\mathscr{W}(0, x)$ is an open cover of $\prod_{\iota \in U(0,x)} (M_\iota, \varrho_\iota)$ so that, as above in the case of $\mathscr{W}(0)$, there exist a special tree $\mathscr{T}_x$ and a $(\mathscr{T}_x, \mathscr{U})$-mapping φ_x such that $\varphi_x = \pi_{U(0,x)}^{-1}\tilde{\varphi}_x$ where $\tilde{\varphi}_x$ is a $(\mathscr{T}_{x'} \prod_{\iota \in U(0,x)} \mathscr{U}_\iota)$-mapping, and $\pi_{U(0,x)}([\mathscr{T}_x, \varphi_x]) < \mathscr{W}(0, x)$.

Put $\mathscr{T}(1) = \langle \mathscr{T}(0) \to \{\mathscr{T}_x \mid x \in E(\mathscr{T}(0))\}\rangle$, $\varphi_1 = \langle \varphi_0 \to \{\varphi_x \mid x \in E(\mathscr{T}(0))\}\rangle$.

Define $\mathscr{V}(1) = [\mathscr{T}(1), \varphi_1]$. Again, we may suppose that φ_x was chosen so that $\mathscr{V}(1)$ consists of basic open sets.

Induction step: Suppose that $\mathscr{V}(n) = [\mathscr{T}(n), \varphi_n]$ and all the other necessary requisites are defined so that

(i) $\mathscr{T}(n) = \langle \mathscr{T}(n-1) \to \{\mathscr{T}_x \mid x \in E(\mathscr{T}(n-1))\}\rangle$;

(ii) $\varphi_n = \langle \varphi_{n-1} \to \{\varphi_x \mid x \in E(\mathscr{T}(n-1))\}\rangle$ and $\varphi_x = \pi_{U(n-1,x)}\varphi_x$ where φ_x is a $(\mathscr{T}_x, \prod_{\iota \in U(n-1,x)} \mathscr{U}_\iota)$-mapping;

(iii) $\pi_{U(n-1,x)}([\mathscr{T}_x, \varphi_x]) \prec \pi_{U(n-1,x)}(\mathscr{D}) \wedge \pi_{U(n-1,x)}(\mathscr{F}) \wedge \prod_{\iota \in U(n-1,x)} \mathscr{K}^{\varrho_\iota}(2^{-n})$; moreover, each $V \in \mathscr{V}(n)$ is basic open;

(iv) if $x \in E(\mathscr{T}(n-1))$ and $G \in \mathscr{A}(R)$ for some $R \in \mathscr{L}_x^{n-1}$ i.e. $G \in \mathscr{A}_x^{n-1}$ then $(P(G) \cap \bigcup(n-1, x)) - Y \neq \emptyset$ where $Y = U(0)$ if $n = 1$ and $Y = U(n-2, r(x))$ where $r(x)$ is an element of $E(\mathscr{T}(n-2))$ such that $x \in \operatorname{End}(\mathscr{T}_{r(x)})$ if $n \geqq 2$.

A restriction to φ_x from (ii) quarantees that $P(V_y^n) \subset U(n-1, x)$ if $y \in \operatorname{End}(\mathscr{T}_x)$.

We will show how to proceed for $n + 1$.

As above, put $E(\mathscr{T}(n)) = \{y \in \operatorname{End}(\mathscr{T}(n)) \mid V_y^n = \bigcap \varphi_n(C(y))$ is not contained in any member of $\mathscr{R}\}$. Suppose $E(\mathscr{T}(n)) \neq \emptyset$ (otherwise the proof is finished). Take $y \in E(\mathscr{T}(n))$. Necessarily, there is $x' \in E(\mathscr{T}(n-1))$ such that $y \in \operatorname{End}(\mathscr{T}_{x'})$.

Put $\mathscr{L}_y^n = \{R \in \mathscr{R} \mid$ there exists $G \in \mathscr{B}(R)$ such that

$$\pi_{U(n-1,x')}(G) \supset \pi_{U(n-1,x')}(V_y^n)\} .$$

$\mathscr{L}_y^n$ is non-void because of (iii). For the same reason, there is $D \in \mathscr{D}$ such that $\pi_{U(n-1,x')}(D) \supset \pi_{U(n-1,x')}(V_y^n)$. As above, there is a finite $\mathscr{L}' \subset \mathscr{L}_y^n$ such that for all $R \in \mathscr{L}_y^n - \mathscr{L}'$, $(P(D) \cap P(G)) - U(n-1, x') \neq \emptyset$ for all $G \in \mathscr{A}(R)$. Furthermore, for each $R \in \mathscr{L}'$ there is a finite set $I(R) \subset A$ such that $(I(R) \cap P(G)) - U(n-1, x') \neq$ $\neq \emptyset$ for each $G \in \mathscr{A}(R) = \mathscr{A}(R, n-1)$.

Put

$$U(n, y) = U(n-1, x') \cup P(D) \cup \bigcup_{R \in \mathscr{L}'} I(R),$$

$$\mathscr{W}(n, y) = \pi_{U(n,y)}(\mathscr{D}) \wedge \pi_{U(n,y)}(\mathscr{F}) \wedge \prod_{\iota \in U(n,y)} \mathscr{K}^{\varrho_\iota}(2^{-n-1}).$$

Now the definition of $\mathscr{T}_y, \varphi_y, \mathscr{T}(n+1), \varphi_{n+1}$ is analogous to that of $\varphi_{x'}, \mathscr{T}_{x'}$, $\mathscr{T}^{(1)}$, φ_1 stated above. Suppose that $\mathscr{T}(n)$ and φ_n are constructed for each $n \in \omega$.

Put $\mathcal{T}(n) = (T(n), \leqq_n)$. As $T(n) \subset T(n+1)$, we may define $\mathcal{T}(\omega) = (\bigcup_{n\in\omega} T(n), \bigcup_{n\in\omega} \leqq_n)$, for the sake of brevity put $T(\omega) = \bigcup_{n\in\omega} T(n)$, $\leqq_\omega = \bigcup_{n\in\omega} \leqq_n$.

$\mathcal{T}(\omega)$ is obviously a tree. We prove that each chain in $\mathcal{T}(\omega)$ is finite:

Suppose that there is an infinite chain $C = \{c(n) \mid n \in \omega\}$ in $\mathcal{T}(\omega)$. We may and will suppose that $c(n) \in \operatorname{End}(\mathcal{T}(n))$ and $c(n+1) \in \mathcal{T}_{c(n)}$ for each $n \in \omega$. Consequently, $c(n) \in E(\mathcal{T}(n))$ for each $n \in \omega$. Put $U = \bigcup_{n\in\omega} U(n, c(n))$. Consider the sets $V^n_{c(n)} \in \mathcal{V}(n)$, $n \in \omega$. By the construction of $\mathcal{V}(n)$'s. $V^n_{c(n)} \supset V^{n+1}_{c(n+1)}$ for each $n \in \omega$ and $\pi_\iota(\{V^n_{c(n)} \mid n \in \omega\})$ is a Cauchy filter for each $\iota \in U$, $P(V^n_{c(n)}) \subset P(V^{n+1}_{c(n+1)}) \subset U$ for each $n \in \omega$. This implies that there is a point $z \in X$ such that $z \in \bigcap_{n\in\omega} \overline{V^n_{c(n)}}$. There is a basic open set Z in X such that $z \in Z \subset R_{z_0}$ for some $R_{z_0} \in \mathcal{R}$. As $\operatorname{diam} \pi_\iota(V^n_{c(n)}) < 2^{-n}$ for each $\iota \in U(n-1, c(n-1))$ and for each $n \in \omega$, we obtain that there is an integer $n_0 \geqq 2$ such that $\pi_{U(n-1,c(n-1))}(Z) \supset \pi_{U(n-1,c(n-1))}(V^n_{c(n)})$ for all $n > n_0$, i.e. $R \in \mathcal{Z}^n_{c(n)}$ for all $n > n_0$. Then (iv) stated in the above induction implies $(P(Z) \cap U(n, c(n)) - U(n-1, c(n-1)) \neq \emptyset$ for $n > n_0$. As $\{U(n, c(n)) \mid n > n_0\}$ is an increasing sequence of sets, we obtain that $P(Z)$ is infinite — a contradiction.

Hence $\mathcal{T}(\omega)$ is a special tree. Define a mapping φ_ω: $T(\omega) \to \exp X$ by $\varphi(t) = \varphi_0(t)$ if $t \in T(0)$, $\varphi(t) = \varphi_n(t)$ if $t \in T(n) - T(n-1)$ where n is a positive integer.

It is easy to see that φ_ω is a $(\mathcal{T}(\omega), \mathcal{U})$-mapping. We show that $[\mathcal{T}(\omega), \varphi(\omega)] \prec \mathcal{R}$.

Indeed, if $x \in \operatorname{End}(\mathcal{T}(\omega))$ then there is an integer n such that $x \in T(n)$, $x \in \operatorname{End}(\mathcal{T}(n))$. Then necessarily $x \notin E(\mathcal{T}(n))$, hence there is $R_0 \in \mathcal{R}$ such that $R_0 \supset V^n_x = \bigcap \varphi_\omega(\tilde{C})$ where $\tilde{C}$ is the maximal chain in $\mathcal{T}(\omega)$ containing x (see Remark 1.1). Q.E.D.

Proof of Corollary is easy as the fine uniformity on $(X, \mathcal{U})$ consists of all normal covers on X and each normal ($=$ metrizable $=$ uniformizable) open cover has a locally finite refinement consisting of regular open sets.

Proof of Theorem 2. Use the well-known facts (see e.g. [I]) that each uniform space can be uniformly embedded into a product of complete metric spaces and that the functor λ preserves subspaces (i.e. if X is a subspace of Y then λX is a subspace of λY). So if $\lambda Z = Z$ then Z is a subspace of $\lambda(\pi M_\iota)$ which is fine if all M_ι are complete metric.

Final remark: We indicated in the proof of Proposition 1 that the usage of special trees in our proofs is based on Lemma VII.8 [I]. One could reformulate proofs using just this lemma. We introduced special trees here as they may be of some importance in the investigation of uniform spaces. Let us give an illustration: Take a special tree $\mathcal{T}$.

First, define $L(0) = \min \mathcal{T}$, $L(n+1) = \bigcup\{S(x) \mid x \in L(n)\}$, $n = 1, 2, \ldots$.

Recall that $S(x)$ denotes the set of all immediate successors of x in $\mathcal{T}$.

If $x \in L(n)$ and $y \in S(x)$ then join points x and y by an interval of length 2^{-n}. We obtain a space which resembles a hedgehog with branching prickles. The complexity of its metric uniformity depends on the combinatorial complexity of the special tree $\mathcal{T}$ we started with. (Some details are contained in [PR].)

Special trees also offer the possibility to give an alternative description of various coreflections in the category of uniform spaces which are defined "locally" (an example of such a coreflective class is the class of all e-locally fine uniform spaces [F3], [R]).

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