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ON COMPLETION OF CYCLICALLY ORDERED SETS

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In [6], a completion of linearly cyclically ordered sets (cycles) is constructed by means of cuts. In this note we give another construction of a completion, which can be applied to a larger class of monodimensional cyclically ordered sets.

1. INTRODUCTORY CONCEPTS AND ASSERTIONS

1.1. Ordered sets. Basic notions on ordered sets are assumed to be known (see e.g. [1] or [2]). Ordinal sum of ordered sets $G, H$ is denoted by $G \oplus H$. If $G = (G, <)$ is an ordered set and $H \subseteq G$, then the induced order $< \cap H^2$ on $H$ is denoted briefly by $<$. A linearly ordered set is called a chain. If $G = (G, <)$ is an ordered set and $H \subseteq G$ is a subset of $G$ such that $(H, <)$ is a chain, then $H$ is called a chain in $G$. A chain $H$ in an ordered set $G$ is maximal iff it is contained in no chain in $G$ as a proper subset. As it is well known, the "Hausdorff maximal principle" Every chain in every ordered set $G$ is contained in a maximal chain in $G$ is equivalent to the Axiom of Choice.

An ordered set $G$ is called complete (or a complete lattice) iff any nonvoid subset of $G$ has the supremum and infimum in $G$. $G$ is said to be conditionally complete iff any nonvoid bounded subset of $G$ has the supremum and infimum in $G$. $G$ is referred to as chain complete iff any maximal chain in $G$ is complete.

A subset $I$ of an ordered set $G$ is called an ideal iff it has the property $x \in I, y \in G, y < x \Rightarrow y \in I$. If $A$ is a nonvoid subset of an ordered set $G$, then $I(A)$ denotes the ideal in $G$ generated by $A$, i.e., $I(A) = \{ y \in G; \text{there exists } x \in A \text{ such that } y \leq x \}$. Note that $\sup A = \sup I(A)$ for any nonvoid subset $A$ of an ordered set $G$, whenever one of the elements $\sup A, \sup I(A)$ exists. If $G$ is a chain, then ideals in $G$ are called initial intervals; the dual notion is a final interval in $G$.

1.2. Cyclically ordered sets. A cyclically ordered set ([5]) is a pair $(G, C)$ where $G$ is a set and $C$ is a cyclic order on $G$, i.e., $C$ is a ternary relation on $G$ which is asymmetric, i.e., $(x, y, z) \in C \Rightarrow (z, y, x) \notin C$, cyclic, i.e., $(x, y, z) \in C \Rightarrow (y, z, x) \in C$, and transitive, i.e., $(x, y, z) \in C, (x, z, u) \in C \Rightarrow (x, y, u) \in C$. 407
If, moreover, card $G \geq 3$ and $C$ is
linear, i.e., $x, y, z \in G$, $x \neq y \neq z \neq x \Rightarrow$ either $(x, y, z) \in C$ or $(z, y, x) \in C$,
then $(G, C)$ is called a linearly cyclically ordered set or a cycle.

If $(G, C)$ is a cyclically ordered set and $H \subseteq G$ is a subset of $G$ such that $(H, C \cap H^3)$
is a cycle, then $H$ is called a cycle in $G$. A cycle $H$ in $G$ is maximal iff it is contained
in no cycle in $G$ as a proper subset. In [5] (Theorem 2.5) we proved using Axiom
of Choice that every cycle in every cyclically ordered set is contained in a maximal
cycle; below we show that this proposition is equivalent to the Axiom of Choice.

Let $(G, <)$ be an ordered set. Let us define a ternary relation $C_<$ on $G$ by
$(x, y, z) \in C_<$ iff either $x < y < z$ or $y < z < x$ or $z < x < y$. Then $(G, C_<)$ is
a cyclically ordered set ([5], Theorem 3.5).

Let $(G, C)$ be a cyclically ordered set and $x \in G$. Let us define a binary relation $<_{C,x}$
on $G$ by
$y <_{C,x} z$ iff either $(x, y, z) \in C$ or $x = y \neq z$. Then $(G, <_{C,x})$ is an ordered set
with the least element $x$ ([5], Theorem 3.1).

The proofs of the following two lemmas are trivial.

1.3. Lemma. Let $(G, <)$ be an ordered set and card $G \geq 3$. Then $(G, <)$ is a chain
iff $(G, C_<)$ is a cycle.

1.4. Lemma. Let $(G, <)$ be an ordered set and $H$ a chain in $G$ with card $H \geq 3$.
Then $H$ is a maximal chain in $(G, <)$ iff $H$ is a maximal cycle in $(G, C_<)$.

1.5. Theorem. The proposition
“Every cycle in every cyclically ordered set $G$ is contained in a maximal cycle
in $G$”
is equivalent to the Axiom of Choice.

Proof. One implication is shown in [5]; we shall prove the other one. Thus, let
every cycle in every cyclically ordered set be contained in a maximal cycle and let
$(G, <)$ be an ordered set. Assume that $H$ is a chain in $G$ with card $H \geq 3$. By Lemma
1.3 $(H, C_< \cap H^3)$ is a cycle in $(G, C_<)$ and hence there exists a maximal cycle
$(K, C_< \cap K^3)$ in $(G, C_<)$ such that $K \supseteq H$. By Lemma 1.4 $(K, <)$ is a maximal
chain in $(G, <)$. If $H$ is a chain in $(G, <)$ with card $H \leq 2$ and if there exists no
3-element chain in $G$ containing $H$, then the existence of a maximal chain in $G$
containing $H$ is trivial. Thus the Hausdorff maximal principle and also the Axiom
of Choice is true.

We shall need the following assertion; its proof can be found in [6] (Theorem 3.6
and Corollary 3.9).

1.6. Theorem. Let $G$ be a set, card $G \geq 3$ and let $<_1, <_2$ be linear orders on $G$.
Then the following statements are equivalent:
(A) $C_{<_1} = C_{<_2}$
(B) There exist disjoint subsets $A, B$ of $G$ such that $A \cup B = G$, $<_1 \cap A^2 =
= <_2 \cap A^2$, $<_1 \cap B^2 = <_2 \cap B^2$ and $(G, <_1) = A \oplus B$, $(G, <_2) = B \oplus A$.
Let $G$ be a set, let $<_1$, $<_2$ be orders on $G$. Put $<_1 \sim <_2$ iff $C_{<_1} = C_{<_2}$. Trivially, it holds

1.7. Lemma. Let $G$ be a set. The binary relation $\sim$ is an equivalence relation on the set of all orders on $G$.

1.8. Theorem. Let $G$ be a set, let $<_1$, $<_2$ be orders on $G$. Then the following statements are equivalent:

(A) $C_{<_1} \subseteq C_{<_2}$

(B) If $H$ is a maximal chain in $(G, <_1)$ with card $H \geq 3$, then $H$ is a chain in $(G, <_2)$ and there exist disjoint subsets $A, B$ of $H$ with $A \cup B = H$ such that $<_1 \cap A^2 = <_2 \cap A^2$, $<_1 \cap B^2 = <_2 \cap B^2$ and $(H, <_1) = A \oplus B$, $(H, <_2) = B \oplus A$.

Proof. 1. Let (A) hold and let $(H, <_1)$ be a maximal chain in $(G, <_1)$ with card $H \geq 3$. By Lemma 1.3 $(H, C_{<_1} \cap H^3)$ is a cycle and as $C_{<_1} \subseteq C_{<_2}$, $(H, C_{<_2} \cap H^3)$ is also a cycle, thus $C_{<_1} \cap H^3 = C_{<_2} \cap H^3$. From this it follows, by Lemma 1.3, that $(H, <_2)$ is also a chain. Applying Theorem 1.6 on the set $H$ and linear orders $<_1 \cap H^2$, $<_2 \cap H^2$, we obtain the validity of (B).

2. Let (B) hold and suppose $(x, y, z) \in C_{<_1}$. Then either $x <_1 y <_1 z$ or $y <_1 z <_1 x$ or $z <_1 x <_1 y$ and there exists a maximal chain $H$ in $(G, <_1)$ containing $\{x, y, z\}$. Thus card $H \geq 3$ and, by (B), $(H, <_2)$ is a chain and there exist subsets $A, B$ of $H$ with the desired properties. Assume that $x <_1 y <_1 z$ holds (in other two cases the proof is analogical). We have four possibilities:

1. $x, y, z \in A$. Then $x <_2 y <_2 z$, for $(A, <_1) = (A, <_2)$, and thus $(x, y, z) \in C_{<_2}$;
2. $x, y \in A$, $z \in B$. Then $x <_2 y$ and $z <_2 x$, for $(H, <_2) = B \oplus A$, thus $z <_2 x <_2 z <_2 y$ and $(x, y, z) \in C_{<_2}$;
3. $x, y \in B$. Then $y <_2 z$ and $z <_2 x$, hence $y <_2 z <_2 x$ and $(x, y, z) \in C_{<_2}$;
4. $x, y \in B$. Then $x <_2 y <_2 z$ and $(x, y, z) \in C_{<_2}$.

We have shown that $(x, y, z) \in C_{<_1} \Rightarrow (x, y, z) \in C_{<_2}$, i.e., $C_{<_1} \subseteq C_{<_2}$ and (A) holds.

1.9. Corollary. Let $G$ be a set and let $<_1$, $<_2$ be orders on $G$. Then $<_1 \sim <_2$ holds iff the sets of maximal, at least three element chains in $(G, <_1)$ and in $(G, <_2)$ are the same and for any such maximal chain $H$ the condition (B) of Theorem 1.8 holds.

2. Completeness

2.1. Cuts on cycles. Let $(G, C)$ be a cycle. A cut on $G ([6])$ is a linear order $<$ on $G$ such that $C = C_<$. Any cycle $(G, C)$ contains cuts, for $<_{c,x}$ is a cut on $G$ for any $x \in G ([6], Theorem 2.5). A cut $<$ on a cycle $(G, C)$ is a jump iff $(G, <)$ has both the least and the greatest element; it is a gap, iff $(G, <)$ has neither the least nor the greatest element; $<$ is Dedekind iff $(G, <)$ has just one of the boundary elements.
A cycle \((G, C)\) is dense iff it contains no jumps; it is complete iff it contains no gaps. A cycle \((G, C)\) is continuous iff it is dense and complete, i.e. iff each cut on \(G\) is Dedekind.

2.2. Definition. A cyclically ordered set \((G, C)\) is called cycle complete iff each maximal cycle in \(G\) is complete.

2.3. Theorem. Let \(G\) be a set, let \(<\) be an order on \(G\). If the ordered set \((G, <)\) is chain complete, then the cyclically ordered set \((G, C_<)\) is cycle complete.

Proof. Let \((H, C_< \cap H^3)\) be a maximal cycle in \((G, C_<)\). By Lemmas 1.3 and 1.4, \((H, <)\) is a maximal chain in \((G, <)\). Hence \((H, <)\) is complete. Assume that the cycle \((H, C_< \cap H^3)\) is not complete. Then there exists a cut \(<\) on \((H, C_< \cap H^3)\) which is a gap, i.e. \(<\) is a linear order on \(H\) without the least and the greatest elements and such that \(C_< = C_< \cap H^3\). By Theorem 1.6, there exist disjoint subsets \(A, B\) of \(H\) such that \(A \cup B = H\), \(< \cap A^2 = < \cap A^2 = < \cap B^2 = < \cap B^2\) and \((H, <) = A \oplus B\). \((H, <) = B \oplus A\). Thus \((B, <) = (B, <)\) contains no least element, \((A, <)\) contains no greatest element. If \(A \not= \emptyset\), then \(A\) has no supremum in \((H, <) = A = B\). If \(A = \emptyset\), then \(B = H\) has no infimum in \((H, <)\). In either case this contradicts the assumption that \((H, <)\) is complete and thus the cycle \((H, C_< \cap H^3)\) is complete.

Theorem 2.3 cannot be reversed, i.e., if a cyclically ordered set \((G, C_<)\) is cycle complete, then the ordered set \((G, <)\) need not be chain complete as the following example shows.

2.4. Example. Let \((G, <) = [0, 1)\) with the natural ordering of reals. \((G, <)\) is not chain complete; we show that \((G, C_<)\) is a complete cycle.

Let \(<\) be any cut on \((G, C_<)\). Then either \(< = < \) or \((G, <) = B \oplus A\) where \(A\) is an initial interval, \(B\) a final interval in \([0, 1)\), and \(A \not= \emptyset\), \(B \not= \emptyset\). In the second case either \(A\) has the greatest element or \(B\) has the least element in \([0, 1)\). Thus \((G, <)\) has one of the boundary elements and \(<\) is not a gap.

For cycles we have, however, this assertion:

2.5. Theorem. Let \((G, <)\) be a chain with \(\text{card} \ G \geq 3\). The cycle \((G, C_<)\) is complete iff the chain \((G, <)\) is conditionally complete and has either the least or the greatest element.

Proof. 1. Let the cycle \((G, C_<)\) be complete. The chain \((G, <)\) must contain either the least or the greatest element; otherwise the cut \(<\) on \((G, C_<)\) would be a gap. Assume that \((G, <)\) is not conditionally complete. Then there exists a subset \(A\) of \(G\), \(A \not= \emptyset\) which is upper bounded and such that sup \(A\) does not exist in \((G, <)\). As sup \(A = \sup I(A)\), we may assume that \(A\) is an initial interval in \((G, <)\). Then \(B = G - A\) is a final interval in \((G, <)\), \(B \not= \emptyset\), we have \((G, <) = A \oplus B\) and \(A\) contains no greatest element, \(B\) contains no least element. Let \(<\) be a linear order on \(G\) such that \((G, <) = B \oplus A\). By Theorem 1.6, \(<\) is a cut on \((G, C_<)\) which is a gap. This contradicts the assumption.
2. Let \((G, \prec)\) be a conditionally complete chain which has either the least or the greatest element. Assume that the cycle \((G, C)\) is not complete. Then there exists a cut \(<\) on \((G, C)\) which is a gap. It must necessarily be \(< \neq \prec\) and thus there exist nonvoid disjoint subsets \(A, B\) of \(G\) such that \(\prec \cap A^2 = \prec \cap A^2, \prec \cap B^2 = \prec \cap B^2\) and \((G, <) = A \oplus B, (G, \prec) = B \oplus A\). This implies that \(B\) has no least element, \(A\) has no greatest element. Therefore sup \(A\) does not exist in \((G, \prec)\), which contradicts the assumption that \((G, <)\) is conditionally complete.

Let us call a cyclically ordered set \((G, C)\) monodimensional iff there exists an order \(<\) on the set \(G\) such that \(C = C_\prec\). This concept agrees with [7], for by suitable definition of the dimension in the class of cyclically ordered sets, the cyclically ordered sets of form \((G, C)\) are just the sets with dimension 1.

### 2.6. Theorem
Let \((G, C)\) be a monodimensional cyclically ordered set. Then there exists a cycle complete cyclically ordered set \((H, D)\) and an isomorphic embedding of \((G, C)\) into \((H, D)\).

**Proof.** By assumption there exists an order \(<\) on \(G\) such that \(C = C_\prec\). To the ordered set \((G, <)\) we can construct a chain complete ordered set \((H, <)\) such that there exists an isomorphic embedding \(i: G \rightarrow H\) of \((G, <)\) into \((H, <)\); for instance the Dedekind-Mac Neille completion of \((G, <)\) ([2] or [4]) has this property. By Theorem 2.3, the cyclically ordered set \((H, C)\) is cycle complete and, clearly, \(i: G \rightarrow H\) is an isomorphic embedding of \((G, C)\) into \((H, C)\).

### 3. APPLICATION TO CYCLES

#### 3.1. Completion of cycles by cuts
Let \((G, C)\) be a cycle. Let \(\mathcal{F}\) be the set of all cuts on \((G, C)\), \(\mathcal{F} = \{\langle x, x \rangle; x \in G\} \cup \{\langle x, y \rangle; x \in G, y \in G\} \cup \{\langle x, < \rangle; x \in G\}\); the elements of \(\mathcal{F}\) are called regular cuts. Let us define a ternary relation \(\mathcal{T}\) on \(\mathcal{F}\) (and also on \(\mathcal{F}_\prec\)) by \((<_1, <_2, <_3) \in \mathcal{T}\) iff there exist nonvoid pairwise disjoint subsets \(A, B, D\) of \(G\) such that \(<_1 \cap A^2 = <_2 \cap B^2 = <_3 \cap D^2,<_1 \cap A^2 = <_2 \cap B^2 = <_3 \cap D^2\)

Let \((G, <) = A \oplus B \oplus D, (G, <) = B \oplus D \oplus A, (G, <) = D \oplus A \oplus B\). In [6] (Theorem 4.2, Corollary 4.5, Theorem 5.2 and Theorem 5.6) there is proved that both \((\mathcal{F}, \prec)\) and \((\mathcal{F}_\prec, \prec)\) are complete cycles and that \(x \rightarrow <_{\mathcal{F}}\) is an isomorphic embedding of \((G, C)\) into \((\mathcal{F}, \mathcal{T})\) and into \((\mathcal{F}_\prec, \prec)\). If, moreover, \((G, C)\) is dense, then \((\mathcal{F}_\prec, \prec)\) is continuous ([6], Theorem 5.9).

The results of preceding paragraph give another possibility of a construction of a completion of a cycle. Directly from Theorem 2.6 and its proof we get

#### 3.2. Theorem
Let \((G, C)\) be a cycle, let \(<\) be a cut on \((G, C)\). Let \((H, <)\) be a complete chain such that there exists an isomorphic embedding of \((G, <)\) into \((H, <)\). Then \((H, C_\prec)\) is a complete cycle and there exists an isomorphic embedding of \((G, C)\) into \((H, C_\prec)\).

Especially, we have
3.3. Corollary. Every cycle can be embedded into a complete cycle.

If we choose in Theorem 3.2 \((H, \prec)\) as the Dedekind-Mac Neille completion of \((G, \prec)\), we get further

3.4. Corollary. Let \((G, C)\) be a cycle, \(\prec\) a cut on \((G, C)\). Let \((H, \prec)\) be the Dedekind-Mac Neille completion of \((G, \prec)\). Then \((H, C_\prec)\) is a complete cycle and there exists an isomorphic embedding of \((G, C)\) into \((H, C_\prec)\).

Further, by Theorem 2.5, the stronger version of Theorem 3.2 holds:

3.5. Theorem. Let \((G, C)\) be a cycle, \(\prec\) a cut on \((G, C)\). Let \((H, \prec)\) be a conditionally complete chain containing either the least or the greatest element such that there exists an isomorphic embedding of \((G, \prec)\) into \((H, \prec)\). Then \((H, C_\prec)\) is a complete cycle and there exists an isomorphic embedding of \((G, C)\) into \((H, C_\prec)\).

There is a close connection between the both constructions (given in 3.1 and in 3.2 or 3.5); we shall describe it.

3.6. Ideal hull of a chain. Let \((G, \prec)\) be a chain, let \(\mathcal{I}(G)\) be the set of all ideals (initial intervals) in \(G\), which is ordered by set inclusion. The ordered set \((\mathcal{I}(G), \subset)\) is a complete chain which we call the ideal hull of the chain \((G, \prec)\). Further, let \(\mathcal{I}_0(G)\) be the set of all nonvoid initial intervals in \(G\), i.e. \(\mathcal{I}_0(G) = \mathcal{I}(G) - \{\emptyset\}\). \((\mathcal{I}_0(G), \subset)\) is a conditionally complete chain with the greatest element which we shall call a reduced ideal hull of \((G, \prec)\).

By Theorem 2.5, if \((G, \prec)\) is a chain with \(\text{card } G \geq 3\) and \((\mathcal{I}_0(G), \subset)\) its reduced ideal hull, then \((\mathcal{I}_0(G), C_\subset)\) is a complete cycle.

In the sequel, we assume that there is given a fixed cycle \((G, C)\) and a fixed cut \(\prec\) on \((G, C)\).

3.7. Notation. Let \(\prec\) be a cut on \((G, C)\), other than \(\prec\). By Theorem 1.6, there exist (uniquely determined) disjoint nonvoid subsets \(A, B\) of \(G\) such that \(A \cup B = G\), \(\prec \cap A^2 = \prec \cap A^2\), \(\prec \cap B^2 = \prec \cap B^2\) and \((G, \prec) = A \oplus B\), \((G, \prec) = B \oplus A\). Thus, \(A\) is a nonvoid initial interval in \((G, \prec)\), i.e. \(A \in \mathcal{I}_0(G, \prec)\). Let us put \(i(\prec) = = A\); further, put \(i(\prec) = G\). We have therefore defined a mapping \(i: \mathcal{I} \to \mathcal{I}_0(G, \prec)\).

3.8. Lemma. The mapping \(i\) is a bijection of \(\mathcal{I}\) onto \(\mathcal{I}_0(G, \prec)\).

Proof. If \(i(\prec_1) = i(\prec_2) = A\) and \(B = G - A\), then \((G, \prec_1) = B \oplus A = = (G, \prec_2)\). Thus, \(i\) is an injection. Let \(A \in \mathcal{I}_0(G, \prec)\) be any element. If \(A = G\), then \(A = i(\prec)\). Otherwise put \(B = G - A\), thus \((G, \prec) = A \oplus B\). Define a linear order \(\prec\) on \(G\) by setting \((G, \prec) = B \oplus A\). By Theorem 1.6, \(\prec\) is a cut on \((G, C)\), i.e. \(\prec \in \mathcal{I}\) and \(i(\prec) = A\). The mapping \(i\) is thus a surjection and therefore a bijection of \(\mathcal{I}\) onto \(\mathcal{I}_0(G, \prec)\).

3.9. Theorem. The mapping \(i: \mathcal{I} \to \mathcal{I}_0(G, \prec)\) is an isomorphism of the cycle \((\mathcal{I}, \subset)\) onto the cycle \((\mathcal{I}_0(G, \prec), C_\subset)\).

Proof. By Lemma 3.8, \(i\) is a bijection. Let \(\prec_1, \prec_2, \prec_3 \in \mathcal{I}\), \((\prec_1, \prec_2, \prec_3) \in C_\subset\).
As $i(<_1)$, $i(<_2)$, $i(<_3)$ are nonvoid pairwise distinct initial intervals in a linearly ordered set $(G, <)$, they are linearly ordered by set inclusion. Let us assume that $i(<_3) \subsetneq i(<_2) \subsetneq i(<_1)$ holds. Choose arbitrary elements $x \in i(<_3)$, $y \in i(<_2) - i(<_3)$, $z \in i(<_1) - i(<_2)$. Then $y <_3 z <_3 x$, $z <_2 x <_2 y$, $x <_1 y <_1 z$. This implies by Lemma 4.3 of [6] that $(<_3, <_2, <_1) \in \mathcal{C}$, which is a contradiction. Analogously we show that both $i(<_2) \subset i(<_1) \subset i(<_3)$ and $i(<_1) \subset i(<_3) \subset i(<_2)$ are impossible. Thus it must hold either $i(<_1) \subset i(<_2) \subset i(<_3)$ or $i(<_2) \subset i(<_3) \subset i(<_1)$ or $i(<_3) \subset i(<_1) \subset i(<_2)$ and this implies $(i(<_1), i(<_2), i(<_3)) \in C_c$ in all cases. Hence $i$ is an isomorphism.

Theorem 3.9 shows that the complete hull $(\mathcal{G}, \mathcal{C})$ of a cycle $(G, C)$ can be constructed in the following way: we choose any cut $<$ on $(G, C)$, find the reduced ideal hull of $(G, <)$ and construct the cycle corresponding to that chain. The regular hull $(\mathcal{G}, \mathcal{C})$ of a cycle $(G, C)$, however, can be obtained by a similar construction. Let us recall that the Dedekind-Mac Neille completion of a chain $(G, <)$ can be defined as the set of all initial intervals $A$ in $(G, <)$ with the property: either $B = G - A$ has the least element or $A$ has no greatest element and $B$ has no least element; this set is ordered by set inclusion ([3], chapter IV, par. 5).

3.10. Reduced Dedekind-Mac Neille completion. Let $(G, <)$ be a chain. Let $\mathcal{J}_1(G)$ be the set of all nonvoid initial intervals $A$ in $(G, <)$ with the property: either $B = G - A$ has the least element or $A$ has no greatest element and $B$ has no least element; further, let $G \in \mathcal{J}_1(G)$. The chain $(\mathcal{J}_1(G), <)$ will be called reduced Dedekind-Mac Neille completion of $(G, <)$.

Clearly, $\mathcal{J}_1(G) \subseteq \mathcal{J}_0(G)$ and $(\mathcal{J}_1(G), <)$ is a conditionally complete chain with the greatest element, so that $(\mathcal{J}_1(G), C_c)$ is a complete cycle whenever card $G \geq 3$.

3.11. Theorem. The mapping $i$ defined in 3.7 is an isomorphism of $(\mathcal{G}, \mathcal{C})$ onto $(\mathcal{J}_1(G, <), C_c)$.

Proof. We show that $i$ maps bijectively $\mathcal{G}$ onto $\mathcal{J}_1(G, <)$. Let $< \in \mathcal{G}$. If $< = <$, then $i(<') = G \in \mathcal{J}_1(G, <)$. Let $< \neq <$. Then either $< = <_{c,x}$ for some $x \in G$ or $< <$ is a gap in $(G, C)$. In the first case we obtain $(G, <) = A \oplus B$, $(G, <) = B \oplus A$ and as $(G, <) = (G, <_{c,x})$ has the least element $x$, $B$ has the least element $x$. Thus, $i(<') = A \in \mathcal{J}_1(G, <)$. In the second case we have $(G, <) = A \oplus B$, $(G, <) = B \oplus A$ and as $<$ is a gap, $B$ has not least element and $A$ has no greatest element. Thus, $i(<') = A \in \mathcal{J}_1(G, <)$. We have shown that $i$ maps $\mathcal{G}$ into $\mathcal{J}_1(G, <)$. Let $A \in \mathcal{J}_1(G, <)$. Put $B = G - A$. Then $(G, <) = B \oplus A$ and either $B$ has the least element $x$ or $A$ has no greatest element and $B$ has no least element. In the first case $(G, <)$ has the least element $x$, $< = <_{c,x}$ and $< \in \mathcal{G}$, $i(<') = A$. In the second case $<$ is a gap, $< \in \mathcal{G}$ and $i(<') = A$. We have shown that $i: \mathcal{G} \rightarrow \mathcal{J}_1(G, <)$ is a surjection; by Lemma 3.8 it is a bijection. The proof that $i$ is an isomorphism of $(\mathcal{G}, \mathcal{C})$ onto $(\mathcal{J}_1(G, <), C_c)$ is the same as that of Theorem 3.9.
References


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