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Persistent URL: http://dml.cz/dmlcz/102168

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ON COMPLETION OF CYCLICALLY ORDERED SETS

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(Received June 28, 1985)

In [6], a completion of linearly cyclically ordered sets (cycles) is constructed by means of cuts. In this note we give another construction of a completion, which can be applied to a larger class of monodimensional cyclically ordered sets.

1. INTRODUCTORY CONCEPTS AND ASSERTIONS

1.1. Ordered sets. Basic notions on ordered sets are assumed to be known (see e.g. [1] or [2]). Ordinal sum of ordered sets $G, H$ is denoted by $G \oplus H$. If $G = (G, <)$ is an ordered set and $H \subseteq G$, then the induced order $< \cap H^2$ on $H$ is denoted briefly by $<$. A linearly ordered set is called a chain. If $G = (G, <)$ is an ordered set and $H \subseteq G$ is a subset of $G$ such that $(H, <)$ is a chain, then $H$ is called a chain in $G$. A chain $H$ in an ordered set $G$ is maximal iff it is contained in no chain in $G$ as a proper subset. As it is well known, the “Hausdorff maximal principle” every chain in every ordered set $G$ is contained in a maximal chain in $G$

is equivalent to the Axiom of Choice.

An ordered set $G$ is called complete (or a complete lattice) iff any nonvoid subset of $G$ has the supremum and infimum in $G$. $G$ is said to be conditionally complete iff any nonvoid bounded subset of $G$ has the supremum and infimum in $G$. $G$ is referred to as chain complete iff any maximal chain in $G$ is complete.

A subset $I$ of an ordered set $G$ is called an ideal iff it has the property $x \in I, y \in G, y < x \Rightarrow y \in I$. If $A$ is a nonvoid subset of an ordered set $G$, then $I(A)$ denotes the ideal in $G$ generated by $A$, i.e., $I(A) = \{ y \in G; \text{there exists } x \in A \text{ such that } y \leq x \}$. Note that $\sup A = \sup I(A)$ for any nonvoid subset $A$ of an ordered set $G$, whenever one of the elements $\sup A, \sup I(A)$ exists. If $G$ is a chain, then ideals in $G$ are called initial intervals; the dual notion is a final interval in $G$.

1.2. Cyclically ordered sets. A cyclically ordered set ([5]) is a pair $(G, C)$ where $G$ is a set and $C$ is a cyclic order on $G$, i.e., $C$ is a ternary relation on $G$ which is asymmetric, i.e., $(x, y, z) \in C \Rightarrow (z, y, x) \notin C$, cyclic, i.e., $(x, y, z) \in C \Rightarrow (y, z, x) \in C$, and transitive, i.e., $(x, y, z) \in C, (x, z, u) \in C \Rightarrow (x, y, u) \in C$. 407
If, moreover, card \( G \geq 3 \) and \( C \) is
linear, i.e., \( x, y, z \in G, x \neq y \neq z \neq x \Rightarrow \) either \( (x, y, z) \in C \) or \( (z, y, x) \in C \),
then \( (G, C) \) is called a linearly cyclically ordered set or a cycle.

If \( (G, C) \) is a cyclically ordered set and \( H \subseteq G \) is a subset of \( G \) such that \( (H, C \cap H^3) \)
is a cycle, then \( H \) is called a cycle in \( G \). A cycle \( H \) in \( G \) is maximal iff it is contained
in no cycle in \( G \) as a proper subset. In [5] (Theorem 2.5) we proved using Axiom
of Choice that every cycle in every cyclically ordered set is contained in a maximal
cycle; below we show that this proposition is equivalent to the Axiom of Choice.

Let \( (G, <) \) be an ordered set. Let us define a ternary relation \( C_\prec \) on \( G \) by
\[
(x, y, z) \in C_\prec \Leftrightarrow \text{either } x < y < z \text{ or } y < z < x \text{ or } z < x < y.
\]
Then \( (G, C_\prec) \) is a cyclically ordered set ([5], Theorem 3.5).

Let \( (G, C) \) be a cyclically ordered set and \( x \in G \). Let us define a binary relation \( <_{C,x} \)
on \( G \) by
\[
y <_{C,x} z \Leftrightarrow \text{either } (x, y, z) \in C \text{ or } x = y \neq z.
\]
Then \( (G, <_{C,x}) \) is an ordered set with the least element \( x \) ([5], Theorem 3.1).

The proofs of the following two lemmas are trivial.

1.3. Lemma. Let \( (G, <) \) be an ordered set and card \( G \geq 3 \). Then \( (G, <) \) is a chain
iff \( (G, C_\prec) \) is a cycle.

1.4. Lemma. Let \( (G, <) \) be an ordered set and \( H \) a chain in \( G \) with card \( H \geq 3 \).
Then \( H \) is a maximal chain in \( (G, <) \) iff \( H \) is a maximal cycle in \( (G, C_\prec) \).

1.5. Theorem. The proposition
"Every cycle in every cyclically ordered set \( G \) is contained in a maximal cycle in \( G \)"
is equivalent to the Axiom of Choice.

Proof. One implication is shown in [5]; we shall prove the other one. Thus, let
every cycle in every cyclically ordered set be contained in a maximal cycle and let
\( (G, <) \) be an ordered set. Assume that \( H \) is a chain in \( G \) with card \( H \geq 3 \). By Lemma
1.3 \( (H, C_\prec \cap H^3) \) is a cycle in \( (G, C_\prec) \) and hence there exists a maximal cycle
\( (K, C_\prec \cap K^3) \) in \( (G, C_\prec) \) such that \( K \supseteq H \). By Lemma 1.4 \( (K, <) \) is a maximal
chain in \( (G, <) \). If \( H \) is a chain in \( (G, <) \) with card \( H \leq 2 \) and if there exists no
3-element chain in \( G \) containing \( H \), then the existence of a maximal chain in \( G \)
containing \( H \) is trivial. Thus the Hausdorff maximal principle and also the Axiom
of Choice is true.

We shall need the following assertion; its proof can be found in [6] (Theorem 3.6
and Corollary 3.9).

1.6. Theorem. Let \( G \) be a set, card \( G \geq 3 \) and let \( <_1, <_2 \) be linear orders on \( G \).
Then the following statements are equivalent:
(A) \( C_{<_1} = C_{<_2} \)
(B) There exist disjoint subsets \( A, B \) of \( G \) such that \( A \cup B = G, \)
\[
<_1 \cap A^2 = \quad \quad \quad = \quad _2 \cap A^2, \quad _1 \cap B^2 = \quad _2 \cap B^2 \quad \text{and} \quad (G, <_1) = A \oplus B, (G, <_2) = B \oplus A.
\]

408
Let \( G \) be a set, let \(<_1, <_2\) be orders on \( G \). Put \(<_1 \sim <_2\) iff \( C_{<_1} = C_{<_2} \). Trivially, it holds

1.7. Lemma. Let \( G \) be a set. The binary relation \( \sim \) is an equivalence relation on the set of all orders on \( G \).

1.8. Theorem. Let \( G \) be a set, let \(<_1, <_2\) be orders on \( G \). Then the following statements are equivalent:

A) \( C_{<_1} \subseteq C_{<_2} \)

B) If \( H \) is a maximal chain in \((G, <_1)\) with \( \text{card} \ H \geq 3 \), then \( H \) is a chain in \((G, <_2)\) and there exist disjoint subsets \( A, B \) of \( H \) with \( A \cup B = H \) such that

\[
<_1 \cap A^2 = <_2 \cap A^2, \quad <_1 \cap B^2 = <_2 \cap B^2 \quad \text{and} \quad (H, <_1) = A \oplus B, (H, <_2) = B \oplus A.
\]

Proof. 1. Let (A) hold and let \((H, <_1)\) be a maximal chain in \((G, <_1)\) with \( \text{card} \ H \geq 3 \). By Lemma 1.3 \((H, C_{<_1} \cap H^3)\) is a cycle and as \( C_{<_1} \subseteq C_{<_2} \), \((H, C_{<_2} \cap H^3)\) is also a cycle, thus \( C_{<_1} \cap H^3 = C_{<_2} \cap H^3 \). From this it follows, by Lemma 1.3, that \((H, <_2)\) is also a chain. Applying Theorem 1.6 on the set \( H \) and linear orders \( <_1 \cap H^2, <_2 \subseteq H^2 \), we obtain the validity of (B).

2. Let (B) hold and suppose \((x, y, z) \in C_{<_1}\). Then either \( x <_1 y <_1 z \) or \( y <_1 z \) \( <_1 x \) or \( z <_1 x <_1 y \) and there exists a maximal chain \( H \) in \((G, <_1)\) containing \( \{x, y, z\} \). Thus \( \text{card} \ H \geq 3 \) and, by (B), \((H, <_2)\) is a chain and there exist subsets \( A, B \) of \( H \) with the desired properties. Assume that \( x <_1 y <_1 z \) holds (in other two cases the proof is analogical). We have four possibilities:

1) \( x, y, z \in A \). Then \( x <_2 y <_2 z \), for \((A, <_1) = (A, <_2)\), and thus \((x, y, z) \in C_{<_2}\);

2) \( x, y \in A, z \in B \). Then \( x <_2 y \) and \( z <_2 x \), for \((H, <_2) = B \oplus A\), thus \( z <_2 <_2 x <_2 y \) and \((x, y, z) \in C_{<_2}\);

3) \( x \in A, y, z \in B \). Then \( y <_2 z \) and \( z <_2 x \), hence \( y <_2 z <_2 x \) and \((x, y, z) \in C_{<_2}\);

4) \( x, y, z \in B \). Then \( x <_2 y <_2 z \) and \((x, y, z) \in C_{<_2}\).

We have shown that \((x, y, z) \in C_{<_1} \Rightarrow (x, y, z) \in C_{<_2}\), i.e., \( C_{<_1} \subseteq C_{<_2} \) and (A) holds.

1.9. Corollary. Let \( G \) be a set and let \(<_1, <_2\) be orders on \( G \). Then \(<_1 \sim <_2\) holds iff the sets of maximal, at least three element chains in \((G, <_1)\) and in \((G, <_2)\) are the same and for any such maximal chain \( H \) the condition (B) of Theorem 1.8 holds.

2. COMPLETENESS

2.1. Cuts on cycles. Let \((G, C)\) be a cycle. A cut on \((G, [6])\) is a linear order \( < \) on \( G \) such that \( C = C_< \). Any cycle \((G, C)\) contains cuts, for \(<_c, x\) is a cut on \( G \) for any \( x \in G \) \([6], \text{Theorem 2.5}\). A cut \( < \) on a cycle \((G, C)\) is a jump iff \((G, <)\) has both the least and the greatest element; it is a gap, if \((G, <)\) has neither the least nor the greatest element; \(< \) is Dedekind iff \((G, <)\) has just one of the boundary elements.
A cycle \((G, C)\) is *dense* iff it contains no jumps; it is *complete* iff it contains no gaps. A cycle \((G, C)\) is *continuous* iff it is dense and complete, i.e., iff each cut on \(G\) is Dedekind.

2.2. **Definition.** A cyclically ordered set \((G, C)\) is called *cycle complete* iff each maximal cycle in \(G\) is complete.

2.3. **Theorem.** Let \(G\) be a set, let \(<\) be an order on \(G\). If the ordered set \((G, <)\) is *chain complete*, then the cyclically ordered set \((G, C_<)\) is cycle complete.

**Proof.** Let \((H, C_< \cap H^3)\) be a maximal cycle in \((G, C_<)\). By Lemmas 1.3 and 1.4, \((H, <)\) is a maximal chain in \((G, <)\). Hence \((H, <)\) is complete. Assume that the cycle \((H, C_< \cap H^3)\) is not complete. Then there exists a cut \(<\) on \((H, C_< \cap H^3)\) which is a gap, i.e., \(<\) is a linear order on \(H\) without the least and the greatest elements and such that \(C_< = C_< \cap H^3\). By Theorem 1.6, there exist disjoint subsets \(A, B\) of \(H\) such that \(A \cup B = H\), \(<\cap A^2 = <\cap A^2\), \(<\cap B^2 = <\cap B^2\) and \((H, <) = = A \oplus B\), \((H, <) = B \oplus A\). Thus \((B, <) = (B, <)\) contains no least element, \((A, <)\) contains no greatest element. If \(A \neq \emptyset\), then \(A\) has no supremum in \((H, <) = = A \oplus B\). If \(A = \emptyset\), then \(B = H\) has no infimum in \((H, <)\). In either case this contradicts the assumption that \((H, <)\) is complete and thus the cycle \((H, C_< \cap H^3)\) is complete.

Theorem 2.3 cannot be reversed, i.e., if a cyclically ordered set \((G, C_<)\) is cycle complete, then the ordered set \((G, <)\) need not be chain complete as the following example shows.

2.4. **Example.** Let \((G, <) = [0, 1)\) with the natural ordering of reals. \((G, <)\) is not chain complete; we show that \((G, C_<)\) is a complete cycle.

Let \(<\) be any cut on \((G, C_<)\). Then either \(<\ = <\ or \((G, <) = B \oplus A\) where \(A\) is an initial interval, \(B\) a final interval in \([0, 1)\), and \(A \neq \emptyset\), \(B \neq \emptyset\). In the second case either \(A\) has the greatest element or \(B\) has the least element in \([0, 1)\). Thus \((G, <)\) has one of the boundary elements and \(<\) is not a gap.

For cycles we have, however, this assertion:

2.5. **Theorem.** Let \((G, <)\) be a chain with card \(G \geq 3\). The cycle \((G, C_<)\) is complete iff the chain \((G, <)\) is conditionally complete and has either the least or the greatest element.

**Proof.** 1. Let the cycle \((G, C_<)\) be complete. The chain \((G, <)\) must contain either the least or the greatest element; otherwise the cut \(<\ on \((G, C_<)\) would be a gap. Assume that \((G, <)\) is not conditionally complete. Then there exists a subset \(A\) of \(G\), \(A \neq \emptyset\) which is upper bounded and such that \(sup A\) does not exist in \((G, <)\). As \(sup A = sup I(A)\), we may assume that \(A\) is an initial interval in \((G, <)\). Then \(B = G - A\) is a final interval in \((G, <)\), \(B \neq \emptyset\), we have \((G, <) = A \oplus B\) and \(A\ contains no greatest element, \(B\ contains no least element. Let \(<\ be a linear order on \(G\ such that \((G, <) = B \oplus A\). By Theorem 1.6, <\ is a cut on \((G, C_<)\) which is a gap. This contradicts the assumption.
2. Let \((G, \prec)\) be a conditionally complete chain which has either the least or the
greatest element. Assume that the cycle \((G, C)\) is not complete. Then there exists
a cut \(\prec\) on \((G, C)\) which is a gap. It must necessarily be \(\prec \neq \prec\) and thus there
exist nonvoid disjoint subsets \(A, B\) of \(G\) such that \(\prec \cap A^2 = \prec \cap A^2, \prec \cap B^2 = \prec \cap B^2\) and
\((G, \prec) = A \oplus B, (G, \prec) = B \oplus A\). This implies that \(B\) has no
least element, \(A\) has no greatest element. Therefore sup \(A\) does not exist in \((G, \prec)\),
which contradicts the assumption that \((G, \prec)\) is conditionally complete.

Let us call a cyclically ordered set \((G, C)\) monodimensional iff there exists an
order \(\prec\) on the set \(G\) such that \(C = C_\prec\). This concept agrees with [7], for by suitable
definition of the dimension in the class of cyclically ordered sets, the cyclically
ordered sets of form \((G, C)\) are just the sets with dimension 1.

2.6. Theorem. Let \((G, C)\) be a monodimensional cyclically ordered set. Then
there exists a cycle complete cyclically ordered set \((H, D)\) and an isomorphic
embedding of \((G, C)\) into \((H, D)\).

Proof. By assumption there exists an order \(\prec\) on \(G\) such that \(C = C_\prec\). To the
ordered set \((G, \prec)\) we can construct a chain complete ordered set \((H, \prec)\) such that
there exists an isomorphic embedding \(i: G \rightarrow H\) of \((G, \prec)\) into \((H, \prec)\); for instance
the Dedekind-Mac Neille completion of \((G, \prec)\) ([2] or [4]) has this property. By
Theorem 2.3, the cyclically ordered set \((H, C_\prec)\) is cycle complete and, clearly,
\(i: G \rightarrow H\) is an isomorphic embedding of \((G, C)\) into \((H, C_\prec)\).

3. APPLICATION TO CYCLES

3.1. Completion of cycles by cuts. Let \((G, C)\) be a cycle. Let \(\mathcal{G}\) be the set of all
cuts on \((G, C)\), \(\mathcal{G} = \{\{c, x\}; x \in G\} \cup \{\prec \in \mathcal{G}; \prec\text{ is a gap}\}\); the elements of \(\mathcal{G}\) are
called regular cuts. Let us define a ternary relation \(\epsilon\) on \(\mathcal{G}\) (and also on \(\mathcal{G}\)) by
\((\prec_1, \prec_2, \prec_3) \epsilon \mathcal{G} \iff\) there exist nonvoid pairwise disjoint subsets \(A, B, D\) of \(G\)
such that \(\prec_1 \cap A^2 = \prec_2 \cap A^2 = \prec_3 \cap A^2, \prec_1 \cap B^2 = \prec_2 \cap B^2 = \prec_3 \cap B^2,
\prec_1 \cap D^2 = \prec_2 \cap D^2 = \prec_3 \cap D^2\) and \((G, \prec) = A \oplus B \oplus D, (G, \prec) = B \oplus
\oplus D \oplus A, (G, \prec) = D \oplus A \oplus B\). In [6] (Theorem 4.2, Corollary 4.5, Theorem 5.2
and Theorem 5.6) there is proved that both \((\mathcal{G}, \epsilon)\) and \((\mathcal{G}, \epsilon)\) are complete cycles
and that \(x \rightarrow \prec_\prec\) is an isomorphic embedding of \((G, C)\) into \((\mathcal{G}, \epsilon)\) and into \((\mathcal{G}, \epsilon)\).
If, moreover, \((G, C)\) is dense. then \((\mathcal{G}, \epsilon)\) is continuous ([6], Theorem 5.9).

The results of preceding paragraph give another possibility of a construction of
a completion of a cycle. Directly from Theorem 2.6 and its proof we get

3.2. Theorem. Let \((G, C)\) be a cycle, let \(\prec\) be a cut on \((G, C)\). Let \((H, \prec)\) be a complete
chain such that there exists an isomorphic embedding of \((G, \prec)\) into \((H, \prec)\).
Then \((H, C_\prec)\) is a complete cycle and there exists an isomorphic embedding of
\((G, C)\) into \((H, C_\prec)\).

Especially, we have
3.3. Corollary. Every cycle can be embedded into a complete cycle.

If we choose in Theorem 3.2 \((H, \prec)\) as the Dedekind-Mac Neille completion of \((G, \prec)\), we get further

3.4. Corollary. Let \((G, C)\) be a cycle, \(\prec\) a cut on \((G, C)\). Let \((H, \prec)\) be the Dedekind-Mac Neille completion of \((G, \prec)\). Then \((H, C_\prec)\) is a complete cycle and there exists an isomorphic embedding of \((G, C)\) into \((H, C_\prec)\).

Further, by Theorem 2.5, the stronger version of Theorem 3.2 holds:

3.5. Theorem. Let \((G, C)\) be a cycle, \(\prec\) a cut on \((G, C)\). Let \((H, \prec)\) be a conditionally complete chain containing either the least or the greatest element such that there exists an isomorphic embedding of \((G, \prec)\) into \((H, \prec)\). Then \((H, C_\prec)\) is a complete cycle and there exists an isomorphic embedding of \((G, C)\) into \((H, C_\prec)\).

There is a close connection between the both constructions (given in 3.1 and in 3.2 or 3.5); we shall describe it.

3.6. Ideal hull of a chain. Let \((G, \prec)\) be a chain, let \(\mathcal{I}(G)\) be the set of all ideals (initial intervals) in \(G\), which is ordered by set inclusion. The ordered set \((\mathcal{I}(G), \subset)\) is a complete chain which we call the ideal hull of the chain \((G, \prec)\). Further, let \(\mathcal{I}_0(G)\) be the set of all nonvoid initial intervals in \(G\), i.e. \(\mathcal{I}_0(G) = \mathcal{I}(G) - \{\emptyset\}\). \((\mathcal{I}_0(G), \subset)\) is a conditionally complete chain with the greatest element which we shall call a reduced ideal hull of \((G, \prec)\).

By Theorem 2.5, if \((G, \prec)\) is a chain with card \(G \geq 3\) and \((\mathcal{I}_0(G), \subset)\) its reduced ideal hull, then \((\mathcal{I}_0(G), C_\subset)\) is a complete cycle.

In the sequel, we assume that there is given a fixed cycle \((G, C)\) and a fixed cut \(\prec\) on \((G, C)\).

3.7. Notation. Let \(\prec\) be a cut on \((G, C)\), other than \(\prec\). By Theorem 1.6, there exist (uniquely determined) disjoint nonvoid subsets \(A, B\) of \(G\) such that \(A \cup B = G\), \(\prec \cap A^2 = \prec \cap A^3\), \(\prec \cap B = \prec \cap B^2\) and \((G, \prec) = A \oplus B\), \((G, \prec) = B \oplus A\). Thus, \(A\) is a nonvoid initial interval in \((G, \prec)\), i.e. \(A \in \mathcal{I}_0(G, \prec)\). Let us put \(\mathfrak{i}(\prec) = = A\); further, put \(\mathfrak{i}(\prec) = G\). We have therefore defined a mapping \(\mathfrak{i}: \mathfrak{I} \to \mathcal{I}_0(G, \prec)\).

3.8. Lemma. The mapping \(\mathfrak{i}\) is a bijection of \(\mathfrak{I}\) onto \(\mathcal{I}_0(G, \prec)\).

Proof. If \(\mathfrak{i}(\prec_1) = \mathfrak{i}(\prec_2) = A\) and \(B = G - A\), then \((G, \prec_1) = B \oplus A = = (G, \prec_2)\). Thus, \(\mathfrak{i}\) is an injection. Let \(A \in \mathcal{I}_0(G, \prec)\) be any element. If \(A = G\), then \(A = \mathfrak{i}(\prec)\). Otherwise put \(B = G - A\), thus \((G, \prec) = A \oplus B\). Define a linear order \(\prec\) on \(G\) by setting \((G, \prec) = B \oplus A\). By Theorem 1.6, \(\prec\) is a cut on \((G, C)\), i.e. \(\prec \in \mathfrak{I}\) and \(\mathfrak{i}(\prec) = A\). The mapping \(\mathfrak{i}\) is thus a surjection and therefore a bijection of \(\mathfrak{I}\) onto \(\mathcal{I}_0(G, \prec)\).

3.9. Theorem. The mapping \(\mathfrak{i}: \mathfrak{I} \to \mathcal{I}_0(G, \prec)\) is an isomorphism of the cycle \((\mathfrak{I}, C)\) onto the cycle \((\mathcal{I}_0(G, \prec), C_\subset)\).

Proof. By Lemma 3.8, \(\mathfrak{i}\) is a bijection. Let \(\prec_1, \prec_2, \prec_3 \in \mathfrak{I}\), \((\prec_1, \prec_2, \prec_3) \in C\).
As \( i(<_1), i(<_2), i(<_3) \) are nonvoid pairwise distinct initial intervals in a linearly ordered set \((G, <)\), they are linearly ordered by set inclusion. Let us assume that 
\[
i(<_3) \subset i(<_2) \subset i(<_1) \quad \text{holds. Choose arbitrary elements } x \in i(<_3), \ y \in i(<_2) - i(<_3), \ z \in i(<_1) - i(<_2). \text{ Then } y <_3 z <_3 x, \ z <_2 x <_2 y, \ x <_1 y <_1 z. \text{ This implies by Lemma 4.3 of [6] that } (<_3, <_2, <_1) \in \mathcal{C}, \text{ which is a contradiction. Analogously we show that both } i(<_2) \subset i(<_1) \subset i(<_3) \text{ and } i(<_1) \subset i(<_3) \subset i(<_2) \text{ are impossible. Thus it must hold either } i(<_1) \subset i(<_2) \subset i(<_3) \text{ or } i(<_2) \subset i(<_3) \subset i(<_1) \text{ or } i(<_3) \subset i(<_1) \subset i(<_2) \text{ and this implies } (i(<_1), i(<_2), i(<_3)) \in C_c \text{ in all cases. Hence } i \text{ is an isomorphism.}
\]

Theorem 3.9 shows that the complete hull \((\mathcal{G}, \mathcal{C})\) of a cycle \((G, C)\) can be constructed in the following way: we choose any cut cut \(<_0 \text{ on } (G, C), \text{ find the reduced ideal hull of } (G, <) \text{ and construct the cycle corresponding to that chain. The regular hull } (\mathcal{G}, \mathcal{C}) \text{ of } (G, C), \text{ however, can be obtained by a similar construction. Let us recall that the Dedekind-Mac Neille completion of a chain } (G, <) \text{ can be defined as the set of all initial intervals } A \text{ in } (G, <) \text{ with the property: either } B = G - A \text{ has the least element or } A \text{ has no greatest element and } B \text{ has no least element; this set is ordered by set inclusion ([3], chapter IV, par. 5).}

3.10. Reduced Dedekind-Mac Neille completion. Let \((G, <)\) be a chain. Let \( \mathcal{I}(G) \) be the set of all nonvoid initial intervals \( A \) in \((G, <)\) with the property: either \( B = G - A \) has the last element or \( A \) has no greatest element and \( B \) has no least element; further, let \( G \in \mathcal{I}(G) \). The chain \( (\mathcal{I}(G), <) \) will be called reduced Dedekind-Mac Neille completion of \((G, <)\).

Clearly, \( \mathcal{I}(G) \subseteq \mathcal{I}(G) \) and \((\mathcal{I}(G), <)\) is a conditionally complete chain with the greatest element, so that \((\mathcal{I}(G), C_c)\) is a complete cycle whenever card \( G \geq 3 \).

3.11. Theorem. The mapping \( i \) defined in 3.7 is an isomorphism of \((\mathcal{G}, \mathcal{C})\) onto \((\mathcal{I}(G), C_c))\).

Proof. We show that \( i \) maps bijectively \( \mathcal{G} \) onto \( \mathcal{I}(G) \). Let \( i(<) = G \in \mathcal{I}(G, <) \). Let \( i(<) = x \in G \) or \( x \in G \) and as \( (G, <) = (G, c_{<x}) \) has the least element \( x \). \( B \) has the least element \( x \). Thus, \( i(<) = A \in \mathcal{I}(G, <) \). In the second case we have \( (G, <) = A \oplus B \), \( (G, <) = B \oplus A \) and as \( i(<) = A \in \mathcal{I}(G, <) \). We have shown that \( i \) maps \( \mathcal{G} \) into \( \mathcal{I}(G, <) \). Let \( A \in \mathcal{I}(G, <) \). Put \( B = G - A \). Then \( (G, <) = B \oplus A \) and either \( B \) has the least element \( x \) or \( A \) has no greatest element and \( A \) has no least element. In the first case \((G, <)\) has the least element \( x \), \( < = c_{<x} \) and \( \in \mathcal{G} \), \( i(<) = A \). In the second case \( < \text{ is a gap, } A \in \mathcal{G} \) and \( i(<) = A \). We have shown that \( i : \mathcal{G} \to \mathcal{I}(G, <) \) is a surjection; by Lemma 3.8 it is a bijection. The proof that \( i \) is an isomorphism of \((\mathcal{G}, \mathcal{C})\) onto \((\mathcal{I}(G, <), C_c)\) is the same as that of Theorem 3.9.

413
References


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