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Radoslav M. Dimitrić

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ON CHAINS OF FREE MODULES OVER VALUATION DOMAINS

RADOSLAV DIMITRIĆ, Beograd

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As the title suggests, this paper deals with smooth ascending chains of modules over valuation domains. Using the definition of $\varkappa$-free modules given in my earlier paper, a notion of an $\mathscr{F}_2$-module is introduced: such a module is the union of a pure continuous chain of length ω_2 of free modules of rank $\aleph_1$ such that every quotient of consecutive terms of the chain is an $\aleph_1$-free module. In the first part we prove that the union of a continuous chain of length ω_1 of free modules, with a property that its every quotient of consecutive terms is an $\mathscr{F}_2$-module, is a free module.

The second part of the article contains a proof of the existence of $\mathscr{F}_2$-modules that are not free and exibits some constructions used to build up new $\mathscr{F}_2$-modules from the ones already constructed. One of the results is that the ω_2-union of an ascending chain of $\mathscr{F}_2$-modules is an $\mathscr{F}_2$-module, under certain additional assumptions.

Though the case of modules over valuation domains differs essentially from that of Abelian groups (the notion of $\varkappa$-free modules being an example to support this), the final results of this type are still the same. For the case of Abelian groups, works of Paul Hill in 1970's gave a complete insight into the subject.

CONVENTIONS

All modules here are torsion free modules over commutative valuation domains.

Our notation is the same as in [1]: $N \leqq_* M$ denotes that N is a pure submodule of M. $rk\ M$ denotes the cardinality of a maximal independent set of M. A submodule N of a free R-module $F = \bigoplus_{i\in I} Rx_i$ is called a *slice* of F if $N = \bigoplus_{i\in J} Rx_i$, $J \subset I$.

A pure ascending chain of R-modules M_α,

$$0 = M_0 \leqq_* M_1 \leqq_* \dots \leqq_* M_\alpha \leqq_* \dots \leqq_* M,\ \alpha < \mu$$

is called *smooth* or *continuous* if $M = \bigcup_{\alpha<\mu} M_\alpha$ and, for every limit ordinal α, $M_\alpha = \bigcup_{i<\alpha} M_i$.

Recall from [1] that, for an infinite cardinal $\varkappa$, an R-module M is $\varkappa$-free if its every submodule K of rank $<\varkappa$ is purely embeddable in a free pure submodule F of M. The $\aleph_0$- and $\aleph_1$-freeness coincide (see [5]).

For more on terminology and modules over valuation domains, the reader may wish to consult a recently published book on the subject [6]. This book contains a great part of the results from our reference list.

$\mathscr{F}_2$-MODULES

To begin with, we prove the following simple but useful lemma:

Lemma 1. *If A is an $\aleph_1$-free module, then its every pure submodule B is $\aleph_1$-free as well.*

Proof. Let $K \leqq_* B \leqq_* A$, rk $K < \aleph_1$. There exists a free module F, $K \leqq_* F \leqq_* \leqq_* A$ with $rk\, F = rk K$. Now, K must be free by Proposition 2 in [1] or Proposition 7 in [3].

While studying smooth chains of free modules we have obtained the following result (Corollary 6, [3]):

Proposition 2. *If $0 = F_0 \leqq_* F_1 \leqq_* \dots \leqq_* F_\alpha \leqq_* \dots M$ $(\alpha < \omega_1)$ is a smooth chain of free modules F_α of rank $\leqq \aleph_1$ and, for every α, $F_{\alpha+1}/F_\alpha$ is $\aleph_0$-free, then the union $M = \bigcup F_\alpha$ is also free.*

Our present aim is to extend the result by changing somewhat the hypotheses. The first option is to prolong the length of the given chain up to ω_2 and leave the other hypotheses unchanged. Our second choice will be to leave the length of the chain unchanged, delete the restriction on the rank of F_α and have some other hypotheses on each $F_{\alpha+1}/F_\alpha$.

The first choice leads to

Definition 3. An R-module M is called an *$\mathscr{F}_2$-module* if it is the union of a continuous chain $0 = F_0 \leqq_* F_1 \leqq_* \dots \leqq_* F_\alpha \leqq_* \dots M$, $\alpha < \omega_2$, of free modules of rank $\leqq \aleph_1$, such that every $F_{\alpha+1}/F_\alpha$ is $\aleph_1$-free.

The proof of the following proposition is in [1], Theorems 11 and 10.

Proposition 4. *Every $\mathscr{F}_2$-module is $\aleph_2$-free.*

Our main goal here is to prove the result arising from the second alternative:

Theorem 6. *Given a smooth chain of free modules F_α,*

$$(*) \qquad 0 = F_0 \leqq_* F_1 \leqq_* \dots \leqq_* F_\alpha \leqq_* \dots F\,, \quad \alpha < \omega_1,$$

such that every $F_{\alpha+1}/F_\alpha$ is an $\mathscr{F}_2$-module, then F is free.

Before proceeding to the proof of this theorem we need an auxiliary tool:

Lemma 5. *Under the hypotheses of Theorem 6, where every $F_{\alpha+1}/F_\alpha$ is a smooth union of free modules*

$$(**) \qquad 0 = F_{\alpha,0} \leqq_* \dots \leqq_* F_{\alpha,\gamma} \leqq_* \dots \leqq_* F_{\alpha+1}/F_\alpha\,,$$

$\gamma < \omega_2$, of rank at most $\aleph_1$ such that, for every $\gamma < \omega_2$, $F_{\alpha,\gamma+1}/F_{\alpha,\gamma}$ is $\aleph_1$-free,

given a submodule H of F of rank $\aleph_1$, there is a free pure submodule $\bar{H}$ of F containing H such that

(i) *for every $\alpha < \omega_1$, $\bar{H} \cap F_\alpha$ is a slice of F_α;*

(ii) *for every $\alpha < \omega_1$ there is a $\gamma = \gamma(\alpha, H) < \omega_2$ such that*

$$((F_{\alpha+1} \cap \bar{H}) + F_\alpha)/F_\alpha = F_{\alpha,\gamma} \leqq_* F_{\alpha+1}/F_\alpha;$$

(iii) *$\bar{H}$ is at most $\aleph_1$-generated.*

Proof. First note that, as a rank $\aleph_1$ submodule of F, H may be assumed to be $\aleph_1$-generated (If it is not, then consider the purification H' of H in F which is also of rank $\aleph_1$. H' is $\aleph_1$-free being a pure submodule of $\aleph_1$-free module F (Lemma 1). By Proposition 6 of [1], $pd\, H' \leqq 1$. By Theorem 2.4 in [5], H' has to be $\aleph_1$-generated. Now apply the procedure below on H' instead of on H.).

For every $\alpha < \omega_1$, $((H \cap F_{\alpha+1}) + F_\alpha)/F_\alpha$ is an at most $\aleph_1$-generated submodule of $F_{\alpha+1}/F_\alpha$, therefore, there is an F_{α,γ_1} from the chain (**) such that $\gamma_1 < \omega_2$ and $((H \cap F_{\alpha+1}) + F_\alpha)/F_\alpha \leqq F_{\alpha,\gamma_1} = (F'_{\alpha,\gamma_1} \oplus F_\alpha)/F_\alpha$ $(F_{\alpha,\gamma_1} \cong F'_{\alpha,\gamma_1} \leqq F_{\alpha+1})$. Define $H_2 = \langle H, \{F'_{\alpha,\gamma_1}\}_{\alpha<\omega_1}\rangle$ which is obviously $\aleph_1$-generated. Thus we have $F_{\alpha,\gamma_1} \leqq ((H_2 \cap F_{\alpha+1}) + F_\alpha)/F_\alpha \leqq F_{\alpha+1}/F_\alpha$; in the same manner we find F_{α,γ_2} from (**) such that $F_{\alpha,\gamma_1} \leqq ((H_2 \cap F_{\alpha+1}) + F_\alpha)/F_\alpha \leqq F_{\alpha,\gamma_2} = (F'_{\alpha,\gamma_2} \oplus F_\alpha)/F_\alpha$, and so on. We inductively get a sequence $H_n = \langle H_{n-1}, \{F'_{\alpha,\gamma_{n-1}}\}_{\alpha<\omega_1}\rangle$ of at most $\aleph_1$-generated submodules of F satisfying $F_{\alpha,\gamma_{n-1}} \leqq ((H_n \cap F_{\alpha+1}) + F_\alpha)/F_\alpha \leqq F_{\alpha,\gamma_n}$, $n < \omega_0$.

If $\bigcup_{n\in N} H_n = \bar{H}$ and $\gamma = \sup_{n<\omega_0} \gamma_n$, then by the smoothness of (**) we obtain from the last inequality $((\bar{H} \cap F_{\alpha+1}) + F_\alpha)/F_\alpha = F_{\alpha,\gamma}$. Thus, by this construction we can achieve (ii) and (iii).

Now combine this process with the first one in the proof of Lemma 2 of [3] to get the desired $\bar{H}$ satisfying (i), (ii), (iii).

Since $\bar{H} = \bigcup_{\alpha<\omega_1} \bar{H}_\alpha$ where $\bar{H}_\alpha = \bar{H} \cap F_\alpha$ are free by (i), and $\bar{H}_{\alpha+1}/\bar{H}_\alpha \cong F_{\alpha,\gamma}$, $\alpha < \omega_1$ by (ii) we derive that $\bar{H}$ is free. There is not difficult to prove that $\bar{H}$ is a pure submodule of F by taking advantage of (i).

Notice that in this proof there has never been used any hypothesis on $F_{\alpha,\gamma+1/\alpha,\gamma}$.

Let us now prove Theorem 6.

Proof (of Theorem 6). In order to prove the theorem, we construct a smooth chain of pure free submodules E_β of the module F:

$$(***) \qquad 0 = E_0 \leqq_* \dots \leqq_* E_\beta \leqq_* \dots \leqq_* E_\lambda = F\,, \quad \beta < \lambda\,,$$

where the following conditions are satisfied for every $\alpha < \omega_1$, $\beta < \lambda$:

(i) $E_{\beta+1}/E_\beta$ is at most $\aleph_1$-generated,

(ii) $E_\beta \cap F_\alpha = E_{\beta\alpha}$ is a slice of F_α,

(iii) there exists $\gamma = \gamma(\alpha, \beta) < \omega_2$ such that $((E_\beta \cap F_{\alpha+1}) + F_\alpha)/F_\alpha = F_{\alpha,\gamma}$,

(iv) $E_{\beta+1}/E_\beta$ is $\aleph_2$-free.

Note that by our hypotheses for the chain (**), each $F_{\alpha+1}/F_\alpha$ is at most $\aleph_2$-generated and, by (*), so is F. Therefore, without loss of generality, we may assume $\lambda = \omega_2$ in (***).

The construction of the chain $\{E_\beta\}_{\beta<\omega_2}$ proceeds by transfinite induction in β. Define $E_0 = 0$ and if β is a limit ordinal, define $E_\beta = \bigcup_{\nu<\beta<\omega_2} E_\nu$ with (ii) and (iii) easily checked. E_β is a free (pure) submodule of F since, by inductive hypothesis, for every $\nu < \beta$, (i) and (iv) are satisfied.

The construction of $E_{\beta+1}$ starts with adding $\leqq\aleph_1$ elements of F (which are not yet in E_β) to E_β, thus generating a submodule H, and then by applying Lemma 5 to get $\bar{H} = E_{\beta+1}$.

Now we only need to prove (iv) which, together with (i), immediately shows that F is free: $E_{\beta+1}/E_\beta = \bigcup_{\alpha<\omega_1} ((F_\alpha \cap E_{\beta+1}) + E_\beta)/E_\beta = \bigcup_{\alpha<\omega_1} F_{\beta\alpha}$, $F_{\beta\alpha} = E_{\beta+1,\alpha}/E_{\beta,\alpha}$ by (ii), i.e. $F_{\beta\alpha}$ is free. Further, $F_{\beta,\alpha+1}/F_{\beta,\alpha} \cong ((E_\beta + F_{\alpha+1}) \cap E_{\beta+1})/((E_\beta + F_\alpha) \cap E_{\beta+1}) \cong$ $\cong (F_\alpha + E_{\beta+1}) \cap (F_{\alpha+1} + E_\beta))/(F_\alpha + E_\beta) \leqq_* $ (purity follows from (iii)) $\leqq_*$ $\leqq_* (F_{\alpha+1} + E_\beta)/(F_\alpha + E_\beta) \cong$ (by (iii)) $\cong F_{\alpha+1}/F_\alpha/F_{\alpha,\gamma}$ $(\gamma = \gamma(\alpha, \beta) < \omega_2)$. The last module is $\aleph_1$-free being an ascending union of $\aleph_1$-free modules $F_{\alpha,\gamma'}/F_{\alpha,\gamma}$ $(\gamma < \gamma' < \omega_2)$ (Corollary 9 and Lemma 7 in [1]). Now, Lemma 1 implies that $F_{\beta,\alpha+1}/F_{\beta,\alpha}$ is $\aleph_1$-free as well, and therefore, by Theorem 11 [1], $E_{\beta+1}/E_\beta$ is $\aleph_2$-free.

EXISTENCE AND CONSTRUCTION OF $\mathscr{F}_2$-MODULES

Now we come to the question whether our Definition 3 has not been too strong, i.e., whether there exist $\mathscr{F}_2$-modules that are not free. Before answering this question, we need help of several lemmas.

I have already mentioned that by a Pontryagin-type theorem ([5], Corollary 2.6), $\aleph_0$- and $\aleph_1$-freeness coincide. On the other hand, we have the following result:

Lemma 7. *There are $\aleph_1$-generated $\aleph_1$-free modules that are not free.*

Proof. If either $gl.d.\ R > 2$ or $pd\ Q > 1$, then $\aleph_1$-free modules that are not free are found among the pure submodules of free modules of rank $\aleph_1$ – a fact derived from Theorem 9, Lemma 10 [3] and/or Corollary 5, Theorem 1, [1]. If both $gl.d.\ R \leqq 2$ and pd $Q = 1$, then the example in [3] gives a non-free $\aleph_1$-free module generated by less than $\aleph_1$ elements.

Lemma 8. *If $M \oplus H$ is a $\varkappa$-free module, with rank of H less than $\varkappa$, then H is free.*

Proof. Since $rk\,H < \varkappa$, there is a free module F' such that $H \leqq_* F' \leqq_* M \oplus H$ and $rk\,F' = rk\,H$. Now $F' = H \oplus (F' \cap M)$, so H has to be free.

The following lemma has several forms from several sources, the latest being [4], Lemma 1.2.

Lemma 9. *For a regular uncountable cardinal $\varkappa$, assume that $M = \bigcup_{\nu<\varkappa} A_\nu = \bigcup_{\nu<\varkappa} B_\nu$, where all A_ν, B_ν are less than $\varkappa$-generated R-modules and the unions are continuous chains. Then the set $\mathscr{C} = \{\nu < \varkappa \mid A_\nu = B_\nu\}$ is closed and unbounded in $\varkappa$.*

Proof. $\mathscr{C}$ is closed since the chains involved are smooth. Let $\mu < \varkappa$ and define $\nu_0 = \mu$ and, inductively,

$$\nu_{n+1} \geqq \nu_n \text{ such that } \begin{cases} A_{\nu_n} \leqq B_{\nu_{n+1}}, & n \text{ even} \\ A_{\nu_{n+1}} \geqq B_{\nu_n}, & n \text{ odd}. \end{cases}$$

(The choices indicated are possible for cardinality reasons.) It is now clear that $\nu \in \mathscr{C}$, where $\nu = \sup \nu_n$, $n < \omega_0$ i.e. $\mathscr{C}$ is also unbounded.

Theorem 10. *There are $\mathscr{F}_2$-modules that are not free.*

Proof. We will construct a smooth ascending chain $0 = F_0 \leqq_* F_1 \leqq_* \dots \dots \leqq_* F_\alpha \leqq_* \dots \leqq_* M_{\omega_2}$, $\alpha < \omega_2$ of pure free modules of rank $\aleph_1$, such that, for every $1 < \alpha < \omega_2$, $F_{\alpha+1}/F_\alpha$ is $\aleph_1$-free but not free. We employ transfinite induction in $\alpha < \omega_2$. If α is a limit ordinal, define $F_\alpha = \bigcup_{i<\alpha} F_i$. By Theorem 11 in [1], F_α is free of rank $\aleph_1$ if all F_i $(i < \alpha)$ are such. If α is a non-limit ordinal, then take an $\aleph_1$-generated module M_α that is $\aleph_1$-free but not free. Its free resolution is $0 \to F'_{\alpha-1} \to \to F_\alpha \to M_\alpha \to 0$; here $F'_{\alpha-1}$ is free since, by Proposition 6 in [1], $pd\, M_\alpha = 1$. $F'_{\alpha-1}$ must be of rank $\aleph_1$; if it were of rank at most $\aleph_0$, than it would be contained in an $\aleph_0$-generated slice F'_α of $F_\alpha = F'_\alpha \oplus F''_\alpha$ and we would have $M_\alpha = F_\alpha/F'_{\alpha-1} \cong \cong F'_\alpha/F'_{\alpha-1} \oplus F''_\alpha$. Here $F'_\alpha/F'_{\alpha-1}$ is of at most countable rank and Lemma 8 forces it to be free, making M_α also free, which is impossible. Therefore we may identify $F'_{\alpha-1}$ with the already constructed $F_{\alpha-1}$.

From the way of forming $M = \bigcup_{\alpha<\omega_2} F_\alpha$, we see that M is an $\mathscr{F}_2$-module. M is not free: if, on the contrary, $M = \bigoplus_{i<\omega_2} Rx_i$, then by Lemma 9 there is a cub $\mathscr{C}$ in ω_2 such that, for every $\alpha \in \mathscr{C}$, $F_\alpha = \bigoplus_{i<\alpha<\omega_2} Rx_i$. Now, for $\alpha, \beta \in C$ and any $\gamma < \omega_2$, $\alpha < \gamma < \beta$ we have $F_\beta = F_\alpha \oplus \bigoplus_{\alpha \leqq i < \beta} Rx_i$, hence $F_\gamma = F_\alpha \oplus \big(F_\gamma \cap \bigoplus_{\alpha \leqq i < \beta} Rx_i\big)$. In particular, for $\gamma = \alpha + 1$ we get that $F_{\alpha+1}/F_\alpha$ is free, which we have already prevented by the choice of M_α.

The rest of my results here consists of building up new $\mathscr{F}_2$-modules from given $\mathscr{F}_2$-modules.

Lemma 11. *Any direct sum of $\aleph_1$-free modules is again $\aleph_1$-free.*

Proof. By Lemma 7 [1], the direct sum of two $\aleph_1$-free modules is $\aleph_1$-free. The proof now becomes a trivial application of transfinite induction and Corollary 9[1].

Lemma 12. *The direct sum of $\aleph_1$ $\mathscr{F}_2$-modules is again an $\mathscr{F}_2$-module.*

Proof. Assume that every $\mathscr{F}_2$-module A_i $(i \in I, |I| \leqq \aleph_1)$ is the union of a continuous chain

(i) $0 = F_{i0} \leqq_* F_{i1} \leqq_* \dots \leqq_* F_{i\alpha} \leqq_* \dots \leqq_* A_i$ $(\alpha \in \omega_2, i \in I)$ of free modules $F_{i\alpha}$ of rank at most $\aleph_1$, such that, for every $\alpha < \omega_2$, $F_{i,\alpha+1}/F_{i\alpha}$ is $\aleph_1$-free. Consider the following chain:

$$0 = \bigoplus_{i\in I} F_{i0} \leqq_* \bigoplus_{i\in I} F_{i1} \leqq_* \dots \leqq_* \bigoplus_{i\in I} F_{i\alpha} \to_* \dots \leqq_* \bigoplus_{i\in I} A_i, \quad \alpha < \omega_2.$$

It is clearly a smooth chain of free modules of rank $\aleph_1$, since all the chains (i) are, and its union is $\bigoplus_{i\in I} A_i$. Since $\bigoplus_{i\in I} F_{i,\alpha+1}/\bigoplus_{i\in I} F_{i,\alpha} \cong \bigoplus_{i\in I} (F_{i,\alpha+1}/F_{i,\alpha})$, we complete the proof that $\bigoplus_{i\in I} A_i$ is an $\mathscr{F}_2$-module by direct application of Lemma 11.

Proposition 13. *If* $0 \to B \to A \to C \to 0$ *is an exact sequence and* B, C *are* $\mathscr{F}_2$-*modules, then* A *is also an* $\mathscr{F}_2$-*module.*

Proof. According to the definition of the $\mathscr{F}_2$-modules we have the following smooth chains of free modules:

(1) $$0 = F_0 \leqq_* \dots \leqq_* F_\alpha \leqq_* \dots \leqq_* B, \quad \alpha < \omega_2,$$

for every $\alpha < \omega_2$, $rk\, F_\alpha \leqq \aleph_1$ and $F_{\alpha+1}/F_\alpha$ is $\aleph_1$-free;

(2) $$0 = F'_0 \leqq_* \dots \leqq_* F'_\alpha \leqq_* \dots \leqq_* C, \quad \alpha < \omega_2,$$

for every $\alpha < \omega_2$, $rk\, F'_\alpha \leqq \aleph_1$ and $F'_{\alpha+1}/F'_\alpha$ is $\aleph_1$-free.

Since B and C are at most $\aleph_2$-generated, so is A and therefore A has a rank filtration

(3) $$0 = A_0 \leqq_* \dots \leqq_* A_\alpha \leqq_* \dots \leqq_* A, \quad \alpha < \omega_2 \quad \text{(smooth)}$$

where every *gen* $A_\alpha \leqq \aleph_1$. For every α, $B_\alpha = B \cap A_\alpha$ is at most $\aleph_1$-generated, so, by Lemma 9, $B \cap A_\alpha = F_\alpha$ on a cub $\mathscr{C}_1 \ni \alpha$; $\bigcup_{\alpha\in\mathscr{C}_1} (B + A_\alpha)/B = (B + \bigcup_{\alpha\in\mathscr{C}_1} A_\alpha)/B = A/B = C$ is an $\aleph_2$-filtration, thus, by the same lemma, there is a cub $\mathscr{C} \subset \mathscr{C}_1$ with $F'_{\nu(\alpha)} = (B + A_\alpha)/B$ ($\mathscr{C}$ is identified here with its normal function $\nu\colon \omega_2 \to \omega_2$). To summarize, the filtration (3) satisfies:

(i) every A_α is at most $\aleph_1$-generated;

(ii) for every α, $B \cap A_\alpha = F_{\nu(\alpha)}$;

(iii) for every α, $(B + A_\alpha)/B = F'_{\nu(\alpha)}$.

By (ii) and (iii) the exact sequence $0 \to B \cap A_\alpha \to A_\alpha \twoheadrightarrow (B + A_\alpha)/B$ is an extension of free by free, therefore every $A_\alpha \in \mathscr{C}$ is free. Because of $A_{\alpha+1}/((A_{\alpha+1} \cap B) + A_\alpha) \cong (B + A_{\alpha+1})/B/(B + A_\alpha)/B$, $((A_{\alpha+1} \cap B) + A_\alpha)/A_\alpha \cong (A_{\alpha+1} \cap B)/(A_\alpha \cap B)$, the exact sequence $0 \to ((A_{\alpha+1} \cap B) + A_\alpha)/A_\alpha \to A_{\alpha+1}/A_\alpha \to A_{\alpha+1}/((A_{\alpha+1} \cap B) + A_\alpha) \to 0$ is the same as $0 \to F_{\nu(\alpha+1)}/F_{\nu(\alpha)} \to A_{\alpha+1}/A_\alpha \to F'_{\nu(\alpha+1)}/F'_{\nu(\alpha)} \to 0$. Now, the conditions on (1) and (2) and Corollary 9 and Lemma 7 in [1] imply that $A_{\alpha+1}/A_\alpha$ is $\aleph_1$-free, which completes the proof.

In the end we give two results with the proofs only outlined:

Proposition 14. *If a smooth chain of* $\mathscr{F}_2$-*modules* $0 = M_0 \leqq_* M_1 \leqq_* \dots \dots \leqq_* M_\alpha \leqq_* \dots \leqq_* M$, $\alpha < \omega_2$ is such that for every $\alpha < \omega_2$, $M_{\alpha+1}/M_\alpha$ is $\aleph_1$-free, then $M = \bigcup M_\alpha$ $(\alpha < \omega_2)$ is also an $\mathscr{F}_2$-module.

Proof (outlined). The proof is similar to that of Theorem 5 [3]: using the technique employed in Lemma 2 [3] (with the tight systems of M_α replaced by the chains of free modules $\{F_i\}_{i<\omega_2}$ involved in Definition 3 of each $\mathscr{F}_2$-module M_α. For tight systems see also [6]) we construct a smooth chain of free modules $0 = A_0 \leqq_* \dots \dots \leqq_* A_\alpha \leqq_* \dots \leqq_* A_{\omega_2} = M$ satisfying:

1) for every $\alpha < \omega_2$, rk $A_\alpha \leqq \aleph_1$;
2) for every isolated $\alpha < \omega_2$, $A_\alpha = F_{\alpha i}$ for some $i < \omega_2$;
3) for every $\beta < \alpha < \omega_2$, $A_\alpha \cap M_\beta = F_{\beta i}$ for some $i < \omega_2$;
4) for every $\beta < \alpha < \omega_2$, $A_\alpha + M_\beta \leqq_* M_\alpha$.

The construction is possible, since $\mathrm{cf}\,\alpha \leqq \aleph_1$ for every $\alpha < \omega_2$. Now we use the exact sequence

$$0 \to (A_{\alpha+1} \cap M_\alpha)/A_\alpha \to A_{\alpha+1}/A_\alpha \to A_{\alpha+1}/(A_{\alpha+1} \cap M_\alpha) \to 0$$

to prove that $A_{\alpha+1}/A_\alpha$ is $\aleph_1$-free for every α. Theorem 10 in [1] and Lemma 1 enable us to complete the proof that M is an $\mathscr{F}_2$-module.

Corollary 15. *The direct sum of $\aleph_2$ $\mathscr{F}_2$-modules is again an $\mathscr{F}_2$-module.*

Proof. The direct sum $A = \oplus A_i$, $i < \omega_2$ of $\mathscr{F}_2$-modules A_i can be represented as the union of a continuous chain of modules $M_\alpha = \bigoplus_{i<\alpha} A_i$, $\alpha < \omega_2$, where every M_α is an $\mathscr{F}_2$-module by Lemma 12 and every $M_{\alpha+1}/M_\alpha$ is $\aleph_2$-free (Proposition 4). Now apply Proposition 14.

More general results on chains of free modules over valuation domains will appear in [2].

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Author's address: 29. Novombra, 108, 11000 Belgrade, Yugoslavia.