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LINEAR DIFFERENTIAL EQUATIONS WITH QUASIPERIODIC COEFFICIENTS

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Let $A(t)$ be a (real or complex) $n \times n$ matrix for $t \in R$, let A depend continuously on t and fulfil

$$(0.1) \qquad A(t) + A^*(t) = 0 \quad \text{for} \quad t \in R$$

(i.e. $A(t)$ is antisymmetric in the real case and $iA(t)$ is Hermitian in the complex case, $t \in R$). Let X_A be the matrix solution of

$$(0.2) \qquad \dot{x} = A(t)\, x\,,$$

$X_A(0) = I$. Then $X_A(t)$ is an orthonormal matrix in the real case and $X_A(t)$ is a unitary matrix in the complex case, $t \in R$. Assume in addition that A is uniformly quasiperiodic with at most $r + 1$ frequencies (see Chapter I, § 2). Then X_A need not be almost periodic (even in the case that $n = 1$, i.e. $A(t) = i\alpha(t)$, $\alpha(t) \in R$).

Problem. *Given A and $\eta > 0$, does there exists such a matrix-valued function C that*

(i) *both C and X_C are uniformly quasiperiodic with at most $r + 1$ frequencies,*
(ii) $\|A(t) - C(t)\| \leqq \eta$ *for* $t \in R$?

An affirmative answer is given (see Theorem I.2.1) for such couples (n, r) that the manifold $SO(n)$ in the real case ($SU(n)$ in the complex case) has the estimation property of homotopies of order $1, 2, \ldots, r$ ($SO(n)$ is the manifold of orthonormal $n \times n$ matrices with determinant equal to 1, $SU(n)$ is the manifold of unitary $n \times n$ matrices with determinant equal to 1).

A Riemannian manifold M is said to have the *estimation property of homotopies of order j* – shortly $M \in EP(j)$, see Definition I.2.1 – if such a $c = c(M, j) > 0$ exists that the following holds:

Assume that $m \in M$, $g_0, g\colon \langle 0, 1\rangle^j \to M$, $g_0(x) = m$ for $x \in \langle 0, 1\rangle^j$, $g(x) = m$ for $x \in \partial(\langle 0, 1\rangle^j)$, g is of class $C^{(1)}$ and is homotopic with g_0. Then there exists such a homotopy $h\colon \langle 0, 1\rangle \times \langle 0, 1\rangle^j \to M$ that $h(1, x) = g(x)$, $h(0, x) = g_0(x)$ for $x \in M$ and

$$\left\|\frac{\partial h}{\partial \beta}\right\| \leqq c\,, \quad \max_{x,i} \max \left\{\left\|\frac{\partial h}{\partial x_i}\right\|, \left\|\frac{\partial^2 h}{\partial \beta\, \partial x_i}\right\|\right\} \leqq c \max\left\{1, \max_{x,i} \left\|\frac{\partial g}{\partial x_i}\right\|\right\}.$$

However, in general it is not known for which couples (n, j) the relations $SO(n) \in \in EP(j)$, $SU(n) \in EP(j)$ hold. It is proved in the Appendix in an elementary way that $SO(n) \in EP(j)$, $SU(n) \in EP(j)$ for $n = 1, 2, 3, \ldots, j = 1, 2$. The proof is based on the very simple structure of the homotopy groups $\pi_1(SO(n))$, $\pi_1(SU(n))$, $\pi_2(SO(n))$, $\pi_2(SU(n))$, $n = 1, 2, 3, \ldots$. So far the conclusion can be drawn that the answer to the problem is affirmative for $n = 1, 2, 3, \ldots, r = 1, 2$.

Theorem I.2.1 is proved in Chapter I, § 5. However, its proof depends on Theorem I.4.1, the proof of which is very lengthy and in fact extends through Chapters II and III. A list of symbols can be found after Appendix.

CHAPTER I

1. Let $\mathbb{R}$ denote real numbers, $\mathbb{C}$ complex numbers, $\mathbb{Z}$ integers and $\mathbb{N}$ natural numbers (excluding 0). The letters n, r, j are used for natural numbers only. $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$ and Matr (n) denotes the set of all $n \times n$ matrices with entries from $\mathbb{K}$. (Mostly we consider both the real and the complex case simultaneously.)

For A from Matr (n), A^* is the adjoint matrix. I denotes the matrix with 1's on the main diagonal and 0's everywhere else. 0 is the matrix with all entries equal to 0. $U(n)$ or $O(n)$ denotes the set of all unitary or orthonormal $n \times n$ matrices, respectively (i.e. matrices A from Matr (n) with complex or real entries satisfying $AA^* = I$) and $SU(n)$ and $SO(n)$ are respectively the sets of those matrices from $U(n)$ and $O(n)$ with determinants equal to 1. When considering both the real and complex cases we use $Y(n)$ for $U(n)$ or $O(n)$ and $SY(n)$ for $SU(n)$ or $SO(n)$.

To simplify the notation we define for $A \in$ Matr (n) and a vector $x = (x_1, \ldots, x_n)$ from $\mathbb{K}^n$ that Ax is the product of A with the $n \times 1$ matrix

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

For $x, y \in \mathbb{K}^n$, (x, y) is the usual inner product and $\|x\| = (x, x)^{1/2}$. For $A \in$ Matr (n), $\|A\| = \sup \{ \|Ax\|;\ x \in \mathbb{K}^n \text{ and } \|x\| = 1\}$ and for a function f with values in $\mathbb{K}^n$ or Matr (n) let $\|f\|$ denote $\sup \{ \|f(x)\| : x \in \operatorname{dom}(f)\}$.

We introduce the following notation: if $J \subset \mathbb{R}$ is an interval containing 0 and $C: J \to$ Matr (n) a continuous function, then the function $X_C: J \to$ Matr (n) is the matrix solution of the system $\dot{x} = C(t)\, x$ satisfying $X_C(0) = I$.

If Λ is a set and $C: J \times \Lambda \to$ Matr (n) is a function continuous in the first variable, then $X_C: J \times \Lambda \to$ Matr (n) is the function such that for each $z \in \Lambda$, $X_C(t, z)$ as the function of t is the matrix solution of $\dot{x} = C(t, z)\, x$ satifying $X_C(0, z) = I$.

We shall investigate the system of ordinary differential equations

$$\dot{x} = A(t)\, x\,, \tag{1.1}$$

where $A: \mathbb{R} \to$ Matr (n) is a uniformly almost periodic function satisfying

$$A(t) + A^*(t) = 0 \quad \text{for} \quad t \in \mathbb{R}\,. \tag{1.2}$$

First, let us clarify the meaning of the condition (1.2) for the solutions of the system (1.1).

Lemma 1.1. *The condition* (1.2) *and each of the following properties are equivalent*:

(1.3) *Let* $x: \mathbb{R} \to \mathrm{Matr}\,(n)$ *be a solution of the system* (1.1). *Then* $\|x(t)\|$ *does not depend on* t.

(1.4) $$X_A(t) \in Y(n) \quad \text{for} \quad t \in \mathbb{R}\,.$$

Proof. Assuming (1.2) we have

$$\mathrm{d}/\mathrm{d}t\,(X_A^*(t)\,X_A(t)) = X_A^*(t)\,(A(t) + A^*(t))\,X_A(t) = 0\,.$$

Since moreover $X_A(0) = I$ we see that for each t, $X_A^*(t)\,X_A(t) = I$. Therefore (1.2) implies (1.4).

Let us assume (1.4) and let $x(t)$ be a solution of (1.1). There is a c in $\mathbb{K}^n$ such that $x(t) = X_A(t)\,c$ for each t. For this c we have

$$\|x(t)\|^2 = (X_A(t)\,c, X_A(t)\,c) = (X_A^*(t)\,X_A(t)\,c, c) = \|c\|^2\,,$$

so (1.4) implies (1.3).

Finally, let us assume (1.3). Let $s \in \mathbb{R}$. For any $c \in \mathbb{K}^n$ there exists a solution $x(t)$ of the system (1.1) such that $x(s) = c$.

By (1.3), $\|x(t)\|^2$ does not depend on t and so

$$0 = \mathrm{d}/\mathrm{d}t\,(\|x(t)\|)^2 = ((A(t) + A^*(t))\,x(t), x(t)) \quad \text{for} \quad t \in \mathbb{R}\,.$$

In particular, $((A(s) + A^*(s))\,c, c) = 0$. Substituting for c the vectors $e_k, e_k + e_j$, $e_k + ie_j$ with $k, j \in \{1, \ldots, n\}$, $k \neq j$, where $e_1, \ldots, e_n$ is the usual basis of $\mathbb{K}^n$, we get that $A(s) + A^*(s) = 0$; therefore (1.3) implies (1.2).

The facts that a function A satisfies (1.2) and is uniformly almost periodic, are not sufficient for the solution $X_A(t)$ to be uniformly almost periodic. Let us introduce the following notation: $AP(n)$ is the set of all uniformly almost periodic functions $A: R \to \mathrm{Matr}\,(n)$ satisfying (1.2) and

$AP_{\mathrm{sol}}(n)$ is the set of all functions A from $AP(n)$ such that X_A is a uniformly almost periodic function.

We shall investigate the problem whether $AP_{\mathrm{sol}}(n)$ is dense in $AP(n)$ in the uniform topology.

We shall use the following results from the theory of real uniformly almost periodic functions:

Lemma 1.2. *Let* $a: \mathbb{R} \to \mathbb{R}$ *be a uniformly almost periodic function and* $\varepsilon > 0$. *Then there is a trigonometric polynomial* $T: \mathbb{R} \to \mathbb{R}$ *that* $\|a - T\| \leqq \varepsilon$.

Lemma 1.3. *Let* $T: \mathbb{R} \to \mathbb{R}$ *be a trigonometric polynomial. Then the functions* T, $\mathrm{i}T$, $\exp\,(\int_0^t T(s)\,\mathrm{d}s)$ *and* $\exp\,(\mathrm{i}\int_0^t T(s)\,\mathrm{d}s)$ *are uniformly almost periodic.*

Let us consider the smallest values of n. In the real case $AP(1)$ is trivial, since it contains only the function which is identically 0. In the complex case $AP(1)$ is the set of all uniformly almost periodic functions $A\colon \mathbb{R} \to \mathbb{C}$ with purely imaginary values. By Lemma 1.2, to any such function A and any $\varepsilon > 0$ it is possible to find a real trigonometric polynomial T such that $\|A - \mathrm{i}T\| < \varepsilon$. Since

$$X_{\mathrm{i}T}(t) = \exp\left(\mathrm{i} \int_0^t T(s)\,\mathrm{d}s\right) \quad \text{for} \quad t \in \mathbb{R}\,,$$

by Lemma 1.3 the function $\mathrm{i}T$ belongs to $AP_{\mathrm{sol}}(1)$. Therefore $AP_{\mathrm{sol}}(1)$ is dense in $AP(1)$.

$AP(2)$ is in the real case equal to the set of functions A such that

$$A(t) = \begin{pmatrix} 0 & a(t) \\ -a(t) & 0 \end{pmatrix},$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is a uniformly almost periodic function. For any such function A and any $\varepsilon > 0$ it is possible to find a function

$$P(t) = \begin{pmatrix} 0 & T(t) \\ -T(t) & 0 \end{pmatrix},$$

where T is a real trigonometric polynomial, such that $\|A - P\| < \varepsilon$. As matrices

$$\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}$$

$(x \in \mathbb{R})$ commute, we have

$$X_P(t) = \begin{pmatrix} 0 & \exp\left(\int_0^t T(s)\,\mathrm{d}s\right) \\ \exp\left(-\int_0^t T(s)\,\mathrm{d}s\right) & 0 \end{pmatrix} \quad \text{for} \quad t \in \mathbb{R}\,.$$

By Lemma 1.3, X_P is an element of $AP_{\mathrm{sol}}(2)$; so in the real case $AP_{\mathrm{sol}}(2)$ is dense in $AP(2)$.

Because of these facts we shall further assume that $n > 1$ in the complex case and $n > 2$ in the real case.

2. We shall mostly work with quasiperiodic functions.

Let $B(s_0, s_1, \ldots, s_k)$ be a function, $k \in \mathbb{N}$, $\operatorname{dom}(B) \subseteq \mathbb{R}^{k+1}$ and $p \neq 0$ a real number. We say that B is periodic with period p in each variable, $B \in PP(p)$, if the following holds: if $(s_0, s_1, \ldots, s_k) \in \operatorname{dom}(B)$, them also $(s_0, \ldots, s_j \pm p, \ldots, s_k) \in \operatorname{dom} B$ and $B(s_0, \ldots, s_j, \ldots, s_k) = B(s_0, \ldots, s_j \pm p, \ldots, s_k)$ for $j \in \{0, 1, \ldots, k\}$.

A function A: $\mathbb{R} \to \operatorname{Matr}(n)$ is called *quasiperiodic with at most* $r + 1$ *frequencies* if there exist real numbers $\lambda_1 > 0, \ldots, \lambda_r > 0$ and a continuous function $B\colon \mathbb{R}^{r+1} \to \operatorname{Matr}(n)$ from $PP(1)$ such that $A(t) = B(\lambda_0 t, \ldots, \lambda_r t)$ for $t \in \mathbb{R}$. Here obviously we could replace the condition $B \in PP(1)$ by the following one: there exists real $p \neq 0$ such that $B \in PP(p)$.

Any quasiperiodic function with at most $r + 1$ frequencies is uniformly almost periodic. Examples of quasiperiodic functions with at most $r + 1$ frequencies are trigonometric polynomials: $\sum_{k=v}^{r} (a_k \sin(\lambda_k t) + b_k \cos(\lambda_k t))$ in the real case and $\sum_{k=0}^{r} c_k \exp(\mathrm{i}\lambda_k t)$ in the complex case.

The following, more precise form of Lemma 1.2 for quasiperiodic functions will be useful.

Lemma 2.1. *Let $B: \mathbb{R}^{r+1} \to \mathbb{R}$ be a continuous function from $PP(1)$ qnd $\lambda_0, \ldots, \lambda_r$ non-negative real numbers. Let $A: \mathbb{R} \to \mathbb{R}$ be the quasiperiodic function with at most $r + 1$ frequencies defined by $A(t) = B(\lambda_0 t, \ldots, \lambda_r t)$ and let $\varepsilon > 0$. Then there is a trigonometric polynomial $T: \mathbb{R} \to \mathbb{R}$ of the form $T(t) = \sum\{b_k \exp 2\pi i t(k_0\lambda_0 + \ldots \ldots k_r\lambda_r)); k = (k_0, \ldots, k_r), |k_0| \leqq m, \ldots, |k_r| \leqq m\}$ (m is a natural number) such that $\|A - T\| \leqq \varepsilon$.*

By $QP(n, r)$ we shall denote the set of all quasiperiodic function $A: \mathbb{R} \to \mathrm{Matr}(n)$ with at most $r + 1$ frequencies which satisfy (1.2) and by $QP_{\mathrm{sol}}(n, r)$ the set of all A from $QP(n, r)$, such that X_A is a quasiperiodic function with at most $r + 1$ frequencies

For each $r \in \mathbb{N}$ the inclusions $QP(n, r) \subseteq AP(n)$ and $QP_{\mathrm{sol}}(n, r) \subseteq AP_{\mathrm{sol}}(n)$ hold. By Lemma 1.2 we see that the set $\bigcup\{QP(n, r); r \in \mathbb{N}\}$ is dense in $AP(n)$. The problem whether $AP_{\mathrm{sol}}(n)$ is dense in $AP(n)$ would be therefore solved if we could show that $QP_{\mathrm{sol}}(n, r)$ is dense in $QP(n, r)$ for each $r \in \mathbb{N}$. To this end we introduce the following concept.

Definition 2.1. Let M be a connected Riemannian manifold and $j \in \mathbb{N}$. We say that M has the *homotopy estimation property of the order* j, $M \in EP(j)$, if there is a constant $c = c(M, j) > 0$ such that the following holds:

Let $m \in M$, $g_0: \langle 0, 1\rangle^j \to M$ a function identically equal to m, $L \geqq 1$ and $g: \langle 0, 1\rangle^j \to M$ a function of the class $C^{(2)}$ such that

$$g(x) = m \text{ for all } x \in \partial(\langle 0, 1\rangle^j),$$

$$g \text{ is homotopic with } g_0 \text{ and}$$

$$\left\|\frac{\partial g}{\partial x_i}\right\| \leqq L \quad \text{for} \quad i = 1, \ldots, j\,.$$

Then there is a homotopy $h(\beta, x)$ of functions g and g_0 of the class $C^{(2)}$ satisfying for $i = 1, \ldots, j$

$$\left\|\frac{\partial h}{\partial \beta}\right\| \leqq c\,, \quad \left\|\frac{\partial h}{\partial x_i}\right\| \leqq cL \quad \text{and} \quad \left\|\frac{\partial^2 h}{\partial \beta\, \partial x_i}\right\| \leqq cL\,.$$

(By a homotopy of functions g_1 and $g_2: \langle 0, 1\rangle^j \to M$ (in this order), where $g_1(x) = g_2(x) = m$ for each $x \in \partial(\langle 0, 1\rangle^j)$, we understand a continuous function $h: \langle 0, 1\rangle \times \langle 0, 1\rangle^j \to M$ satisfying

$$h(1, x) = g_1(x) \quad \text{and} \quad h(0, x) = g_2(x) \quad \text{for} \quad x \in \langle 0, 1\rangle^j\,,$$

$$h(\beta, x) = m \quad \text{for} \quad x \in \partial(\langle 0, 1\rangle^j) \quad \text{and} \quad \beta \in \langle 0, 1\rangle\,.$$

The main result of this paper is the following theorem:

Theorem 2.1. *Let $r, n \in \mathbb{N}$. If $SY(n)$ has the homotopy estimation properties of* *orders* 1 *up to* r, *then* $QP_{\text{sol}}(n, r)$ *is dense in* $QP(n, r)$. In the Appendix we show, that for all n in question ($n > 1$ in the complex case and $n > 2$ in the real case) $SY(n)$ have the estimation properties of orders 1 and 2.

3. In this paragraph we shall state several useful lemmas.

Lemma 3.1. *$QP_{\text{sol}}(n, r)$ is dense in $QP(n, r)$ iff the following condition holds: Let $D: \mathbb{R}^{r+1} \to \operatorname{Matr}(n)$ be a function of the class $C^{(2)}$ belonging to $PP(1)$ and satisfying*

$$D(s) + D^*(s) = 0 \quad \text{for} \quad s \in \mathbb{R}^{r+1}, \tag{3.1}$$

$\omega_1, \ldots, \omega_r$ non-negative numbers such that $1, \omega_1, \ldots, \omega_r$ are independent over rational numbers and $C: \mathbb{R} \to \operatorname{Matr}(n)$ the function

$$C(t) = D(t, \omega_1 t, \ldots, \omega_r t) \quad \text{for} \quad t \in \mathbb{R}. \tag{3.2}$$

Then C belongs to the closure of $QP_{\text{sol}}(n, r)$.

Proof. Assume the above condition holds. Let $B: \mathbb{R}^{r+1} \to \mathbb{R}$ be a continuous function from $PP(1)$ and $\lambda_0, \ldots, \lambda_r$ non-negative real numbers. We must show that the function $A: \mathbb{R} \to \operatorname{Matr}(n)$, $A(t) = B(\lambda_0 t, \ldots, \lambda_r t)$ for $t \in \mathbb{R}$, belongs to the closure of $QP_{\text{sol}}(n, r)$.

To B we can find an arbitrarily close function which is from $PP(1)$ and of the class $C^{(2)}$. Hence we can assume that B is of the class $C^{(2)}$. Further, without loss of generality we can assume that $B(s) + B^*(s) = 0$ for each $s \in \mathbb{R}^{r+1}$, $\lambda_0 \neq 0$, and $\lambda_0, \ldots, \lambda_k$ are independent over rational numbers for some $k \in \{0, \ldots, r\}$, while $\lambda_{k+1}, \ldots, \lambda_r$ are their rational combinations,

$$\lambda_i = \sum_{j=0}^{k} a_j^i \lambda_j \quad \text{for} \quad i \in \{k + 1, \ldots, r\}.$$

For $s_0, \ldots, s_k \in \mathbb{R}$ let us define

$$D_1(s_0, \ldots, s_k) = B(s_0, \ldots, s_k, \sum_{j=0}^{k} a_j^{k+1} s_j, \ldots, \sum_{j=0}^{k} a_j^r s_j).$$

Since B belongs to $PP(1)$ and all a_j^i are rational, there is $q \in \mathbb{N}$ such that D_1 belongs to $PP(q)$. Let

$$D(s_0, \ldots, s_r) = D_1(q s_0, \ldots, q s_k) \quad \text{for} \quad s_0, \ldots, s_r \in \mathbb{R},$$

$\omega_1 = \lambda_1/\lambda_0, \ldots, \omega_k = \lambda_k/\lambda_0$ and let $\omega_{k+1}, \ldots, \omega_r$ be non-negative real numbers such that $1, \omega_1, \ldots, \omega_r$ are independent over rational numbers. Then D is of the class $C^{(2)}$, belongs to $PP(1)$ and satisfies (3.1). Because of our assumption the function C defined by (3.2) belongs to the closure of $QP_{\text{sol}}(n, r)$. We have

$$C\left(\frac{\lambda_0}{q} t\right) = D\left(\frac{\lambda_0}{q} t, \frac{\lambda_1}{q} t, \ldots, \frac{\lambda_k}{q} t, \omega_{k+1} t, \ldots, \omega_r t\right) =$$
$$= D_1(\lambda_0 t, \ldots, \lambda_k t) = B(\lambda_0 t, \ldots, \lambda_r t) = A(t) \quad \text{for} \quad t \in \mathbb{R}.$$

It is easily verified that for each real f and each function F from the closure of

$QP_{sol}(n, r)$ also the function $F(ft)$ belongs there. Therefore A is an element of the closure of $QP_{sol}(n, r)$.

We shall use the following notation. Let $a = (a_1, \ldots, a_r)$, $b = (b_1, \ldots, b_r)$ be elements of $\mathbb{R}^r$. Then $a \equiv b \pmod 1$ denotes that $a_i = b_i \pmod 1$ for each $i = 1, \ldots, r$; $a \cdot b = \sum_{i=1}^{r} a_i b_i$; $ua = (ua_1, \ldots, ua_r)$ for $u \in \mathbb{R}$, and $a + b = (a_1 + b_1, \ldots, a_r + b_r)$. Moreover, ω denotes $(\omega_1, \ldots, \omega_r)$, p denotes $(p_1, \ldots, p_r)$ and l denotes $(l_1, \ldots, l_r)$.

Lemma 3.2. *Let $\omega_1, \ldots, \omega_r$ be real numbers such that $1, \omega_1, \ldots, \omega_r$ are independent over rational numbers. Then the set*

$$\{x \in \mathbb{R}^r : x \equiv k\omega \pmod 1;\ k \in \mathbb{Z}\} \text{ is dense in } \mathbb{R}^r .$$

Proof can be found in [CA], ch. III, § 5.

Lemma 3.3. *Let $\omega_1, \ldots, \omega_r$ be irrational numbers and $Q > 0$. Then there are integers $p_1, \ldots, p_r, q$ such that $q > Q$ and*

$$\left|\omega_k - \frac{p_k}{q}\right| \leqq q^{-(r+1)/r} \quad \text{for} \quad k = 1, \ldots, r . \tag{3.3}$$

Proof can again be found in [CA], ch. I, § 5.

Lemma 3.4. *Let $\omega_1, \ldots, \omega_r$ be real numbers such that $1, \omega_1, \ldots, \omega_r$ are independent over rational numbers, $Q > 0$ and $\varepsilon > 0$. Then there are integers $p_1, \ldots, p_r, q, l_1, \ldots, l_r$ and a real $r \times r$-matrix S such that $q > Q$, (3.3) holds, and if σ_k is the vector equal to the k^{th} column of the matrix S, the following holds:*

$$\sigma_k \equiv l_k (p/q) \pmod 1 \quad \text{for} \quad k = 1, \ldots, r , \tag{3.4}$$

$$\varepsilon/4 \leqq \|\sigma_k\| \leqq \varepsilon \quad \text{for} \quad k = 1, \ldots, r , \tag{3.5}$$

$$\|S - (\varepsilon/2) I\| \leqq \varepsilon/4 . \tag{3.6}$$

Proof. By Lemma 3.2 we see that there is an integer k_0 such that the set $\{x \in \mathbb{R}^r : x \equiv k\omega \pmod 1;\ k \in \mathbb{Z}$ and $|k| \leqq k_0\}$ is an $\varepsilon/8r$ – net for $\langle 0, 1\rangle^r$. By Lemma 3.3 we can find integers $p, \ldots, p_r, q$ such that $q \geqq Q$,

$$q \geqq \left(\frac{8rk_0\sqrt{r}}{\varepsilon}\right)^{r/(r+1)}, \quad \text{i.e.} \quad \frac{\sqrt{r}}{q^{(1+1/r)}} \leqq \frac{\varepsilon}{8k_0 r},$$

and such that (3.3) holds. For each $k \in \mathbb{Z}$, $|k| \leqq k_0$ we have

$$\left\|k\omega - k\frac{p}{q}\right\| \leqq \frac{\sqrt{r}k}{q^{(1+1/r)}} \leqq \frac{\varepsilon}{8r} .$$

Therefore the set $\{x \in \mathbb{R}^r : x \equiv k(p/q) \pmod 1;\ k \in \mathbb{Z}$ and $|k| \leqq k_0\}$ is an $\varepsilon/4r$ – net for $\langle 0, 1\rangle^r$. Consequently, it is possible to find vectors $\sigma_k \in \mathbb{R}^r$ and integers l_k,

$$\left\|\sigma_k - \frac{\varepsilon}{2} e_k\right\| \leqq \frac{\varepsilon}{4r} \quad \text{for} \quad k = 1, \ldots, r \tag{3.7}$$

($e_1, \ldots, e_r$ is the usual coordinate system in $\mathbb{R}^r$). We can estimate

$$\frac{\varepsilon}{2} - \frac{\varepsilon}{4r} \leqq \left\|\frac{\varepsilon}{2} e_k\right\| - \left\|\sigma_k - \frac{\varepsilon}{2} e_k\right\| \leqq \|\sigma_k\| \leqq \left\|\frac{\varepsilon}{2} e_k\right\| + \left\|\sigma_k - \frac{\varepsilon}{2} e_k\right\| \leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{4r}$$

for $k = 1, \ldots, r$; therefore (3.5) holds.

Let $x \in \mathbb{R}^r$, $\|x\| = 1$ and $x = (x_1, \ldots, x_r)$. Since $Se_k = \sigma_n$ for every k, (3.7) yields

$$\left\|Sx - \frac{\varepsilon}{2} x\right\| = \left\|\sum_{k=1}^{r} x_k\right\| \left(\sigma_k - \frac{\varepsilon}{2} e_k\right)\Big\| \leqq \frac{\varepsilon}{4}.$$

Therefore also (3.6) holds.

Lemma 3.5. *A real matrix S satisfying* (3.4) *and* (3.6) *is regular and its entries are rational numbers from* (0, 1) *reducible to the common denominator q. Since* Det S *can be written in the form h/q^r with $h \in \mathbb{Z}$, $|h| \leqq r!\,(q-1)^r$, the entries of the matrix S^{-1} are also rational numbers reducible to the common denominator h. Moreover,*

$$\|S^{-1}\| \leqq 4/\varepsilon\,. \tag{3.8}$$

We leave the proof to the reader.

Lemma 3.6. *Suppose a real matrix S and integers $p_1, \ldots, p_r, q, l_1, \ldots, l_r$ satisfy* (3.4) *(σ_k is again the k^{th} column of the matrix S). Let $E: \mathbb{R}^{r+1} \to$* Matr (n) *be a function from $PP(1)$. Then the function $F: \mathbb{R}^{r+1} \to$* Matr (n), *$F(t, \alpha) = E(t, (p/q)\, t + S\alpha)$ for $t \in \mathbb{R}$ and $\alpha \in \mathbb{R}^r$ satisfies*

$$F(t + q, \alpha) = F(t, \alpha) \qquad \text{for} \quad t \in \mathbb{R} \quad \text{and} \quad \alpha \in \mathbb{R}^r\,,$$

$$F(t, \alpha + \beta) = F(t + l\,.\,\beta, \alpha) \quad \text{for} \quad t \in \mathbb{R}\,, \quad \alpha \in \mathbb{R}^r \quad \text{and} \quad \beta \in \mathbb{Z}^r\,.$$

Proof. The first equality is obvious. Let $\beta = (\beta_1, \ldots, \beta_r) \in \mathbb{Z}^r$. By (3.4), $S\beta = \sum_{k=1}^{r} \beta_k \sigma_k \equiv \sum_{k=1}^{r} \beta_k l_k (p/q) \pmod 1$, so that $F(t, \alpha + \beta) = E(t, (p/q)\, t + S\alpha + S\beta) = E(t + \sum_{k=1}^{r} l_k \beta_k, (p/q)\,(t + \sum l_k \beta_k) + S\alpha) = F(t + l\,.\,\beta, \alpha)$, which proves the second equality.

4. Let functions $D(s_0, \ldots, s_r)$, $C(t)$ and numbers $\omega_1, \ldots, \omega_r$ be the same as in Lemma 3.1 and $0 < \eta < 1$. We want to show that there is a function $A \in QP_{\text{sol}}(n, r)$ such that $\|C - A\| \leqq \eta$.

It will be useful to consider separately the trace $\operatorname{Tr}(C(t))$ of the function C, and the function C_1 whose trace identically vanishes and which is defined by the following relations:

$$D_1(s) = D(s) - (1/n) \operatorname{Tr}(D(s))\, I\,, \tag{4.1}$$

$$C_1(t) = D_1(t, \omega t) \quad \text{for} \quad t \in \mathbb{R}\,. \tag{4.2}$$

Since D satisfies (3.1), the function $\operatorname{Tr}(D(s))$ in the real case identically vanishes, i.e. D equals D_1 and C equals C_1. In the complex case, $\operatorname{Tr}(D(s))$ is a function with purely imaginary values. We have $\mathrm{Tr}\,(C(t)) = \mathrm{Tr}\,(D(t, \omega t))$. By Lemma 2.1 there exists a trigonometric polynomial $T: \mathbb{R} \to \mathbb{R}$ such that

$$\|(-i/n)\,\mathrm{Tr}\,(C) - T\| \leqq \eta/3 \tag{4.3}$$

and

$$T(t) = \sum\{b_k \exp(2\pi i t(k_0 + k_1\omega_1 + \ldots + k_r\omega_r))\,;\ k = (k_0, \ldots, k_r) \text{ and } |k_0| \leqq m, \ldots, |k_r| \leqq m\} \text{ for } t \in \mathbb{R}\,. \tag{4.4}$$

We shall shortly write that we take the sum over $|k| \leqq m$.

In both the real and complex cases the function $D_1: \mathbb{R}^{r+1} \to \mathrm{Matr}\,(n)$ is of the class $C^{(2)}$, belongs to $PP(1)$ and

$$C(t) = C_1(t) + (1/n)\,\mathrm{Tr}\,(C(t))\,I \quad \text{for} \quad t \in \mathbb{R}\,, \tag{4.5}$$

$$X_C(t) = \exp(1/n \textstyle\int_0^t \mathrm{Tr}\,C(\sigma)\,d\sigma)\,X_{C_1}(t) \quad \text{for} \quad t \in \mathbb{R}\,, \tag{4.6}$$

$$D_1(s) + D_1^*(s) = 0 \quad \text{for} \quad s \in \mathbb{R}^{r+1}\,, \tag{4.7}$$

$$\mathrm{Tr}\,D_1(s) = 0 \quad \text{for} \quad s \in \mathbb{R}^{r+1}\,, \tag{4.8}$$

$$\|\partial D_1/\partial s_k\| \leqq 2M \quad \text{for} \quad k = 1, \ldots, r\,. \tag{4.9}$$

Later we shall apply to D_1 the coordinate transformation mentioned in Lemma 3.6. Thus we shall work with functions which have the properties introduced in the following definition.

Definition 4.1. Let $l_1, \ldots, l_r, q$ be integers. We shall denote by $P(n, r, l, q)$ the set of all functions f with values in $\mathrm{Matr}\,(n)$ such that $\mathrm{Dom}\,(f) = R \times G$, where G satisfies the condition

$$\text{if} \quad g \in G \quad \text{and} \quad \beta \in \mathbb{Z}^r \quad \text{then also} \quad g + \beta \in G\,,$$

$$f(t, x) + f^*(t, x) = 0 \quad \text{for} \quad (t, x) \in \mathrm{Dom}\,(f)\,. \tag{4.10}$$

$$\mathrm{Tr}\,(f(t, x)) = 0 \quad \text{for} \quad (t, x) \in \mathrm{Dom}\,(f)\,, \tag{4.11}$$

$$f(t + q, x) = f(t, x) \quad \text{for} \quad (t, x) \in \mathrm{Dom}\,(f)\,, \tag{4.12}$$

$$f(t, x + \beta) = f(t + l\,.\,\beta, x) \quad \text{for} \quad (t, x) \in \mathrm{Dom}\,(f)\,, \quad \beta \in \mathbb{Z}^r\,. \tag{4.13}$$

We shall state several lemmas making the meaning of the conditions (4.10)–(4.13) more transparent.

Lemma 4.1. *Let $J \subseteq \mathbb{R}$ be an interval containing* 0 *and $p: J \to \mathrm{Matr}\,(n)$ a continuous function. Then $p(t) + p^*(t) = 0$ and $\mathrm{Tr}\,(p(t)) = 0$ for each $t \in J$ iff $X_p(t) \in SY(n)$ for each $t \in J$.*

Proof. As in Lemma 1.1 we can show that $p(t) + p^*(t) = 0$ for each $t \in \mathbb{R}$ iff $X_p(t) \in Y(n)$ for each $t \in J$. The rest of our assertion follows from the fact that for $t \in J$

$$\mathrm{Det}\,(X_p(t)) = \mathrm{Det}\,(X_p(0)) \exp(\textstyle\int_0^t \mathrm{Tr}\,(p(\tau))\,d\tau) = \exp \int_0^t \mathrm{Tr}\,(p(\tau))\,d\tau\,,$$

and from the continuity of the function $\mathrm{Tr}\,(p)$.

Lemma 4.2. *Let $l_1, \ldots, l_r, q$ be integers and f a continuous function from $P(n, r, l, q)$. Then f belongs to $PP(q)$, the values of X_f are from $SY(n)$ and*

(4.14) $\quad X_f(t + q, x)\, X_f^*(q, x) = X_f(t, x)$ *for* $(t, x) \in \mathrm{Dom}\,(f)$,

(4.15) $\quad X_f(t + l\,.\,\beta, x)\, X_f^*(l\,.\,\beta, x) = X_f(t, x + \beta)$ *for* $(t, x) \in \mathrm{Dom}\,(f)$, $\beta \in \mathbb{Z}^r$.

Lemma 4.3. *Let $l, \ldots, l_r, q$ be integers, f a continuous function from $P(n, r, l, q)$ and $E \subseteq \mathbb{R}^r$, $\mathbb{R} \times E \subseteq \mathrm{Dom}\,(f)$.*

Let E be such that for each $g \in E$ and $\beta \in Z^r$ also $g + \beta \in E$. If $X_f(q, x) = I$ for all $x \in E$ then the function $X_f|_{\mathbb{R} \times E}$ belongs to $PP(q)$.

Proofs of these lemmas are easy and we omit them.

Now we shall need a theorem whose proof is rather lengthy. Therefore we shall only present the result here and postpone its proof to Chapters II and III.

Theorem 4.1. *If $SY(n) \in EP(1) \cap \ldots \cap EP(r)$ then there are numbers $W(n, r) > 1$ and $V(n, r) > 1$ depending only on n and r such that the following holds:*

Let $l_1, \ldots, l_r, q$ be integers, $\xi(t, x_1, \ldots, x_r)$: $\mathbb{R}^{r+1} \to \mathrm{Matr}\,(n)$ a function of the class $C^{(2)}$ belonging to $P(n, r, l, q)$, and $L > 0$ a real number such that

$$q > V(n, r)\frac{1}{L} \quad \textit{and} \quad \left\|\frac{\partial \xi}{\partial x_k}\right\| \leqq L$$

for $k = 1, \ldots, r$. Then there is a function $\varrho(t, x_1, \ldots, x_r)$: $\mathbb{R}^{r+1} \to \mathrm{Matr}\,(n)$ of the class $C^{(2)}$ belonging to $P(n, r, l, q)$ and satisfying

$$\|\varrho - \xi\| \leqq W(n, r)\, L, \quad \left\|\frac{\partial \varrho}{\partial x_k}\right\| \leqq W(n, r)\, L \quad \textit{for} \quad k = 1, \ldots, r \quad \textit{and}$$

$$X_\varrho(q, x) = I \quad \textit{for all} \quad x \in \mathbb{R}^r .$$

5. Let us return to the functions D_1 and C_1 defined by (4.1) and (4.2). Let us pick $Q > 0$ and $\varepsilon > 0$ so that

$$(5.1) \qquad \frac{\eta}{6W(n, r)\, Mr} > \varepsilon, \quad Q > \frac{V(n, r)}{2M\varepsilon r} \quad \text{and} \quad Q > \left(\frac{24r^{5/2}M\, W(n, r)}{\eta}\right)^r .$$

For these Q and ε and for our $\omega_1, \ldots, \omega_r$ there are, by Lemma 3.4, integers $p_1, \ldots, p_r, q, l_1, \ldots, l_r$ and a real matrix $S \in \mathrm{Matr}\,(r)$ such that $q > Q$ and (3.3)–(3.6) hold. Let the function $D_2(t, x_1, \ldots, x_r)$: $\mathbb{R}^{r+1} \to \mathrm{Matr}\,(n)$ be defined as follows:

$$(5.2) \qquad D_2(t, x) = D_1\left(t, \frac{p}{q}\, t + Sx\right) \quad \text{for} \quad t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^r .$$

Then D_2 is of the class $C^{(2)}$; by (4.7), (4.8) and Lemma 3.6, D_2 belongs to $P(n, r, l, q)$ and by (4.9), (3.5) and the equality $Sx = \sum_{k=1}^{r} x_k \sigma_k$ we have

$$(5.3) \qquad \left\|\frac{\partial D_2}{\partial x_k}\right\| \leqq r\, 2M\varepsilon \quad \text{for} \quad k = 1, \ldots, r .$$

Notice that

$$(5.4)\qquad C_1(t) = D_1(t, \omega t) = D_2\left(t, S^{-1}\left(\left(\omega - \frac{p}{q}\right)t\right)\right) \quad \text{for} \quad t \in R\,.$$

Further, by (5.1) and $q \geqq Q$ we have $q \geqq V(n, r)/2M\varepsilon r$.

By Theorem 4.1 there is a function $B(t, x)$: $R^{r+1} \to \text{Matr}\,(n)$ of the class $C^{(2)}$ belonging to $P(n, r, l, q)$ and such that

$$(5.5)\qquad \|D_2 - B\| \leqq 2M\varepsilon r\, W(n, r)\,,$$

$$(5.6)\qquad \left\|\frac{\partial B}{\partial x_k}\right\| \leqq 2M\varepsilon r\, W(n, r) \quad \text{for} \quad k = 1, \ldots, r\,,$$

$$(5.7)\qquad X_B(q, x) = I \quad \text{for} \quad x \in R^r\,.$$

By Lemmas 4.2 and 4.3 the functions B and X_B belong to $PP(q)$. Therefore also the functions $\partial X_B/\partial x_k$ for $k = 1, \ldots, k$ belong to $PP(q)$. The periodicty of these functions in t and (5.6) imply

$$(5.8)\qquad \left\|\frac{\partial X_B}{\partial x_k}\right\| \leqq 2M\varepsilon r\, W(n, r)\, q \quad \text{for} \quad k = 1, \ldots, r\,.$$

Let us define the function A_1: $R \to \text{Matr}\,(n)$ by which we want to approximate the function C_1, as follows:

$$(5.9)\qquad X_{A_1}(t) = X_B\left(t, S^{-1}\left(\left(\omega - \frac{p}{q}\right)t\right)\right) \quad \text{for} \quad t \in R\,,$$

$$(5.10)\qquad A_1(t) = \left[\frac{\mathrm{d}}{\mathrm{d}t} X_{A_1}(t)\right] X^*_{A_1}(t) \quad \text{for} \quad t \in R\,.$$

Denoting $y = (y_1, \ldots, y_r) = S^{-1}(\omega - p/q)$ we can rewrite (5.10) as

$$(5.11)\qquad A_1(t) = B(t, yt) + \sum_{k=1}^{r} y_k \left[\frac{\partial}{\partial x_k} X_B(t, yt)\right] X^*_B(t, yt) \quad \text{for} \quad t \in R\,.$$

Let us define in the real case $A = A_1$ and in the complex case $A = A_1 + \mathrm{i}TI$, where T is a real trigonometric polynomil with the properties (4.3) and (4.4). In the real case we have $X_A = X_{A_1}$ and in the complex case

$$X_A(t) = \exp\,(\mathrm{i} \textstyle\int_0^t T(\tau)\, \mathrm{d}\tau)\, X_{A_1}(t) =$$

$$= \exp\left(\sum_{|k| \leqq m} b_k \frac{\exp\,(2\pi \mathrm{i} t(k_0 + \omega_1 k_1 + \ldots + \omega_r k_r)) - 1}{2\pi(k_0 + \omega_1 k_1 + \ldots + \omega_r k_r)}\right) X_B(t, yt)\,.$$

We want to show that A belongs to $QP_{\text{sol}}(n, r)$ and $\|A - C\| \leqq \eta$.

Since B belongs to $P(n, r, l, q)$, by Lemma 4.2 the values of the function X_B are from $SY(n)$. By (5.9) also the values of the function X_{A_1} are from $SY(n)$; therefore by Lemma 4.1 $A_1(t) + A_1^*(t) = 0$ for each $t \in R$. Considering moreover that the values of T are real we see that A satisfies (1.2) in both the real and complex cases.

Define functions $F_1, F_2, F_3, F_4: \mathbb{R}^{r+1} \to \text{Matr}(n)$ as follows:

$$F_1(t, x) = B\left(t, S^{-1}\left(x - \frac{p}{q}t\right)\right) +$$

$$+ \sum_{k=1}^{r} y_k \left[\frac{\partial}{\partial x_k} X_B\left(t, S^{-1}\left(x - \frac{p}{q}t\right)\right)\right] X_B^*\left(t, S^{-1}\left(x - \frac{p}{q}t\right)\right),$$

$$F_2(t, x) = \left(\sum_{|k| \leq m} ib_k \exp\left(2\pi i(k_0 t + k_1 x_1 + \ldots + k_r x_r)\right)\right) I,$$

$$F_3(t, x) = X_B\left(t, S^{-1}\left(x - \frac{p}{q}t\right)\right),$$

$$F_4(t, x) = \exp\left(\sum_{|k| \leq m} b_k \frac{\exp\left(2\pi i(k_0 t + k_1 x_1 + \ldots + k_r x_r)\right) - 1}{2\pi(k_0 + k_1\omega_1 + \ldots + k_r\omega_r)}\right) I$$

for $t \in \mathbb{R}$ and $x = (x_1, \ldots, x_r) \in \mathbb{R}^r$.

The functions B, X_B and $(\partial/\partial x_k) X_B$ belong to $PP(q)$. By Lemma 3.5 the matrix hS^{-1} has integer entries; therefore the functions F_1 and F_3 are periodic with the period q^{2r} in $x_1, \ldots, x_r$ and with the period h in t. The functions F_2 and F_4 obviously belong to $PP(1)$. Therefore all the functions $F_1, F_3, F_1 + F_2$ and F_3F_4 belong to $PP(h)$. Moreover, they are continuous. In the real case we have $A(t) = (t, \omega t)$ and $X_A(t) = F_3(t, \omega t)$ and in the complex case $A(t) = F_1(t, \omega t) + F_2(t, \omega t)$ and $X_A(t) = F_3(t, \omega t) F_4(t, \omega t)$ for $t \in \mathbb{R}$.

Consequently, A and X_A are quasiperiodic functions with at most $r + 1$ frequencies, i.e. $A \in QP_{sol}(n, r)$.

By (3.3) and (3.8) we have $|y_k| \leq \|y\| \leq 4\varepsilon^{-1} r^{1/2} q^{-(r+1)/r}$ for $y = S^{-1}(\omega - p/q)$, where $k = 1, \ldots, r$. The norm of the function X_B is bounded by 1 since its values belong to $SY(n)$. By (5.4), (5.11) and (5.8) we can estimate $\|C_1 - A_1\| \leq \|D_2 - B\| + 8r^{5/2} M\, W(n, r)\, q^{-1/r}$, therefore by (5.5), (5.1) and $q > Q$ we have $\|C_1 - A_1\| \leq \frac{2}{3}\eta$. In the real case, this means $\|A - C\| \leq \frac{2}{3}\eta < \eta$. In the complex case we can conclude $\|A - C\| \leq \eta$ by considering, moreover, (4.3) and (4.5).

CHAPTER II

1. Now we shall describe a method for extending functions defined on certain subsets of $\mathbb{R}^r$ and with values in $\text{Matr}(n)$ to functions defined on the whole $\mathbb{R}^r$, where the bounds of norms of derivatives of the original function are preserved except for multiplying by a constant. We shall need this method for the proof of Theorem I.4.1.

We shall use the following notation:

$\# u$ is the number of elements of the set u,

$\mathscr{P}(r)$ denotes the set of all subsets of $\{1, \ldots, r\}$,

$\mathscr{P}_j(r)$ is the set of all subsets of $\{1, \ldots, r\}$ which have j elements,

$\mathscr{P}_{\geqq j}(r)$ is the set of all subsets of $\{1, \ldots, r\}$ which have at least j elements; $\emptyset$ denotes the empty set,

$\{x\}$ and $[x]$ denote the fractional and the integer part of $x \in \mathbb{R}$, i.e. $[x] \in \mathbb{Z}$ and $0 \leqq \{x\} < 1$; for $x = (x_1, \ldots, x_r) \in \mathbb{R}^r [x] = ([x_1], \ldots, [x_r])$ and $\{x\} = (\{x_1\}, \ldots \ldots, \{x_r\})$; $x = [x] + \{x\}$.

For $x = (x_1, \ldots, x_r)$ a set $U \subseteq R^r$ is called an *x_i-neighbourhood* of x iff there is $\varepsilon_0 > 0$ such that

$$U = \{(x_1, \ldots, x_i + \varepsilon, \ldots, x_r) ; \quad -\varepsilon_0 < \varepsilon < \varepsilon_0 .$$

In the natural way we define also the right and left x_i-neighbourhoods of x.

Let us define $z: \mathbb{R}^r \times \mathbb{R}^r \times \mathscr{P}(r) \to \mathbb{R}^r$ to be the function such that for $x = (x_1, \ldots, x_r)$ and $y = (y_1, \ldots, y_r)$ the i^{th} coordinate of $z(x, y, a)$, i.e. $z_i(x, y, a)$, equals y_i if $i \in a$ and equals x_i otherwise. Fixing y and a, $z(x, y, a)$ as a function of x is defined on $\mathbb{R}^r$, is of the class $C^{(\infty)}$ and the derivative $\partial z_i / \partial x_k$ is either identically equal to 0 if $i \neq k$ or $i = k \in a$, or identically equal to 1 for $i = k \notin a$.

Further, let us define $a: \mathbb{R}^r \to \mathscr{P}(r)$ to be the function such that $a(x) = \{i; x_i \in \mathbb{Z}\}$ for $x = (x_1, \ldots, x_r)$. For a natural number $j \leqq r$ let $S_j^r = \{x \in \mathbb{R}^r;\ a(x) \geqq j\}$ and $S_{r+1}^r = \emptyset$; (i.e. S_j^r is the set of all $x \in \mathbb{R}^r$ with at least j integer coordinates).

Throughout the whole chapter $\varphi: \mathbb{R} \to \langle 0, 1 \rangle$ will be an even function with a continuous second derivative, non-increasing on $\langle 0, \infty)$, and $F \geqq 1$ a constant such that (see Fig. 1)

(1.1) $$\varphi(t) + \varphi(1 - t) = 1 \quad \text{for} \quad t \in \langle 0, 1 \rangle ,$$

(1.2) $$\varphi(t) = 1 \quad \text{for} \quad |t| \leqq \tfrac{1}{10} \quad \text{and} \quad \varphi(t) = 0 \quad \text{for} \quad |t| \geqq \tfrac{9}{10} ,$$

(1.3) $$\left\| \frac{d\varphi}{dt} \right\| \leqq F ,$$

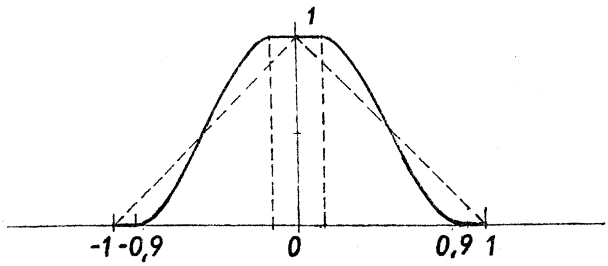

Fig. 1

and $f: \mathbb{R}^r \times \mathbb{Z}^r \to \mathbb{R}$ will be the function

(1.4) $$f(x, \alpha) = \prod_{i=1}^{r} \varphi(\alpha_i - x_i)$$

for $x = (x_1, \ldots, x_r) \in \mathbb{R}^r$ and $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$.

Lemma 1.1. *Let $x \in \mathbb{R}^r$ and $\alpha \in \mathbb{Z}^r$. Then $f(\alpha, \alpha) = 1$. If $f(x, \alpha) \neq 0$ then $\alpha_i = x_i$ for $i \in a(x)$ and $\alpha_i \in \{[x_i], [x_i] + 1\}$ for $i \notin a(x)$.*

Proof. $f(\alpha, \alpha) = 1$ because $\varphi(0) = 1$. Suppose there is an i either in $a(x)$ and such that $\alpha_i \neq x_i$ or in the complement of $a(x)$ and such that $\alpha_i \notin \{[x_i], [x_i] + 1\}$. Then for this i we have $|\alpha_i - x_i| \geqq 1$, therefore $\varphi(\alpha_i - x_i) = 0$ and $f(x, \alpha) = 0$.

Lemma 1.2. *Let $x \in \mathbb{R}^r$ and $\alpha, \beta \in \mathbb{Z}^r$. Then $f(x, \alpha) = f(x - \beta, \alpha - \beta)$.* This lemma is obvious.

2. We can extend a function $b: \mathbb{Z}^r \to \operatorname{Matr}(n)$ to the whole $\mathbb{R}^r$ putting $\hat{b}(x) = \sum_{\alpha \in \mathbb{Z}^r} f(x, \alpha)\, b(\alpha)$. Lemma 1.1 guarantees that the sum always has at most 2^r non-zero summands and that for $x \in \mathbb{Z}^r$, $\hat{b}(x) = b(x)$.

We shall generalize this method in order to be able to extend functions with domains S_j^r for each $j \in \{1, \ldots, r\}$. First we define for a function $b: S_j^r \to \operatorname{Matr}(n)$

(2.1) $$\tilde{b}(x) = \sum_{\alpha \in \mathbb{Z}^r} f(x, \alpha) \sum_{a \in \mathcal{P}_j(r)} b(z(x, \alpha, a)) \quad \text{for} \quad x \in \mathbb{R}^r .$$

By Lemma 1.1 we have again finitely many non-zero summands. Obviously, $\tilde{b}$ is continuous if b is continuous and $\tilde{b}$ is identically 0 if b is such.

Let us present some examples. For $r = j$ we have $S_j^r = \mathbb{Z}^r$ and the definition of $\tilde{b}$ coincides with the above definition of $\hat{b}$. Therefore in this case $\tilde{b}$ is ab extension of b. Let $b: S_1^2 \to \operatorname{Matr}(n)$. S_1^2 is the set $(\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R})$ and for each $x = (x_1, x_2) \in \mathbb{R}^2$ we have

$$\begin{aligned} \tilde{b}(x) = {} & \varphi(\{x_1\})\, \varphi(\{x_2\})\, (b([x_1], x_2) + b(x_1, [x_2])) + \\ & + \varphi(\{x_1\})\, \varphi(1 - \{x_2\})\, (b([x_1], x_2) + b(x_1, [x_2] + 1)) + \\ & + \varphi(1 - \{x_1\})\, \varphi(\{x_2\})\, (b([x_1] + 1, x_2) + b(x_1, [x_2])) + \\ & + \varphi(1 - \{x_1\})\, \varphi(1 - \{x_2\})\, (b([x_1] + 1, x_2) + b(x_1, [x_2] + 1)) . \end{aligned}$$

Suppose moreover that b is equal to 0 on $\mathbb{Z}^2$. Let $x \in S_1^2$, say $x_1 \in \mathbb{Z}$. Then $\varphi(1 - \{x_1\}) = \varphi(1) = 0$, $\varphi(\{x_1\}) = \varphi(0) = 1$ and $b(x_1, [x_2]) = b(x_1, [x_2] + 1) = 0$; therefore using the property (1.1) of the function φ we get $\tilde{b}(x) = (\varphi(\{x_2\}) + \varphi(1 - \{x_2\}))\, b([x_1], x_2) = b(x)$. In this case $\tilde{b}$ is again an extension of b.

Theorem 2.1. *Let $b: S_j^r \to \operatorname{Matr}(n)$, $j \in \{1, \ldots, r\}$, $L > 0$, $K > 0$ and $m \in \{1, 2\}$.*

a) If b is equal to 0 on S_{j+1}^r then $\tilde{b}$ extends b, $\tilde{b} \supseteq b$.

b) If b has continuous m^{th} derivatives w.r.t. its domain and $\|b\| \leqq L$, $\|\partial b / \partial x_i\| \leqq K$ for $i = 1, \ldots, r$, then $\tilde{b}$ is of the class $C^{(m)}$ and

(2.2) $$\|\tilde{b}\| \leqq 2^r \binom{r}{j} L ,$$

(2.3) $$\left\| \frac{\partial \tilde{b}}{\partial x_i} \right\| \leqq 2^r \binom{r}{j} (K + LF) \quad \textit{for} \quad i = 1, \ldots, r .$$

For the proof of this theorem we need the following lemma:

Lemma 2.1. For each $x \in R^r$

$$\sum_{\alpha \in Z^r} f(x, \alpha) = 1 . \tag{2.4}$$

Proof. If $r = 1$ then the left hand side of (2.4) equals $\varphi(\{x_1\}) + \varphi(1 - \{x_1\})$ which, by (1.1), equals 1. Assume that $r > 1$ and (2.4) holds for $r - 1$. We can rewrite the left hand side of (2.4) as

$$(\varphi(\{x_r\}) + \varphi(1 - \{x_r\}))\left(\sum_{\alpha \in Z^{r-1}} \left(\prod_{i=1}^{r-1} \varphi(\alpha_i - x_i)\right)\right).$$

By (1.1) again and the induction hypothesis this equals 1.

Proof of Theorem 2.1. a) Assume b is equal to 0 on S^r_{j+1}. Let $x \in S^r_j$. If x belongs to S^r_{j+1} then for each $a \in \mathscr{P}_j(r)$ and $\alpha \in Z^r$ the vector $z(x, \alpha, a)$ belongs to S^r_{j+1}, therefore $b(z(x, \alpha, a)) = 0$ and $\tilde{b}(x) = 0 = b(x)$. If x is not an element of S^r_{j+1} then $a(x) \in \mathscr{P}_j(r)$ and for every $\alpha \in Z^r$

$$\sum_{a \in \mathscr{P}_j(r)} b(z(x, \alpha, a)) = b(z(x, \alpha, a(x))) ,$$

since for $a \neq a(x)$, $z(x, \alpha, a)$ belongs to S^r_{j+1} and thus $b(z(x, \alpha, a)) = 0$. If α is such that there is $i \in a(x)$ with $\alpha_i \neq x_i$ then $|\alpha_i - x_i| \geqq 1$ and $\varphi(\alpha_i - x_i) = 0$. If there is no i no $a(x)$ with $\alpha_i \neq x_i$ then $z(x, \alpha, a(x)) = x$. Thus $\tilde{b}(x) = \sum_{\alpha \in Z^r} \left(\prod_{i=1}^{r} \varphi(\alpha_i - x_i)\right) .$ $. \, b(x)$. By (2.4) this equals $b(x)$. We proved the assertion a.

b) Let $\alpha \in Z^r$, $a \in \mathscr{P}_j(r)$. Let us consider $b(z(x, \alpha, a))$ as a function of x. Let $x \in R^r$. If $i \notin a$ then $z(x, \alpha, a)$ belongs to S^r_j and some x_i-neighbourhood of x is included in S^r_j; therefore $z(x, \alpha, a)$ belongs to the domain of $\partial b/\partial x_i$. We easily see that

$$\frac{\partial}{\partial x_2} [b(z(x,/\alpha, a))] = \frac{\partial b}{\partial x_i} (z(x, \alpha, a)) .$$

For $i \in a$ the function $z(\cdot, \alpha, a)$ is constant on some x_i-neighbourhood of x; therefore $(\partial/\partial x_i) [b(z(x, \alpha, a))] = 0$.

Similarly we can see that in the case $m = 2$, $(\partial^2/\partial x_i \, \partial x_k) [b(z(x, \alpha, a))]$ equals either $(\partial^2 b/\partial x_i \, \partial x_k) (z(x, \alpha, a))$ if i, k are not elements of a, or 0 if i or k is an element of a. Since the function f is of the class $C^{(2)}$, we see from (2.1) that $\tilde{b}$ is of the class $C^{(m)}$.

From the estimates of norms of b and its derivatives we have

$$\Big\| \sum_{a \in \mathscr{P}_j(r)} b(z(x, \alpha, a)) \Big\| \leqq \binom{r}{j} L \quad \text{for} \quad x \in R^r , \quad \alpha \in Z^r , \tag{2.5}$$

$$\left\| \frac{\partial}{\partial x_i} \left(\sum_{a \in \mathscr{P}_j(r)} b(z(x, \alpha, a)) \right) \right\| \leqq \binom{r}{j} K \quad \text{for} \quad x \in R^r, \quad \alpha \in Z^r \text{ and } i = 1, \dots, r . \tag{2.6}$$

The next two estimates follow from (1.2), (1.4) and $|\varphi| \leqq 1$.

$$|f(x, \alpha)| \leqq 1 \quad \text{for} \quad x \in R^r , \quad \alpha \in Z^r , \tag{2.7}$$

$$\left| \frac{\partial f}{\partial x_i} (x, \alpha) \right| \leqq F \quad \text{for} \quad x \in R^r , \quad \alpha \in Z^r \quad \text{and} \quad i = 1, \dots, r . \tag{2.8}$$

Let $x \in \mathbb{R}^r$, $U = \{y \in \mathbb{R}^r;\ \|x - y\| \leqq \frac{1}{10}\}$. For $y \in U$ the sum (2.1) defining $\tilde{b}(y)$ may contain only such summands $f(y, \alpha) \sum_{a \in \mathscr{P}_j(r)} b(z(y, \alpha, a))$ different from 0, for which $\alpha_k = [x_k]$ or $\alpha_k = [x_k] + 1$ for each $k = 1, \ldots, r$ (otherwise $|y_k - \alpha_k| \geqq |x_k - \alpha_k| - |x_k - y_k| \geqq \frac{9}{10}$ for some k, i.e. $f(y, \alpha) = 0$. Therefore when estimating $\|b(x)\|$ and $\|(\partial b/\partial x_i)(x)\|$, $i = 1, \ldots, r$, we can consider only these summands. There are 2^r of them. Thus we get from (2.1), (2.5) and (2.7) the estimate (2.2) and from (2.1), (2.5)–(2.8) using the product rule the estimate (2.3).

3. Now we shall construct extensions of functions b defined on S_j^r and not necessarily equal to 0 on S_{j+1}^r ($j = 1, \ldots, r$). Let $b\colon S_j^r \to \operatorname{Matr}(n)$. We define by induction functions $b_i\colon S_i^r \to \operatorname{Matr}(n)$ for $j \leqq i \leqq r$:

$$(3.1) \qquad b_r = b|_{Z^r} \quad\text{and}\quad b_i = b|_{S_i^r} - \Big(\sum_{k=i+1}^{r} \tilde{b}_k\Big)\Big|_{S_i^r} \quad\text{for}\quad r > i \geqq j\,.$$

We shall show by induction that the function $\tilde{b}_i$ extends b_i, $\tilde{b}_i \supseteq b_i$ and

$$(3.2) \qquad \Big(\sum_{k=i}^{r} \tilde{b}_k\Big)\Big|_{S_i^r} = b|_{S_i^r}\,.$$

For $i = r$ this follows from Theorem 2.1 a). Suppose that for some i, $r \geqq i \geqq j$, $\tilde{b}_i \supseteq b_i$ holds. Then $\tilde{b}_i|_{S_i^r} = b_i$ and (3.2) follows from (3.1). Moreover, if $i > j$ we have by (3.2) and the definition of b_{i-1} ($b_{i-1} = b|_{S_{i-1}^r} - (\sum_{k=i}^{r} \tilde{b}_k)|_{S_{i-1}^r}$) that b_{i-1} is equal to 0 on S_i^r, and therefore by Theorem 2.1 a, $\tilde{b}_{i-1} \supseteq b_{i-1}$.

We proved our claim.

Define $\hat{b} = \sum_{k=j}^{r} \tilde{b}_k$ where $b_r, \ldots, b_j$ are defined by (3.1).

Lemma 3.1. *If b is equal to 0 on S_{j+1}^r then $\hat{b} = \tilde{b}$.*

Proof. From (3.1) we see that for $j < i \leqq r$ the functions b_i are identically 0, therefore $\hat{b} = \tilde{b}_j = \tilde{b}$.

Let us define by induction constants $K(r, j)$ for $j = r, \ldots, 0$:

$$(3.3) \qquad \begin{aligned} &K(r, r) = 2^r\,,\\ &K(r, j) = K(r, j + 1) + 2^r\binom{r}{j}(1 + K(r, j + 1)) \quad\text{for}\quad r > j > 0\,,\\ &K(r, 0) = 0\,.\end{aligned}$$

Theorem 3.1. *Let $b\colon S_j^r \to \operatorname{Matr}(n)$, $j = 1, \ldots, r$, $L > 0$, $K > 0$ and $m \in \{1, 2\}$. Then the above defined function $\hat{b}$ is an extension of b. If b has continuous m^{th} derivatives w.r.t. S_j^r and $\|b\| \leqq L$, $\|\partial b/\partial x_i\| \leqq K$ for $i = 1, \ldots, r$ hold, then $\hat{b}$ is of class $C^{(m)}$ and the following holds:*

$$(3.4) \qquad \|\hat{b}\| \leqq K(r, j)\, L,$$

$$(3.5) \qquad \left\|\frac{\partial \hat{b}}{\partial x_i}\right\| \leqq K(r, j)\,(K + (r - j + 1)\, LF) \quad\text{for}\quad i = 1, \ldots, r\,.$$

Proof. The fact that $\hat{b}$ is an extension of b follows from (3.2) for $i = j$. The rest will be proved by induction. The case when $j = r$ is solved by Theorem 2.1 since by Lemma 3.1 $\hat{b} = \tilde{b}$ when $j = r$. Let $1 \leqq j < r$ and assume that the theorem holds for $j + 1$. Define $c = b|_{S_{j+1}^r}$. Then c has continuous m^{th} derivatives w.r.t. S_{j+1}^r, $\|c\| \leqq L$ and $\|\partial c/\partial x_i\| \leqq K$ for $i = 1, \ldots, r$. Let $c_{j+1}, \ldots, c_r$ be defined by (3.1). Obviously, $c_i = b_i$ for $j + 1 \leqq i \leqq r$, and so $\hat{c} = \sum_{k=j+1}^{r} \tilde{b}_k$. By the induction hypothesis $\hat{c}$ is of the class $C^{(m)}$, $\|\hat{c}\| \leqq K(r, j + 1) L$ and for $i = 1, \ldots, r$, $\|\partial \hat{c}/\partial x_i\| \leqq$ $\leqq K(r, j + 1)\ (K + (r - j) LF)$. By the definition of b_j, $b_j = b - \hat{c}|_{S_j^r}$. Consequently, b_j has continuous m^{th} derivatives w.r.t. S_j^r and

$$\|b_j\| \leqq L(K(r, j + 1) + 1),$$

$$\left\|\frac{\partial b_j}{\partial x_i}\right\| \leqq K + K(r, j + 1)(K + (r - j) LF) \leqq (K + (r - j) LF)(K(r, j + 1) + 1)$$

for $i = 1, \ldots, r$.

By Theorem 2.1, $\tilde{b}_j$ is of the class $C^{(m)}$ and

$$\|\tilde{b}_j\| \leqq 2^r \binom{r}{j} (1 + K(r, j + 1)) L,$$

$$\left\|\frac{\partial \tilde{b}_j}{\partial x_i}\right\| \leqq 2^r \binom{r}{j} (1 + K(r, j + 1))(K + (r - j + 1) LF) \quad \text{for} \quad i = 1, \ldots, r.$$

Since $\hat{b} = \hat{c} + \tilde{b}_j$, we get (3.4) and (3.5). Since, moreover, both $\hat{c}$ and $\tilde{b}_j$ are of the class $C^{(m)}$, also $\hat{b}$ is of the class $C^{(m)}$.

The theorem is proved.

4. We shall often work with functions which have the properties described in the following definition.

Definition 4.1. Let b be a function, Dom $(b) \subseteq \mathbb{R}^r$, and $0 < \varepsilon \leqq \frac{1}{10}$. We say that b is *coordinatewise constant* in the ε-neighbourhood of integers, $b \in KZ(\varepsilon)$, if for all $x = (x_1, \ldots, x_r)$, $y = (y_1, \ldots, y_r)$ from the domain of b, which satisfy $|x_i - y_i| \leqq \varepsilon$ for all $i \in a(x)$ and $x_i = y_i$ for all $i \notin a(x)$, the equality $b(x) = b(y)$ holds.

For example, let b be defined on $\mathbb{R}^2$. Then b belongs to $KZ(\frac{1}{10})$ iff for all $\alpha_1, \alpha_2 \in \mathbb{Z}$

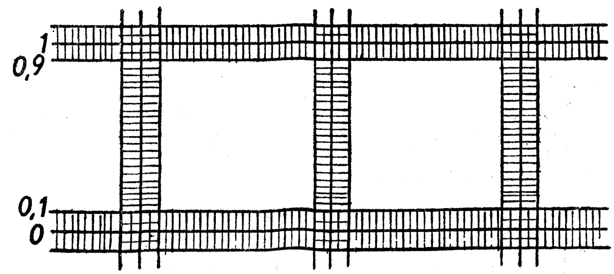

Fig. 2

and $x_1, x_2 \in R$ the following holds:
if $|x_1 - \alpha_1| \leqq \frac{1}{10}$ then $b(x_1, x_2) = b(\alpha_1, x_2)$ and symmetrically
if $|x_2 - \alpha_2| \leqq \frac{1}{10}$ then $b(x_1, x_2) = b(x_1, \alpha_2)$.
Further, if both $|x_1 - \alpha_1|$ and $|x_2 - \alpha_2|$ are smaller than $\frac{1}{10}$ then $b(x_1, x_2) = b(\alpha_1, \alpha_2)$. (See Fig.2)

Lemma 4.1. *Let b be a function,* $\mathrm{Dom}(b) \subseteq R^r$, $0 \leqq \varepsilon \leqq \frac{1}{10}$ *and $b \in KZ(\varepsilon)$. If x and y are vectors from the domain of b such that for each i either $x_i = y_i$ or there is an integer m_i for which $|x_i - m_i| \leqq \varepsilon$ and $|y_i - m_i| \leqq \varepsilon$, then $b(x) = b(y)$.*

The proof is easy and we omit it.

Lemma 4.2. *Let $0 < \varepsilon \leqq \frac{1}{10}$ and $0 < j \leqq r$. Let $b: S^r_j \to$* Matr (n) *be a function from $KZ(\varepsilon)$. Then also $\tilde{b}$ (defined by (2.1)) belongs to $KZ(\varepsilon)$.*

Proof. Let x, y be vectors from R^r such that $|x_i - y_i| \leqq \varepsilon$ for all $i \in a(x)$ and $x_i = y_i$ for all $i \notin a(x)$. We need to show that $\tilde{b}(x) = \tilde{b}(y)$.

For any $\alpha \in Z^r$ and $a \in \mathscr{P}_j(r)$ the vectors $z(x, \alpha, a)$ and $z(y, \alpha, a)$ belong to $S^r_j = \mathrm{Dom}(b)$. Since b is in $KZ(\varepsilon)$, $b(z(x, \alpha, a)) = b(z(y, \alpha, a))$ holds.

Let $\alpha = (\alpha_1, \ldots, \alpha_r)$. For $i \notin a(x)$ obviously $\varphi(\alpha_i - x_i) = \varphi(\alpha_i - y_i)$ and for $i \in a(x)$ either $|\alpha_i - x_i| \geqq 1$ and therefore $|\alpha_i - y_i| \geqq \frac{9}{10}$, or $\alpha_i = x_i$ and therefore $|\alpha_i - y_i| \leqq \frac{1}{10}$. By (1.2) in both cases $\varphi(\alpha_i - x_i) = \varphi(\alpha_i - y_i)$. Consequently, $f(x, \alpha) = f(y, \alpha)$.

These two facts imply $\tilde{b}(x) = \tilde{b}(y)$.

Let b, c be functions with values in Matr (n), $\mathrm{Dom}(b) \subseteq R^r$, $\mathrm{Dom}(c) \subseteq R^r$, such that b and c belong to $KZ(\varepsilon)$. Then also $b \pm c$ (defined on the intersection of the domains of b and c) and the functions which we get by restricting b or c to any subset of their domains, belong to $KZ(\varepsilon)$.

This observation and Lemma 4.2 imply

Theorem 4.1. *Let $0 < \varepsilon \leqq \frac{1}{10}$ and $0 < j \leqq r$. Let $b: S^r_j \to$* Matr (n) *be a function which belongs to $KZ(\varepsilon)$. Then also the function $\hat{b}$ (defined at the beginning of* § 3) *belongs to $KZ(\varepsilon)$.*

5. Further, we shall need a different formula for $\tilde{b}$. For $u \in \mathscr{P}(r)$ let $E(u) = \{\alpha = (\alpha_1, \ldots, \alpha_r);\ \alpha_i = 0 \text{ for } i \in u \text{ and } \alpha_i \in \{0, 1\} \text{ for } i \notin u\}$.

Lemma 5.1. *Let $b: S^r_j \to$* Matr (n), $j \in \{1, \ldots, r\}$. *Then*

$$\tilde{b}(x) = \sum_{\alpha \in E(a(x))} \sum_{u \in \mathscr{P}_j(r)} f(\{x\}, \alpha)\, b(z(x, \alpha + [x], u))\,.$$

Proof. Using (2.1) and Lemmas 1.2, 1.1 we get

$$\tilde{b}(x) = \sum_{\alpha \in Z^r} \sum_{u \in \mathscr{P}_j(r)} f(x, \alpha)\, b(z(x, \alpha, u)) =$$
$$= \sum_{\alpha \in Z^r} \sum_{u \in \mathscr{P}_j(r)} f(\{x\}, \alpha)\, b(z(x, \alpha + [x], u)) =$$
$$= \sum_{\alpha \in E(a(x))} \sum_{u \in \mathscr{P}_j(r)} f(\{x\}, \alpha)\, b(z(x, \alpha + [x], u))\,.$$

Lemma 5.2. *There are functions* d_j: $\langle 0, 1)^r \times \{0, 1\}^r \times \mathscr{P}_{\geqq j}(r) \to \mathbb{R}$ *for* $j = 1, \ldots, r$ *such that if* b: $S^r_j \to \mathrm{Matr}\,(n)$, *then the following holds for* $x \in \mathbb{R}^r$:

$$\hat{b}(x) = \sum_{\alpha \in E(a(x))} \sum_{u \in \mathscr{P}_{\geqq j}(r)} d_j(\{x\}, \alpha, u)\, b(z(x, \alpha + [x], u)) \,. \tag{5.1}$$

The proof is rather long and only technically difficult. We omit it.

Now we shall extend functions with parameters. Let b: $\mathbb{R} \times S^r_j \to \mathrm{Matr}\,(n)$, $j = 1, \ldots, r$. For $t \in \mathbb{R}$ let us denote by b_t: $S^r_j \to \mathrm{Matr}\,(n)$ the function defined by $b_t(x) = b(t, x)$ for $x \in S^r_j$. Define

$$\hat{b}(t, x) = \hat{b}_t(x) \quad \text{for} \quad t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^r \,. \tag{5.2}$$

By the previous lemma,

$$\hat{b}(t, x) = \sum_{\alpha \in E(a(x))} \sum_{u \in \mathscr{P}_{\geqq j}(r)} d_j(\{x\}, \alpha, u)\, b(t, z(x, \alpha + [x], u)) \,. \tag{5.3}$$

Theorem 5.1. *Let* $b(t, x)$: $\mathbb{R} \times S^r_j \to \mathrm{Matr}\,(n)$, $L > 0$, $K > 0$. *The function* $\hat{b}$ *defined by* (5.2) *is an extension of* b. *If* b *has continuous second derivatives w.r.t.* $\mathbb{R} \times S^r_j$ *and* $\|b\| \leqq L$, $\|\partial b/\partial x_i\| \leqq K$ *for* $i = 1, \ldots, r$, *then* $\hat{b}$ *is of the class* $C^{(2)}$ *and the following estimates hold*:

$$\|\hat{b}\| \leqq K(r, j)\, L \,, \tag{5.4}$$

$$\left\| \frac{\partial \hat{b}}{\partial x_i} \right\| \leqq K(r, j)\,(K + (r - j + 1)\, LF) \quad \text{for} \quad i = 1, \ldots, r \,. \tag{5.5}$$

Proof. From (5.3) we can see that there exists a continuous second derivative of the function $\hat{b}$ w.r.t. t, and also that $\partial \hat{b}/\partial t = (\partial b/\partial t)^\wedge$. Theorem 5.1 follows from Theorem 3.1 applied to the functions b_t and for $\partial b/\partial t$.

Theorem 5.2. *Let* $b(t, x)$: $\mathbb{R} \times S^r_j \to \mathrm{Matr}\,(n)$ *and let* $l_1, \ldots, l_r, q$ *be integers. If* b *belongs to* $P(n, r, l, q)$ *then also* $\hat{b}$ *belongs there.*

Proof. (I.4.10), (I.4.11) and (I.4.12) for $\hat{b}$ are easily verified by using (5.3) and the corresponding properties of b.

Let $t \in \mathbb{R}$, $x \in \mathbb{R}^r$ and $\beta \in \mathbb{Z}^r$. We have $a(x + \beta) = a(x)$, $\{x + \beta\} = \{x\}$, $[x + \beta] = [x] + \beta$ and for each $u \in \mathscr{P}_{\geqq j}(r)$, $z(x + \beta, \alpha + [x + \beta], u) = z(x, \alpha + [x], u) + \beta$. Therefore, considering (5.3) and the property (I.4.13) of b we see that $\hat{b}(t, x + \beta) = \sum_{\alpha \in E(a(x))} \sum_{u \in \mathscr{P}_{\geqq j}(r)} d_j(\{x\}, \alpha, u)\, b(t, z(x, \alpha + [x], u) + \beta) = b(t + \beta \,.\, l, x)$; i.e., $\hat{b}$ satisfies (I.4.13), too.

CHAPTER III

We shall need the following combinatorial concept. Let $M = M_1, M_2, \ldots, M_{2m}$ be a sequence of subsets of the set $\{1, \ldots, r\}$. We shall call M simple iff there is a permutation p of the set $\{1, 2, \ldots, 2m\}$ with the following properties:

$$p(2i - 1) < p(2i) \quad \text{for} \quad i = 1, \ldots, m \,, \tag{1.1}$$

(1.2) $$M_{p(2i-1)} = M_{p(2i)} \quad \text{for} \quad i = 1, \ldots, m\,,$$

(1.3) if $i \in \{1, \ldots, m\}$ and $k \in \{1, 2, \ldots, 2m\}$ are such that $p(2i-1) < k < p(2i)$, then $p^{-1}(k) < 2i - 1$.

Roughly speaking, M is simple iff we can get to the empty sequence as follows: we find j such that $M_j = M_{j+1}$ and cross out from M both M_j and M_{j+1}. In the resulting sequence we again find two identical adjacent sets, cross them out and so on. The reason for introducing the permutation p is that first we cross out $M_{p(1)}, M_{p(2)}, \ldots$ and last $M_{p(2m-1)}, M_{p(2m)}$.

Let us give some examples for $r = 2$. We write p as $p = (p(1), \ldots, p(2m))$.

1. $m = 2$ $M = \{1\}, \{1, 2\}, \{1, 2\}, \{1\}$; $p = (2, 3, 1, 4)$,
2. $m = 4$ $M = \{2\}, \{1, 2\}, \{1, 2\}, \{1\}, \{2\}, \{2\}, \{1\}, \{2\}$; $p = (2, 3, 5, 6, 4, 7, 1, 8)$,
3. $m = 3$ $M = \emptyset, \{2\}, \{1\}, \{1\}, \{2\}, \emptyset$; $p = (3, 4, 2, 5, 1, 6)$.

Lemma 1.1. *Let $M = M_1, M_2, \ldots, M_{2m}$ be a simple sequence and $Q \in \mathscr{P}(r)$. Then also $M_1 - Q, M_2 - Q, \ldots, M_{2m} - Q$ is simple.*

The proof is easy and we omit it.

Lemma 1.2. *For every natural number r and $j \in \{1, \ldots, r\}$ there exists a sequence $M = M_1, M_2, \ldots, M_{2m}$ of subsets of the set $\{1, \ldots, r\}$ satisfying*

(1.4) $$M_1 = \{1, \ldots, r\}\,,$$

(1.5) $$\# M_k \leqq r - j \quad \textit{for} \quad k = 2, \ldots, 2m\,,$$

and such that for all $Q \subseteq \{1, \ldots, r\}$ with j elements the seqeunce $M_1 - Q, \ldots, M_{2m} - Q$ is simple.

Proof. Let $s = \binom{r}{j}$ and let $Q_0, \ldots, Q_{s-1}$ be a sequence consisting of all subsets of $\{1, \ldots, r\}$ with j elements.

By induction we define a sequence M: $M_1 = \{1, \ldots, r\}$, and if $M_1, \ldots, M_{(2^k)}$ are the first 2^k members of M $(0 \leqq k < s)$ then the next 2^k members are obtained by subtracting Q_k from all $M_1, \ldots, M_{(2^k)}$ and putting them in the inverse order after $M_1, \ldots, M_{(2^k)}$, i.e.

(1.6) $$M_{(2^k+t+1)} = M_{(2^k-t)} - Q_k \quad \text{for} \quad 0 \leqq k < s\,, \quad 0 \leqq t < 2^k\,.$$

It can be easily seen that if $0 \leqq k < s$ and $0 < i < 2^{s-k}$ then there is a set $G \subseteq \subseteq \{1, \ldots, r\}$ such that either

$$M_{(i2^k+t)} = M_t - G \quad \text{for} \quad 1 \leqq t \leqq 2^k \quad \text{or} \quad M_{(i2^k+t)} = M_{(2^k-t+1)} - G$$

for $1 \leqq t \leqq 2^k$.

M obviously satisfies (1.4) and (1.5). Let Q be a subset of $\{1, \ldots, r\}$ with j elements. There is k_0, $0 \leqq k_0 < s$ such that $Q_{k_0} = Q$. From (1.6) we get

(1.7) $$M_{(2^{k_0}+t+1)} - Q_{k_0} = M_{(2k_0-t)} - Q_{k_0} \quad \text{for} \quad 0 \leqq t < 2^{k_0}\,.$$

This fact and the previous observation used for $k = k_0 + 1$ imply that for each

$0 < i < 2^{(s-k_0-1)}$ and $0 \leqq t < 2^{k_0}$

$$(1.8) \qquad M_{(i2^{(k_0+1)}+2^{k_0}+t+1)} - Q_{k_0} = M_{(i2^{(k_0+1)}+2^{k_0}-t)} - Q_{k_0} .$$

Let us define a permutation p as follows:

$$\left.\begin{aligned} p(i2^{(k_0+1)} + 2t + 1) &= i2^{(k_0+1)} + 2^{k_0} - t \\ p(i2^{(k_0+1)} + 2t + 2) &= i2^{(k_0+1)} + 2^{k_0} + t + 1 \end{aligned}\right. \quad \text{for } 0 \leqq i < 2^{s-k_0-1}, \ 0 \leqq t < 2^{k_0} .$$

This permutation and the sequence $M_1 - Q_{k_0}, \ldots, M_{2m} - Q_{k_0}$ obviously satisfy (1.1) and (1.3) and by (1.7) and (1.8) also (1.2). The lemma is proved.

2. Let $l_1, \ldots, l_r, q$ be integers. We shall study homotopic properties of functions $X_\xi(q, x)$ for ξ from $P(n, r, l, q)$, which will be helpful in the proof of Theorem I.4.1. Let $\bar{0}$ denote the zero vector from $\mathbb{R}^r$.

Theorem 2.1. *Let* $\xi: \mathbb{R} \times \mathbb{R}^r \to \operatorname{Matr}(n)$ *be a continuous function from* $P(n, r, l, q)$ *and* $j \in \{1, \ldots, r\}$. *Suppose*

$$(2.1) \qquad X_\xi(q, x) = I \quad \text{for} \quad x \in S_j^r .$$

Then there is a continuous function $G: \langle 0, 1\rangle \times R^r \to SY(n)$ *such that*

$$(2.2) \qquad G(1, x) = X_\xi^*(q, x) \quad \textit{for} \quad x \in \mathbb{R}^r ,$$

$$(2.3) \qquad G(0, x) = I \quad \textit{for} \quad x \in \mathbb{R}^r ,$$

$$(2.4) \qquad G(\beta, x) = I \quad \textit{for} \quad \beta \in \langle 0, 1\rangle \quad \textit{and} \quad x \in S_j^r .$$

Proof. For $t \in \mathbb{R}$, $x \in \mathbb{R}^r$ and $u \in \mathscr{P}(r)$ let us define

$$(2.5) \qquad T(t, x, u) = X_\xi(t + l \,.\, z(x, \bar{0}, u), z(\bar{0}, x, u))\, X_\xi^*(l \,.\, z(x, \bar{0}, u), z(\bar{0}, x, u)) .$$

By Lemma I.4.1 the values of X_ξ are from $SY(n)$, therefore

$$(2.6) \qquad T(0, x, u) = I \quad \text{for} \quad x \in \mathbb{R}^r \quad \text{and} \quad u \in \mathscr{P}(r) .$$

Denoting $u_1 = \{1, \ldots, r\}$ we have

$$(2.7) \qquad T(q, x, u_1) = X_\xi(q, x) \quad \text{for} \quad x \in \mathbb{R}^r .$$

If $u \in \mathscr{P}(r)$, $\# u \leqq r - j$ then $z(\bar{0}, x, u) \in S_j^r$ for each $x \in \mathbb{R}^r$. By (2.1) and Lemma I.4.3 the function $X_\xi|_{R \times S_j^r}$ belongs to $PP(q)$, therefore

$$(2.8) \qquad T(q, x, u) = I \quad \text{for} \quad x \in \mathbb{R}^r \quad \text{and} \quad u \in \mathscr{P}(r) , \quad \# u \leqq r - j .$$

Let $u, v \in \mathscr{P}(r)$, $u \supseteq v$. Then for $x \in \mathbb{R}^r$

$$(2.9) \qquad z(\bar{0}, x, u) = z(\bar{0}, x, v) + z(\bar{0}, x, u - v) ,$$

$$(2.10) \qquad z(x, \bar{0}, u) + z(\bar{0}, x, u - v) = z(x, \bar{0}, v) .$$

If, moreover, $u - v \subseteq a(x)$ then $z(\bar{0}, x, u - v) \in \mathbb{Z}^r$ and, by Lemma I.4.2, considering (2.9) we get

$$(2.11) \qquad X_\xi(t + l \,.\, z(\bar{0}, x, u - v), z(\bar{0}, x, v))\, X_\xi^*(l \,.\, z(\bar{0}, x, u - v), z(\bar{0}, x, v)) = \\ = X_\xi(t, z(\bar{0}, x, u)) \quad \text{for} \quad t \in \mathbb{R} .$$

Using (2.11) once with $t + l \,.\, z(x, \bar{0}, u)$ and once with $l \,.\, z(x, \bar{0}, u)$ in place of t and considering (2.10) we get from (2.5)

$$(2.12) \qquad T(t, x, u) = T(t, x, v) \quad \text{for} \quad t \in \mathbb{R}\,, \quad x \in \mathbb{R}^r \quad \text{and} \quad u, v \in \mathcal{P}(r)\,,$$
$$u \supseteq v\,, \quad u - v \subseteq a(x)\,.$$

Let $u_1, \ldots, u_{2m}$ be a simple sequence from Lemma 1.2. We shall show that the function $G: \langle 0, 1 \rangle \times \mathbb{R}^r \to SY(n)$,

$$G(\beta, x) = T^*(q\beta, x, u_1)\, T(q\beta, x, u_2) \ldots T^*(q\beta, x, u_{2m-1})\, T(q\beta, x, u_{2m})$$

(the odd members have the stars), has the properties stated in the theorem.

Obviously, G is continuous and its values are in $SY(n)$ since the values of X_ξ are in $SY(n)$. (2.6) implies (2.3), and (2.7), (2.8) imply (2.2).

Let $x \in S^r_j$. There is $w \subseteq a(x)$ with j elements. By (2.12) we have for $\beta \in \langle 0, 1 \rangle$:

$$G(\beta, x) = T^*(q\beta, x, u_1 - w)\, T(q\beta, x, u_2 - w) \ldots T(q\beta, x, u_{2m} - w)\,.$$

(2.4) follows from the simplicity of the sequence $u_1 - w, \ldots, u_{2m} - w$.

3. Later it will be useful to approximate various functions by functions which are coordinatewise constant in some neighbourhood of integers (see Definition I.4.1). To do this, we shall need functions defined in the following way: for $s = -1, 0, 1, 2, \ldots$ (i.e. $s + 2 \in \mathbb{N}$) let

$$(3.1) \qquad \varepsilon_s = \frac{1}{10 \,.\, 2^{s+1}}\,,$$

and let ψ_s be non-decreasing functions from $\langle 0, 1 \rangle$ to $\langle 0, 1 \rangle$ with continuous second derivatives, and $P_s \geqq 1$ constants such that

$$(3.2) \qquad \left\| \frac{\mathrm{d}\psi_s}{\mathrm{d}x} \right\|^2 \leqq P_s\,, \quad \left\| \frac{\mathrm{d}^2\psi_s}{\mathrm{d}x^2} \right\| \leqq P_s\,,$$

$$(3.3) \qquad \psi_s(x) = 0 \quad \text{for} \quad x \in \langle 0, \varepsilon_{s+1} \rangle\,;$$
$$\psi_s(x) = 1 \quad \text{for} \quad x \in \langle 1 - \varepsilon_{s+1}, 1 \rangle\,,$$

$$(3.4) \qquad \psi_s(x) = x \quad \text{for} \quad x \in \langle \varepsilon_s, 1 - \varepsilon_s \rangle\,.$$

In order to avoid the subscript -1 let us notice that $\varepsilon_{-1} = \frac{1}{10}$ and denote $\psi_{-1} = \psi$ and $P_{-1} = P$.

Lemma 3.1. *For each s and each $x \in \langle 0, 1 \rangle$, $|\psi_s(x) - x| \leqq \varepsilon_s$.*

Proof. Lemma follows from (3.3), (3.4) and the fact that ψ_s are non-decreasing.

For a vector $x = (x_1, \ldots, x_r)$ let us denote by $\Psi_s(x)$ the vector $(\psi_s(x_1), \ldots, \psi_s(x_r))$, writing again Ψ instead of Ψ_{-1}. We identify Ψ_s, Ψ with ψ_s, ψ provided $r = 1$.

Lemma 3.2. *Let g be a function,* $\mathrm{Dom}(g) \subseteq \mathbb{R}^r$ *and* $s = -1, 0, 1, 2, \ldots$. *Suppose* $\mathrm{Dom}(g)$ *has the property that for* $x \in \mathrm{Dom}(g)$ *also* $[x] + \Psi_s(\{x\}) \in \mathrm{Dom}(g)$. *Define* h: $h(x) = g([x] + \Psi_s(\{x\}))$ *for* $x \in \mathrm{Dom}(g)$. *Then h belongs to* $KZ(\varepsilon_{s+1})$ *and if, moreover, g is an element of* $KZ(\varepsilon_s)$ *then* $g = h$.

The proof is easy and we omit it.

Let us notice that examples of subsets of R^r containing with each x also $[x] + \Psi_s(\{x\})$ are R^r, S^r_j, $S^r_j \cap \langle 0, 1\rangle^r$.

It will be useful to introduce the following notation: For $u \in \mathscr{P}(r)$ let $D^r(u) = \{x = (x_1, \ldots, x_r) \in \langle 0, 1\rangle^r;\ x_i = 0 \text{ for } i \in u\}$ and for $j = 0, 1, \ldots, r$ let $D^r_j = \bigcup\{D^r(u);\ u \in \mathscr{P}_j(r)\}$. In particular, $D^r_r = \{\bar{0}\}$ and $D^r_0 = \langle 0, 1\rangle^r$.

4. Let again $l_1, \ldots, l_r, q$ be integers. We shall further investigate the problem of transforming functions $X_\xi(q, x)$ into the function identically equal to I (where ξ belongs to $P(n, r, l, q)$).

In the rest of this chapter we assume $SY(n) \in EP(1) \cap \ldots \cap EP(r)$. Let us denote $c_j = c(SY(n), j)$ for $j = 1, \ldots, r$, and define $c_0 = 3\pi$. If b is a function with domain $d \times A$, where $d \subseteq R$ and $A \subseteq R^r$, then b_t for $t \in d$ denotes the function with the domain A: $b_t(x) = b(t, x)$.

Lemma 4.1. *Let* $j \in \{0, 1, \ldots, r\}$. *Let* $H(\beta, x)\colon \langle 0, 1\rangle \times D^r_j \to SY(n)$ *be a function satisfying*

(4.1) $$H(\beta, x) = I \quad \text{for} \quad \beta \in \langle 0, 1\rangle \quad \text{and} \quad x \in D^r_j \cap S^r_{j+1}\,,$$

(4.2) $$H_\beta \in KZ(\varepsilon_{r-j+1}) \quad \text{for} \quad \beta \in \langle 0, 1\rangle\,.$$

Then

(4.3) $$H(\beta, x) = I \quad \text{for} \quad \beta \in \langle 0, 1\rangle \quad \text{and} \quad x \in D^r_j\,, \quad \operatorname{dist}(x, S^r_{j+1}) \leqq \varepsilon_{r-j+1}\,.$$

Proof. Let $x \in D^r_j$ and $y \in S^r_{j+1}$, $\|x - y\| \leqq \varepsilon_{r-j+1}$. The vector $z(x, y, a(y))$ belongs to S^r_{j+1} and to D^r_j since its i^{th} coordinate is equal to 0 for each i for which $x_i = 0$. Moreover, the distance of this vector from x is less or equal to the distance of y and x. Therefore by (4.2) and (4.1), $H(\beta, x) = H(\beta, z(x, y, a(y))) = I$ for $\beta \in \langle 0, 1\rangle$, which proves (4.3).

Theorem 4.1. *Let* $j \in \{0, 1, \ldots, r\}$. *Let* $\xi(t, x)\colon R^{r+1} \to \operatorname{Matr}(n)$ *be a function of the class* $C^{(2)}$ *from* $P(n, r, l, q)$ *such that*

(4.4) $$\xi_t \in KZ(\varepsilon_{r-j}) \quad \text{for} \quad t \in R$$

and let $M > 0$ *be a constant such that* $Mq \geqq 1$ *and*

(4.5) $$\left\|\frac{\partial \xi}{\partial x_i}\right\| \leqq M \quad \text{for} \quad i = 1, \ldots, r\,,$$

(4.6) $$X_\xi(q, x) = I \quad \text{for} \quad x \in S^r_{j+1}\,.$$

Then there is a function $H(\beta, x)\colon \langle 0, 1\rangle \times D^r_j \to SY(n)$ *with continuous second derivatives w.r.t. its domain, which satisfies*

(4.7) $$H(\beta, x) = X^*_\xi(q, x) \quad \text{for} \quad \beta \in \langle 1 - \varepsilon_{r-j+1}, 1\rangle\,, \quad x \in D^r_j\,,$$

(4.8) $$H(\beta, x) = I \qquad \text{for} \quad \langle 0, \varepsilon_{r-j+1}\rangle\,, \quad x \in D^r_j\,,$$

and such that (4.1), (4.2) *and the following estimates hold*:

$$(4.9) \qquad \left\| \frac{\partial H}{\partial \beta} \right\| \leqq P_{r-j} c_{r-j} ,$$

$$(4.10) \qquad \left\| \frac{\partial H}{\partial x_i} \right\| \leqq P_{r-j} c_{r-j} q M \quad for \quad i = 1, \ldots, r ,$$

$$(4.11) \qquad \left\| \frac{\partial^2 H}{\partial \beta \, \partial x_i} \right\| \leqq P_{r-j} c_{r-j} q M \quad for \quad i = 1, \ldots, r .$$

To prove this theorem we shall need a lemma about matrices from $Y(n)$. For $\alpha \in R$ let us denote

$$Z(\alpha) = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} .$$

Lemma 4.2. *Let Q be a matrix from $Y(n)$. Then in the complex case there is a matrix $V \in SU(n)$ and numbers $\alpha_1, \ldots, \alpha_n \in \langle -\pi, \pi \rangle$ so that E if is the diagonal matrix*

$$E = \begin{pmatrix} \exp(i\alpha_1) & & 0 \\ & \ddots & \\ 0 & & \exp(i\alpha_n) \end{pmatrix}$$

then $Q = VEV^*$, and in the real case there is a matrix $V \in SO(n)$, integers k, s, t, where $0 \leqq k, s, t \leqq n$, and $\alpha_1, \ldots, \alpha_k \in \langle -\pi, \pi)$ so that if I_s and I_t are the unit matrices of the orders s and t and

$$F = \begin{pmatrix} I_s & & & & 0 \\ & -I_t & & & \\ & & Z(\alpha_1) & & \\ & & & \ddots & \\ 0 & & & & Z(\alpha_k) \end{pmatrix}$$

then $Q = VFV^*$.

Proofs can be found in [MA], Chapter V, § 19.

Lemma 4.3. *Let Q be a matrix from $SY(n)$. Then there is a function $g(\beta)$: $\langle 0, 1 \rangle \to \to SY(n)$ with a continuous second derivative such that*

$$(4.13) \qquad g(\beta) = Q \quad for \quad \beta \in \langle 1 - \varepsilon_1, 1 \rangle \quad and \quad g(\beta) = I \quad for \quad \beta \in \langle 0, \varepsilon_1 \rangle ,$$

$$(4.13) \qquad \left\| \frac{\partial g}{\partial \beta} \right\| \leqq 3P_0\pi \quad and \quad \left\| \frac{\partial^2 g}{\partial \beta^2} \right\| \leqq 6P_0\pi^2 .$$

Proof. Since $Q \in Y(n)$ and $\operatorname{Det}(Q) = 1$, we can apply the previous lemma and have, moreover, $\alpha_1 + \ldots + \alpha_n = 0$ in the complex case and t even in the real case — then $-I_t$ can be written as the matrix consisting of $\frac{1}{2}t$ matrices $Z(\pi)$ diagonaly situated. For $\beta \in \langle 0, 1 \rangle$ we denote

$$E(\beta) = \begin{pmatrix} \exp(i\alpha_1 \psi_0(\beta)) & & 0 \\ & \ddots & \\ 0 & & \exp(i\alpha_n \psi_0(\beta)) \end{pmatrix}$$

and

$$F(\beta) = \begin{pmatrix} I_s & & & & & & 0 \\ & Z(\pi\psi_0(\beta)) & & & & & \\ & & \ddots & & & & \\ & & & Z(\pi\psi_0(\beta)) & & & \\ & & & & Z(\alpha_1\,\psi_0(\beta)) & & \\ & & & & & \ddots & \\ 0 & & & & & & Z(\alpha_k\,\psi_0(\beta)) \end{pmatrix}$$

Then obviously the values of the function E belong to $SU(n)$ and the values of the function F to $SO(n)$. Define $g(\beta) = VE(\beta)\,V^*$ in the complex case and $g(\beta) = VF(\beta)\,V^*$ in the real case, for $\beta \in \langle 0, 1\rangle$. (3.2) and (3.3) imply that g has the desired properties.

5. Now we shall prove Theorem 4.1. First let $r = j$. By Lemma I.4.2 the values of X_ξ are from $SY(n)$, thus $X_\xi(q, \bar{0}) \in SY(n)$. By Lemma 4.3 we can find a function $H: \langle 0, 1\rangle \times \{\bar{0}\} = \langle 0, 1\rangle \times D^r_r \to SY(n)$ with continuous second derivative w.r.t. β, such that $\|\partial H/\partial\beta\| \leqq 3P_0\pi$, $H(\beta, \bar{0}) = X^*_\xi(q, \bar{0})$ for $\beta \in \langle 1 - \varepsilon_1, 1\rangle$ and $H(\beta, \bar{0}) = I$ for $\beta \in \langle 0, \varepsilon_1\rangle$. H has all properties required in the theorem (the rest of them is trivial since D^r_r has only one element).

Let now $j < r$. Let us define the function $Y: \mathbb{R}^r \to SY(n)$ by $Y(x) = X^*_\xi(q, x)$ for $x \in \mathbb{R}^r$. (4.5) implies

$$(5.1) \qquad \left\|\frac{\partial Y}{\partial x_i}\right\| \leqq qM \quad \text{for} \quad i = 1, \ldots, r$$

and (4.4) implies that

$$(5.2) \qquad Y \in KZ(\varepsilon_{r-j})\,.$$

Denote $J = \langle 0, 1\rangle^{r-j}$.

Let $u \in \mathscr{P}_j(r)$ and $\{1, \ldots, r\} - u = \{i_1, \ldots, i_{r-j}\}$, where $i_1 < \ldots < i_{r-j}$. Let $p_u: D^r(u) \to J$ be defined by $p_u(x) = (x_{i_1}, \ldots, x_{i_{r-j}})$ for $x = (x_1, \ldots, x_r) \in D^r(u)$. Then p_u is an isometric mapping of $D^r(u)$ onto J which maps $D^r(u) \cap S^r_{j+1}$ onto ∂J.

The function $(Y(p_u)^{-1}): J \to SY(n)$ is of the class $C^{(2)}$; by (4.6) we have

$$(5.3) \qquad (Y(p_u)^{-1})(y) = I \quad \text{for} \quad y \in \partial J$$

and by (5.1),

$$(5.4) \qquad \left\|\frac{\partial(Y(p_u)^{-1})}{\partial y_i}\right\| \leqq qM \quad \text{for} \quad i = 1, \ldots, r - j\,.$$

By Theorem 2.1 there exists a continuous function $G: \langle 0, 1\rangle \times \mathbb{R}^r \to SY(n)$ such that $G(1, x) = Y(x)$ and $G(0, x) = I$ for $x \in \mathbb{R}^r$ and $G(\beta, x) = I$ for $\beta \in \langle 0, 1\rangle$ and $x \in S^r_{j+1}$. Let us define $g_0: J \to SY(n)$ by $g_0(y) = I$ for every $y \in J$. It is easily verified that the function $G(\beta, (p_u)^{-1}(y)): \langle 0, 1\rangle \times J \to SY(n)$ is a homotopy of the functions $Y(p_u)^{-1}$ and g_0.

Since by our assumption $SY(n)$ has the property $EP(r - j)$ and because of the estimates (5.4), there exists a homotopy $h_u(\beta, y)$: $\langle 0, 1\rangle \times J \to SY(n)$ of the functions $Y(p_u)^{-1}$ and g_0, which is of the class $C^{(2)}$ and satisfies

$$(5.5) \qquad h_u(1, y) = (Y(p_u)^{-1})(y) \quad \text{for} \quad y \in J,$$

$$(5.6) \qquad h_u(0, y) = g_0(y) = I \quad \text{for} \quad y \in J,$$

$$(5.7) \qquad h_u(\beta, y) = I \quad \text{for} \quad \beta \in \langle 0, 1\rangle \quad \text{and} \quad y \in \partial J,$$

$$(5.8) \quad \left\|\frac{\partial h_u}{\partial \beta}\right\| \leqq c_{r-j} \quad \text{and} \quad \left\|\frac{\partial h_u}{\partial y_i}\right\|, \left\|\frac{\partial^2 h_u}{\partial \beta\, \partial y_i}\right\| \leqq c_{r-j} qM \quad \text{for} \quad i = 1, \ldots, r-j.$$

Let us define H_u: $\langle 0, 1\rangle \times D^r(u) \to SY(n)$ by

$$(5.9) \qquad H_u(\beta, x) = h_u(\psi_{r-j}(\beta), \Psi_{r-j}(p_u(x))).$$

By (3.3) and (5.6) we have

$$(5.10) \qquad H_u(\beta, x) = I \quad \text{for} \quad \beta \in \langle 0, \varepsilon_{r-j+1}\rangle \quad \text{and} \quad x \in D^r(u).$$

If $x \in D^r(u) \cap S^r_{j+1}$ then $p_u(x) \in \partial J$ and therefore also $\Psi_{r-j}(p_u(x)) \in \partial J$; consequently by (5.7)

$$(5.11) \qquad H_u(\beta, x) = I \quad \text{for} \quad \beta \in \langle 0, 1\rangle \quad \text{and} \quad x \in D^r(u) \cap S^r_{j+1}.$$

We shall show that

$$(5.12) \qquad H_u(\beta, x) = Y(x) \quad \text{for} \quad \beta \in \langle 1 - \varepsilon_{r-j+1}, 1\rangle \quad \text{and} \quad x \in D^r(u).$$

By (5.9), (3.3) and (5.5) such β and x satisfy

$$H_u(\beta, x) = h_u(1, \Psi_{r-j}(p_u(x))) = (Y(p_u)^{-1})(\Psi_{r-j}(p_u(x)));$$

and from the definition of p_u we see that for $x \in D^r(u)$

$$(p_u)^{-1}(\Psi_{r-j}(p_u(x))) = z(\Psi_{r-j}(x), x, u).$$

(3.3) and (3.4) imply that for each $i \in \{1, \ldots, r\}$ either $\psi_{r-j}(x_i) = x_i$ or $0 \leqq x_i, \psi_{r-j}(x_i) \leqq \varepsilon_{r-j}$ or $1 - \varepsilon_{r-j} \leqq x_i, \psi_{r-j}(x_i) \leqq 1$. Consequently, by (5.2) and Lemma II 4.1, $Y((p_u)^{-1}(\Psi_{r-j}(p_u(x)))) = Y(x)$ for $x \in D^r(u)$. We proved (5.12).

(5.11) justifies the following definition of the function $H(\beta, x)$: $\langle 0, 1\rangle \times D^r_j \to SY(n)$:

$$(5.13) \quad H(\beta, x) = H_u(\beta, x) \quad \text{for} \quad \beta \in \langle 0, 1\rangle, \quad u \in \mathscr{P}_j(r) \quad \text{and} \quad x \in D^r(u),$$

since if x is an element of both $D^r(u_1)$, $D^r(u_2)$ and $u_1 \neq u_2$, then $x \in S^r_{j+1}$ and $H(\beta, x) = H_{u_1}(\beta, x) = H_{u_2}(\beta, x) = I$ for $\beta \in \langle 0, 1\rangle$.

(5.12), (5.10) and (5.11) imply that H satisfies (4.7), (4.8) and (4.1). Let us verify that H satisfies (4.2). Let $\beta \in \langle 0, 1\rangle$ and $x = (x_1, \ldots, x_r)$, $y = (y_1, \ldots, y_r) \in D^r_j$ such that for $i \in a(x)$ the inequality $|x_i - y_i| \leqq \varepsilon_{r-j+1}$ holds and for $i \notin a(x)$, x_i equals y_i. We need to show that $H(\beta, x) = H(\beta, y)$. If $u \in \mathscr{P}_j(r)$ is such that $y \in D^r(u)$, then also $x \in D^r(u)$. For any $i \notin u$ either $x_i = y_i$ or $0 \leqq x_i, y_i \leqq \varepsilon_{r-j+1}$ or $1 - \varepsilon_{r-j+1} \leqq x_i, y_i \leqq 1$. By (3.3) and (3.4) we have $\Psi_{r-j}(p_u(x)) = \Psi_{r-j}(p_u(y))$; by (5.9), $H_u(\beta, x) = H_u(\beta, y)$ and therefore $H(\beta, x) = H(\beta, y)$.

It remains to show that H has continuous second derivatives and (4.9), (4.10) and (4.11) hold. Let $\beta \in \langle 0, 1\rangle$ and $x \in D^r_j$. If $x \in D^r(u) - S^r_{j+1}$ for some $u \in \mathscr{P}_j(r)$ then there is a neighbourhood U of (β, x) such that the function H equals the function H_u on $U \cap (\langle 0, 1\rangle \times D^r_j)$. In this case the continuity and the estimates of derivatives of the function H in (β, x) follow from (5.9), from the continuity and estimates of derivatives of the function h_u (it is of the class $C^{(2)}$ and satisfies (5.8)), and from the fact that Ψ_{r-j} has a continuous second derivative and (3.2) holds. If $x \in S^r_{j+1} \cap D^r_j$ then by Lemma 4.1 there is a neighbourhood U of (β, x) such that the function H, is on $U \cap (\langle 0, 1\rangle \times D^r_j)$ identically equal to I. All derivatives of H at this point are therefore equal to 0. Theorem 4.1 is proved.

6. Let us define by induction the constants $Q^n(r, j) \geqq 1$ for $j = r + 1, r, \dots, 1, 0$. Since n is fixed, we usually omit the upper index n. Let $Q(r, r + 1) = P$ and for $j = r, \dots, 1, 0$,

$$Q(r, j) = Q(r, j + 1)\,(1 + K(r, j)\,(3 + 3P_{r-j}c_{r-j} + (P_{r-j}c_{r-j})^2))\,.$$

(The constants c_{r-j} are defined at the beginning of § 4, the constants P_{r-j} and P at the beginning of § 3 and the constants $K(r, j)$ in the second chapter by (II.3.3).

Let us denote

$$W(n, r) = Q^n(r, 0) + r \quad \text{and} \quad V(n, r) = \left(\max\left\{\frac{P_{r-j}c_{r-j}Fr}{Q^n(r, j + 1)}\,;\; j = 0, \dots, r\right\}\right) + 1$$

(F is the constant defined in § 1 of Chapter II).

We shall show that for these $W(n, r)$ and $V(n, r)$ Theorem I.4.1 holds.

Let $l_1, \dots, l_r, q$ be integers, $\xi(t, x_1, \dots, x_r)$: $\mathbb{R}^{r+1} \to \operatorname{Matr}(n)$ a function of the class $C^{(2)}$ belonging to $P(n, r, l, q)$, and $L > 0$ a constant such that

$$q \geqq V(n, r)/L$$

and

$$\left\|\frac{\partial \xi}{\partial x_i}\right\| \leqq L \quad \text{for} \quad i = 1, \dots, r\,. \tag{6.1}$$

We shall construct by induction for $j = r + 1, r, \dots, 1, 0$ functions $\xi_j(t, x_1, \dots, x_r)$: $\mathbb{R}^{r+1} \to \operatorname{Matr}(n)$ of the class $C^{(2)}$, belonging to $P(n, r, l, q)$ and satisfying

$$\|\xi_j - \xi\| \leqq (Q(r, j) + r^{1/2})\,L\,, \tag{6.2}$$

$$\left\|\frac{\partial \xi_j}{\partial x_i}\right\| \leqq Q(r, j)\,L \quad \text{for} \quad i = 1, \dots, r\,, \tag{6.3}$$

$$X_{\xi_j}(q, x) = I \quad \text{for} \quad x \in S^r_j\,, \tag{6.4}$$

$$(\xi_j)_t \in KZ(\varepsilon_{r-j+1}) \quad \text{for} \quad t \in \mathbb{R}\,. \tag{6.5}$$

By doing this, we shall prove Theorem I.4.1 since $\varrho = \xi_0$ has all desired properties.

First, let us define $\xi_{r+1}(t, x_1, \dots, x_r)$: $\mathbb{R}^{r+1} \to \operatorname{Matr}(n)$ by $\xi_{r+1}(t, x) = \xi(t, [x] + \Psi(\{x\}))$. It is easily verified that ξ_{r+1} belongs to $P(n, r, l, q)$. By Lemma 3.2 for each $t \in R$ the function $(\xi_{r+1})_t$ is an element of $KZ(\varepsilon_0)$. Further, ξ_{r+1} is of the class $C^{(2)}$ and due to (6.1) and (3.2) satisfies for $i = 1, \ldots, r$ $\|\partial \xi_{r+1}/\partial x_i\| \leqq PL$. Finally, (6.1) and the inequality $\|x - [x] - \Psi(\{x\})\| \leqq r^{1/2}$ imply $\|\xi_{r+1} - \xi\| \leqq rL$. Therefore ξ_{r+1} meets all our requirements (6.4) is trivial because $S^r_{r+1} = \emptyset$).

Let us suppose now that we have constructed a function ξ_{j+1}, $0 \leqq j < r + 1$, of the class $C^{(2)}$, which belongs to $P(n, r, l, q)$ and satisfies

$$\|\xi_{j+1} - \xi\| \leqq (Q(r, j + 1) + r^{1/2})\, L, \tag{6.6}$$

$$\left\|\frac{\partial \xi_{j+1}}{\partial x_i}\right\| \leqq Q(r, j + 1)\, L \quad \text{for} \quad i = 1, \ldots, r\,, \tag{6.7}$$

$$X_{\xi_{j+1}}(q, x) = I \quad \text{for} \quad x \in S^r_{j+1}\,, \tag{6.8}$$

$$(\xi_{j+1})_t \in KZ(\varepsilon_{r-j}) \quad \text{for} \quad t \in R\,. \tag{6.9}$$

Lemma I.4.1 implies that the values of the function $X_{\xi_{j+1}}$ are from $SY(n)$, and from (6.9) and (6.7) we get

$$(X_{\xi_{j+1}})_t \in KZ(\varepsilon_{r-j}) \quad \text{for} \quad t \in R\,, \tag{6.10}$$

$$\left\|\frac{\partial}{\partial x_i}\,[X_{\xi_{j+1}}(t, x)]\right\| \leqq Q(r, j + 1)\, Lq \quad \text{for} \quad t \in \langle 0, q\rangle\,, \quad x \in R^r \quad \text{and} \quad i = 1, \ldots, r\,. \tag{6.11}$$

The function ξ_{j+1} satisfies the assumptions of Theorem 4.1 (where $M = Q(r, j + 1)\, L$ so that $Mq \geqq 1$ since $Q(r, j + 1) \geqq 1$), hence there is a function $H(\beta, x)\colon \langle 0, 1\rangle \times D^r_j \to SY(n)$ with continuous second derivatives w.r.t. $\langle 0, 1\rangle \times D^r_j$, satisfying (4.8), (4.1), (4.2), (4.9) and such that

$$H(\beta, x) = X^*_{\xi_{j+1}}(q, x) \quad \text{for} \quad \beta \in \langle 1 - \varepsilon_{r-j+1}, 1\rangle \quad \text{and} \quad x \in D^r_j\,, \tag{6.12}$$

$$\left\|\frac{\partial H}{\partial x_i}\right\|, \left\|\frac{\partial^2 H}{\partial \beta\, \partial x_i}\right\| \leqq P_{r-j} c_{r-j} q Q(r, j + 1)\, L \quad \text{for} \quad i = 1, \ldots, r\,. \tag{6.13}$$

Let us now define a function B using X_B first for $t \in \langle 0, q\rangle$ and $x \in D^r_j$:

$$X_B(t, x) = X_{\xi_{j+1}}(t, x)\, H\left(\frac{t}{q}, x\right), \tag{6.14}$$

$$B(t, x) = \frac{\partial}{\partial t}\,[X_B(t, x)]\, X^*_B(t, x)\,. \tag{6.15}$$

Lemma I.4.1 implies that (I.4.10) and (I.4.11) hold for B. Further,

$$B(t, x) = \xi_{j+1}(t, x) + \frac{1}{q}\, X_{\xi_{j+1}}(t, x) \frac{\partial}{\partial \beta}\left[H\left(\frac{t}{q}, x\right)\right] H^*\left(\frac{t}{q}, x\right) X^*_{\xi_{j+1}}(t, x) \quad \text{for} \quad t \in \langle 0, q\rangle \quad \text{and} \quad x \in D^r_j\,. \tag{6.16}$$

The function B has continuous second derivatives w.r.t. its domain and (6.7), (6.11) and (6.13) yield for $i = 1, \ldots, r$

$$\left\|\frac{\partial B}{\partial x_i}\right\| \leqq Q(r, j + 1) L(1 + 3P_{r-j}c_{r-j} + (P_{r-j}c_{r-j})^2). \tag{6.17}$$

From (6.9), (6.10) and (4.2) we obtain for $t \in \langle 0, q \rangle$

$$B_t \in KZ(\varepsilon_{r-j+1}). \tag{6.18}$$

From (6.16) and (4.9) we get

$$\|B - \xi_{j+1}\| \leqq (1/q) P_{r-j}c_{r-j}, \tag{6.19}$$

and the assumption $q \geqq V(n, r)/L$ and the definition $V(n, r)$ yield

$$\frac{1}{q} P_{r-j}c_{r-j} \leqq \frac{Q(r, j + 1) L}{Fr} \leqq Q(r, j + 1) L. \tag{6.20}$$

Now we shall extend B to $R \times D^r_j$. By (4.8) and (6.12) we have $(\partial/\partial\beta) [H(t/q, x)] = 0$ for $t \in (\langle 0, q\varepsilon_{r-j+1}\rangle \cup \langle q - q\varepsilon_{r-j+1}, q\rangle)$ and $x \in D^r_j$. Therefore (6.16) implies that $B(t, x) = \xi_{j+1}(t, x)$ for such t and x; consequently, we can extend B for $t \in \mathbb{R}$ and $x \in D^r_j$ by demanding that B is periodic in t with the period q. Then B satisfies (I.4.10) and (I.4.11) and the periodicity of the function ξ_{j+1} in t with the period q guarantees that the extended B is a continuous function with continuous second derivatives w.r.t. $R \times D^r_j$, satisfies (6.17), (6.19) and for each $t \in \mathbb{R}$, (6.18) holds.

Finally, let us extend B to $\mathbb{R} \times S^r_j$. Lemma 4.1 implies that H satisfies (4.3). The periodicity of the functions B and ξ_{j+1} in t with the period q and (6.16) imply

$$B(t, x) = \xi_{j+1}(t, x) \tag{6.21}$$

$$\text{for} \quad t \in \mathbb{R} \quad \text{and} \quad x \in D^r_j, \quad \operatorname{dist}(x, S^r_{j+1}) \leqq \varepsilon_{(r-j+1)}.$$

We shall show that

$$B(t, x) = B(t + l \,.\, [x], \{x\}) \quad \text{for} \quad t \in \mathbb{R} \quad \text{and} \quad x \in D^r_j. \tag{6.22}$$

Let $x \in D^r_j$, $[x] \neq \bar{0}$. Then some coordinate of x must be equal to 1 and therefore x and $\{x\}$ are elements of $S^r_{j+1} \cap D^r_j$. Since ξ_{j+1} belongs to $P(n, r, l, q)$, we have by (I.4.13)

$$\xi_{j+1}(t, x) = \xi_{j+1}(t + l \,.\, [x], \{x\}) \quad \text{for} \quad t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^r. \tag{6.23}$$

This and (6.21) imply (6.22).

If x is an element of S^r_j then $\{x\}$ is an element of D^r_j. Therefore we can define $B(t, x) = B(t + l \,.\, [x], \{x\})$ for $t \in \mathbb{R}$ and $x \in S^r_j$. It is easily verified that B belongs to $P(n, r, l, q)$ and that for all t, (6.18) holds. (6.23) and (6.19) imply that also this extended B satisfies (6.19). From (6.21) and (6.23) we conclude

$$B(t, x) = \xi_{j+1}(t, x) \tag{6.24}$$

$$\text{for} \quad t \in \mathbb{R} \quad \text{and} \quad x \in S^r_j, \quad \operatorname{dist}(x, S^r_{j+1}) \leqq \varepsilon_{(r-j+1)}.$$

If $x \in S^r_{j+1}$ then by (6.24) there exists a neighbourhood U of x such that on $R \times (U \cap S^r_j)$ the function B equals ξ_{j+1}. If $x \in S^r_j - S^r_{j+1}$ then there exists a neighbourhood U of x such that for $y \in U \cap S_j^r$, $y - [x] \in D_j^r$ and therefore for $t \in \mathbb{R}$ the equality $B(t, y) = B(t + l\,.\,[x], y - [x])$ holds. In both cases we see that B is continuous, has continuous second derivatives w.r.t. its domain and the estimates (6.17) hold.

By (6.14) and (6.12) the function $X_B\colon \mathbb{R} \times S_j^r \to SY(n)$ satisfies $X_B(q, x) = I$ for $x \in D_j^r$. Lemma I.4.2 implies that X_B satisfies (I.4.14) and (I.4.15). If $x \in S_j^r$ then $\{x\} \in D_j^r$ and therefore $X_B(q, \{x\}) = I$. Henceforth, considering (I.4.15) we see that $X_B(q, x) = X_B(q + l\,.\,[x], \{x\})\, X_B^*(q, \{x\})\;\; X_B^*(l\,.\,[x], \{x\})$. Since (I.4.14) holds, $X_B(q + l\,.\,[x], \{x\})\, X_B^*(q, \{x\}) = X_B(l\,.\,[x], \{x\})$. We conclude that

$$X_B(q, x) = I \quad \text{for} \quad x \in S_j^r\,. \tag{6.25}$$

For $j > 0$ we define the function $E\colon \mathbb{R} \times S_j^r \to \operatorname{Matr}(n)$ by $E = B - \xi_{j+1}$. Then E belongs to $P(n, r, l, q)$, is continuous and has continuous second derivatives w.r.t. its domain. By (6.7) and (6.17) the following inequality holds for $i = 1, \ldots, r$:

$$\left\|\frac{\partial E}{\partial x_i}\right\| \leqq Q(r, j + 1)\, L(2 + 3P_{r-j}c_{r-j} + (P_{r-j}c_{r-j})^2)\,.$$

(6.19) and (6.20) imply that $\|E\| \leqq Q(r, j + 1)\, L/rF$, and (6.18) and (6.9) imply that for each $t \in \mathbb{R}$ the function E_t is an element of $KZ(\varepsilon_{r-j+1})$.

Let $\hat{E}(t, x_1, \ldots, x_r)\colon \mathbb{R}^{r+1} \to \operatorname{Matr}(n)$ be the extension of E as defined in Chapter II. By Theorem II.5.1, $\hat{E}$ is of the class $C^{(2)}$ and for $i = 1, \ldots, r$ the inequalities

$$\|\hat{E}\| \leqq K(r, j)\,\frac{Q(r, j + 1)\, L}{rF} \quad \text{and}$$

$$\left\|\frac{\partial \hat{E}}{\partial x_i}\right\| \leqq K(r, j)\; Q(r, j + 1)\; L(3 + 3P_{r-j}c_{r-j} + (P_{r-j}c_{r-j})^2)$$

hold. By Theorem II.5.2, $\hat{E}$ belongs to $P(n, r, l, q)$, and by Theorem II.4.1 for each $t \in \mathbb{R}$ the function $\hat{E}_t$ is an element of $KZ(\varepsilon_{r-j+1})$.

Let us define the function $\xi_j(t, x_1, \ldots, x_r)\colon \mathbb{R}^{r+1} \to \operatorname{Matr}(n)$ by $\xi_j = \hat{E} + \xi_{j+1}$. Then ξ_j is obviously of the class $C^{(2)}$, belongs to $P(n, r, l, q)$ and satisfies (6.5). Since $\hat{E}$ extends E, $\xi_j|_{\mathbb{R} \times S_j r} = B$, therefore (6.25) implies (6.4).

By (6.6) and the estimate for $\|\hat{E}\|$ we have

$$\|\xi_j - \xi\| \leqq \|\hat{E}\| + \|\xi_{j+1} - \xi\| \leqq Q(r, j + 1)\, L(K(r, j) + 1) + r^{1/2}L,$$

hence we see, considering the definition of $Q(r, j)$, that (6.2) holds. Finally we have by (6.7) and the estimates for $\|\partial\hat{E}/\partial x_i\|$:

$$\left\|\frac{\partial \xi_j}{\partial x_i}\right\| \leqq Q(r, j + 1)\, L(1 + K(r, j)\,(3 + 3P_{r-j}c_{r-j} + (P_{r-j}c_{r-j})^2))$$

for $i = 1, \ldots, r$. Considering again the definition of $Q(r, j)$, we see that (6.3) holds.

If $j = 0$, we define $\xi_0 = B$. Then $\xi_0\colon \mathbb{R}^{r+1} \to \operatorname{Matr}(n)$ is of the class $C^{(2)}$ and belongs to $P(n, r, l, q)$. From (6.25) and (6.18) respectively it follows that ξ_0 satisfies

(6.4) and (6.5). Considering the definition of $Q(r, 0)$ we see that (6.17) implies (6.3), and (6.6), (6.19) and (6.20) imply (6.2).

We have found functions ξ_j for $j = r + 1, r, \ldots, 1, 0$ with the desired properties. Theorem I.4.1 is proved.

APPENDIX

Let again n be a fixed natural number, $n > 2$ in the real case and $n > 1$ in the complex case. We shall show that $SY(n)$ has the homotopy estimation properties (see Chapter I, § 2) of orders 1 and 2.

1. Let r be a natural number. In this section we shall show that for any continuous function from $\langle 0, 1\rangle^r$ to $SY(n)$ it is possible to find an arbitrarily close function of the class $C^{(\infty)}$ from $\langle 0, 1\rangle^r$ to $SY(n)$ again.

We shall need some facts about Matr (n) and $SY(n)$ which we shall not prove in detail.

Observe that U^{-1} exists, if $U \in$ Matr (n) and dist $(U, Y(n)) < 1$; the map $U \mapsto U^{-1}$ is analytic.

For $U \in$ Matr (n), $\|U - I\| \leqq 1/3$ put

(1.1) $$V = I + \tfrac{1}{2}(U - I) + \binom{1/2}{2}(U - I)^2 + \ldots .$$

We find easily that the series converges, $\|V - I\| < \frac{1}{3}$, $V^2 = U$. Therefore we shall write $U^{1/2}$ instead of V. $U^{1/2}$ will be used only in case that $\|U - I\| < \frac{1}{3}$; the map $U \mapsto U^{1/2}$ is analytic.

Lemma 1.1. *Let $U, W \in$ Matr (n), $\|U - I\| \leqq \frac{1}{3}$, $\|W - I\| \leqq \frac{1}{3}$. Then*

a) $\|U^{1/2} - W^{1/2}\| \leqq \|U - W\|$,

b) $(U^{1/2})^* = (U^*)^{1/2}$.

Lemma 1.2. *Let U be an element of* Matr (n) *and* dist $(U, SY(n)) < \frac{1}{9}$. *Then $\|UU^* - I\| < \frac{1}{3}$, the matrix $(UU^*)^{1/2}(U^*)^{-1}$ belongs to $Y(n)$ and*

(1.2) $$\|(UU^*)^{1/2}(U^*)^{-1} - U\| \leqq 6 \operatorname{dist}(U, SY(n)) .$$

Proof. Let us denote $d = \operatorname{dist}(U, SY(n))$. Let $S \in SY(n)$, $\|U - S\| = d$. Obviously $S = (S^*)^{-1}$, $\|S^*\| = \|S\| = 1$ and $\|U^* - S^*\| = d$. The inequalities $\|S\| - \|U - S\| \leqq \|U\| = \|U^*\| \leqq \|S\| + \|U - S\|$ imply that $1 - d \leqq \|U\| = \|U^*\| \leqq 1 + d$. Since $d < \frac{1}{9}$, we have

(1.3) $$\|UU^* - I\| = \|UU^* - SS^*\| \leqq \|U\|\,\|U^* - S^*\| + \|U - S\|\,\|S^*\| , \text{ i.e.}$$
$$\|UU^* - I\| \leqq (2 + d)\, d .$$

Hence $\|UU^* - I\| < \frac{1}{3}$. Consequently, the matrix $(UU^*)^{1/2}(U^*)^{-1}$ is well defined; using Lemma 1.1 b) we see easily that this matrix belongs to $Y(n)$. Further,

$$\|(U^*)^{-1} - S\| = \|(U^*)^{-1} - (S^*)^{-1}\| \leqq \|(U^*)^{-1}\|\,\|S^* - U^*\|\,\|(S^*)^{-1}\| \leqq d/(1-d) .$$

Hence $\|U - (U^*)^{-1}\| \leqq \|U - S\| + \|S - (U^*)^{-1}\| \leqq d(2 - d)/(1 - d)$. Finally, by Lemma 1.1 a) and (1.3) we have $\|I - (UU^*)^{1/2}\| \leqq (2 + d)\, d$ and $\|(UU^*)^{1/2}\| \leqq \leqq 1 + (2 + d)\, d$, therefore

$$\|U - (UU^*)^{1/2} (U^*)^{-1}\| \leqq$$
$$\leqq \|I - (UU^*)^{1/2}\| \, \|U\| + \|(UU^*)^{1/2}\| \, \|U - (U^*)^{-1}\| \leqq 6d\,.$$

This proves (1.2).

Let D be a constant such that if U, V are elements of Matr (n) with $\|U\| \leqq 2$, $\|V\| \leqq 2$, then $|\text{Det}\,(U) - \text{Det}\,(V)| \leqq D\|U - V\|$. Let $\Lambda = \{U; U \in \text{Matr}\,(n)$ and $\text{dist}\,(U, SY(n)) \leqq \min\,(\frac{1}{9}, \frac{1}{14D})\}$. Let $U \in \Lambda$. By (1.2) we have $|\text{Det}\,((UU^*)^{1/2}\,(U^*)^{-1}) - \text{Det}\,(U)| \leqq D\; 6\; \text{dist}\,(U, SY(n))$. Hence

$$\text{(1.4)} \qquad |\text{Det}\,((UU^*)^{1/2}\,(U^*)^{-1}) - 1| \leqq 7D\,\text{dist}\,(U, SY(n)) \leqq \tfrac{1}{2}\,.$$

Since $(UU^*)^{1/2}\,(U^*)^{-1}$ belongs to $Y(n)$, $|\text{Det}\,((UU^*)^{1/2}\,(U^*)^{-1})| = 1$. Let γ be the real number with the smallest absolute value satisfying $\text{Det}((UU^*)^{1/2}\,(U^*)^{-1}) = e^{i\gamma}$. Due to (1.4), γ belongs to $\langle -\frac{1}{2}\pi, \frac{1}{2}\pi\rangle$. Let us denote $(\text{Det}\,((UU^*)^{1/2}\,(U^*)^{-1}))^{-1/n} = e^{i\gamma/n}$. Obviously, $|e^{-i\gamma/n} - 1| \leqq |e^{i\gamma} - 1|$, and therefore we have by (1.4):

$$\text{(1.5)} \qquad |(\text{Det}\,((UU^*)^{1/2}\,(U^*)^{-1}))^{-1/n} - 1| \leqq 7D\,\text{dist}\,(U, SY(n))\,.$$

Let us define a function $\mathscr{W}\colon \Lambda \to \text{Matr}\,(n)$ as follows.

$$\text{(1.6)} \qquad \mathscr{W}(U) = (\text{Det}\,((UU^*)^{1/2}\,(U^*)^{-1}))^{-1/n}\,.\,(UU^*)^{1/2}\,(U^*)^{-1}\,.$$

Lemma 1.3. *The values of $\mathscr{W}$ belong to $SY(n)$ and for all $U \in \Lambda$*

$$\text{(1.7)} \qquad \|U - \mathscr{W}(U)\| \leqq 13\,D\,\text{dist}\,(U, SY(n))\,.$$

(Of course, in the real case (1.6) reduces to $\mathscr{W}(U) = (UU^*)^{1/2}\,(U^*)^{-1}$.)

Proof. By the previous lemma, the matrix $(UU^*)^{1/2}\,(U^*)^{-1}$ is an element of $Y(n)$, consequently $\mathscr{W}(U)$ is an element of $SY(n)$ and the norm of $(UU^*)^{1/2}\,(U^*)^{-1}$ equals 1. By (1.2) and (1.5) the following inequality holds:

$$\|U - \mathscr{W}(U)\| \leqq \|U - (UU^*)^{1/2}\,(U^*)^{-1}\| +$$
$$+ |(\text{Det}\,((U^*)^{1/2}\,(U^*)^{-1}))^{-1/n} - 1|\; \|(UU^*)^{1/2}\,(U^*)^{-1}\| \leqq$$
$$\leqq 6\,\text{dist}\,(U, SY(n)) + 7D\,\text{dist}\,(U, SY(n))\,;$$

this implies (1.7) since D is greater or equal to 1.

Theorem 1.1. *Let $F\colon \langle 0, 1\rangle^r \to SY(n)$ be a continuous function. For each $\eta > 0$ there exists a function $H\colon \langle 0, 1\rangle^r \to SY(n)$ which is of the class $C^{(\infty)}$ and satisfies $\|H - F\| \leqq \eta$.*

Proof. We can find a function $F_0\colon \langle 0, 1\rangle^r \to \text{Matr}\,(n)$ which is of the class $C^{(\infty)}$ and satisfies

$$\|F - F_0\| \leqq \min\left(\frac{1}{9}, \frac{1}{14D}, \frac{\eta}{26D}\right).$$

Let us define $H = \mathscr{W}(F_0)$. Then H has values in $SY(n)$, it is of the class $C^{(\infty)}$ and for each $x \in \langle 0, 1 \rangle^r$, by (1.7) we have

$$\|H(x) - F_0(x)\| \leqq 13D \operatorname{dist}(F_0(x), SY(n)).$$

Further, $\operatorname{dist}(F_0(x), SY(n)) \leqq \|F_0(x) - F(x)\| \leqq \eta/26D$. Hence $\|H - F\| \leqq \leqq \|H - F_0\| + \|F_0 - F\| \leqq \eta$.

2. Let m be a natural number greater that 1. We shall prove some theorems about extensions of functions, which are defined on $\partial\langle 0, 1 \rangle^m$ and have values in $SY(n)$, to the whole $\langle 0, 1 \rangle^m$.

Let $\mathscr{L}$ be a function and $\sigma > 0$ a constant, $\mathscr{L}: \langle 0, 1 \rangle \times \{U: U \in \operatorname{Matr}(n)$ and $\|U - I\| \leqq \sigma\} \to \operatorname{Matr}(n)$, such that for each $U \in \operatorname{Matr}(n)$, $\|U - I\| \leqq \sigma$,

$$\mathscr{L}(1, U) = U \quad \text{and} \quad \mathscr{L}(0, U) = I; \tag{2.1}$$

for all $\beta \in \langle 0, 1 \rangle$ the equality $\mathscr{L}(\beta, I) = I$ holds and whenever U is an element of $SY(n)$, also $\mathscr{L}(\beta, U)$ is an element of $SY(n)$; further, $\mathscr{L}$ is of the class $C^{(2)}$.

Let $S > 1$ be a constant bounding the norms of the first and second differential of the function $\mathscr{L}$ on $\operatorname{Dom}(\mathscr{L})$.
(For example, $\mathscr{L}(\beta, U) = \mathscr{W}(I + \beta(U - I)) + \beta(U - \mathscr{W}(U))$, $\sigma = \min(1/9, 1/14D)$.)

Theorem 2.1. *Let $\frac{1}{10} > \varepsilon > 0$. There exists a number $Q = Q(\varepsilon, m)$ (depending on ε and m only), such that if $L > 1$ and $F_1: \partial(\langle 0, 1 \rangle^m) \to SY(n)$, $F_2: \langle 0, 1 \rangle^m \to SY(n)$ are functions from $KZ(\varepsilon)$ with continuous second derivatives w.r.t. their domains and such that*

$$\left\|\frac{\partial F_i}{\partial x_j}\right\| \leqq L \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad j = 1, \ldots, m, \tag{2.2}$$

$$\|F_1 - F_2|_{\partial(\langle 0,1 \rangle^m)}\| \leqq \sigma, \tag{2.3}$$

then there exists a function $F: \langle 0, 1 \rangle^m \to SY(n)$ from $KZ(\frac{1}{2}\varepsilon)$ with continuous second derivatives, extending F_1 and satisfying

$$\left\|\frac{\partial F}{\partial x_j}\right\| \leqq QL \quad \text{for} \quad j = 1, \ldots, m, \tag{2.4}$$

$$\left\|\frac{\partial^2 F}{\partial x_i \partial x_j}\right\| \leqq Q\left(L^2 + \left\|\frac{\partial^2 F_1}{\partial x_i \partial x_j}\right\| + \left\|\frac{\partial^2 F_2}{\partial x_i \partial x_j}\right\|\right) \quad \text{for} \quad i, j = 1, \ldots, m. \tag{2.5}$$

Proof. Let us denote $\Delta = \{x \in \langle 0, 1 \rangle^m;\ \operatorname{dist}(x, \partial(\langle 0, 1 \rangle^m)) \leqq \varepsilon\}$. Let $g: \Delta \to \langle 0, 1 \rangle$ be a function such that (see Fig. 3)

$$\begin{aligned} g(x) &= 1 \quad \text{for} \quad x \in \Delta: \operatorname{dist}(x, \partial(\langle 0, 1 \rangle^m)) \leqq \varepsilon/2, \\ g(x) &= 0 \quad \text{for} \quad x \in \Delta: \operatorname{dist}(x, \partial(\langle 0, 1 \rangle^m)) \geqq 9\varepsilon/10, \end{aligned} \tag{2.6}$$

and g has continuous first and second derivatives on Δ bounded by a constant $G = G(\varepsilon, m)$ (depending on ε and m).

Fig. ($m = 2$):

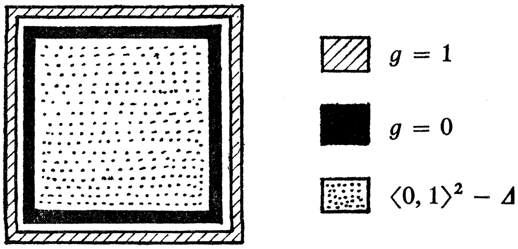

Fig. 3

We define a function $\bar{F}_1: \Delta \to SY(n)$ by

(2.7) $$\bar{F}_1 \in KZ(\varepsilon) \quad \text{and} \quad \bar{F}_1(x) = F_1(x) \quad \text{for} \quad x \in \partial(\langle 0, 1\rangle^m)\,.$$

It is easily verified that $\bar{F}_1$ is well defined, has continuous first and second derivatives on Δ, norms of the first derivatives are bounded by L, the second derivatives satisfy

(2.8) $$\left\|\frac{\partial^2 \bar{F}_1}{\partial x_i\, \partial x_j}\right\| = \left\|\frac{\partial^2 F_1}{\partial x_i\, \partial x_j}\right\| \quad \text{for} \quad i, j = 1, \ldots, m\,,$$

and that the properties $F_1, F_2 \in KZ(\varepsilon)$ and (2.3) imply the inequality $\|\bar{F}_1 - F_2|_\Delta\| \leqq \leqq \sigma$.

The last inequality implies that for each $x \in \Delta$, $\|\bar{F}_1(x)\, F_2^*(x) - I\| \leqq \|\bar{F}_1(x) - - F_2(x)\|\, \|F_2^*(x)\| \leqq \sigma$; thus we can define

(2.9) $$F(x) = F_2(x) \quad \text{for} \quad x \in \langle 0, 1\rangle^m - \Delta\,,$$
$$\mathscr{L}(g(x), \bar{F}_1(x)\, F_2^*(x))\, F_2(x) \quad \text{for} \quad x \in \Delta\,.$$

Then F is a function from $\langle 0, 1\rangle^m$ to $SY(n)$ and for $x \in \langle 0, 1\rangle^m$, $\operatorname{dist}(x, \partial(\langle 0, 1\rangle^m)) \leqq \leqq \frac{1}{2}\varepsilon$ we have by (2.6) and (2.1), $F(x) = \mathscr{L}(1, \bar{F}_1(x)\, F_2^*(x))\, F_2(x) = \bar{F}_1(x)$. Considering (2.7) we see that F extends F_1 and belongs to $KZ(\frac{1}{2}\varepsilon)$.

Obviously, the function F has continuous second derivatives at all points x which satisfy $\operatorname{dist}(x, \partial(\langle 0, 1\rangle^m)) \neq \varepsilon$, since for such x the function F is defined on some neighbourhood by only one of the equalities (2.9), and the functions F_2, $\mathscr{L}$, g and $\bar{F}_1$ have continuous second derivatives. If $\operatorname{dist}(x, \partial(\langle 0, 1\rangle^m)) = \varepsilon$, then we see by (2.6) and (2.1) that on the set $\{y \in \langle 0, 1\rangle^m;\ \|x - y\| \leqq \varepsilon/10\}$ the function F equals F_2; therefore F has continuous second derivatives everywhere.

Considering that the norms of the first and second differential of $\mathscr{L}$ are bounded by S and the first and second derivatives of g by G, we get from (2.9):

$$\left\|\frac{\partial F}{\partial x_i}\right\| \leqq L + S \,.\, 2L + SG \quad \text{for} \quad i = 1, \ldots, m\,,$$

$$\left\|\frac{\partial^2 F}{\partial x_i\, \partial x_j}\right\| \leqq \left\|\frac{\partial^2 F_2}{\partial x_i\, \partial x_j}\right\| + SG^2 + SG(6L + 1) + 10SL^2 +$$

$$+ S\left(\left\|\frac{\partial^2 \bar{F}_1}{\partial x_i \, \partial x_j}\right\| + \left\|\frac{\partial^2 F_2^*}{\partial x_i \, \partial x_j}\right\|\right)$$

for $i, j = 1, \ldots, m$.

Since the norms of the second derivatives of F_2^* are bounded in the same way as the norms of the corresponding second derivatives of F_2, considering (2.8) and the inequalities $L^2 > L > 1$ we see that (2.4) and (2.5) hold for $Q = 10S + 7SG + SG^2 + 1$.

Theorem 2.2. *Let $\frac{1}{10} > \varepsilon > 0$ and let $F_1 \colon \partial(\langle 0, 1\rangle^m) \to SY(n)$ be a function from $KZ(\varepsilon)$ which has continuous second derivatives w.r.t. its domain and such that it is possible to extend F_1 to a continuous function defined on $\langle 0, 1\rangle^m$ and with values in $SY(n)$. Then there is an extension $F \colon \langle 0, 1\rangle^m \to SY(n)$ of the function F_1 which belongs to $KZ(\varepsilon/8)$ and has continuous second derivatives on $\langle 0, 1\rangle^m$.*

Proof. Let $E \colon \langle 0, 1\rangle^m \to SY(n)$ be any continuous extension of F_1. By Theorem 1.1 there is a function $G \colon \langle 0, 1\rangle^m \to SY(n)$ of the class $C^{(\infty)}$ satisfying $\|E - G\| \leqq \sigma$. Let s be the natural number such that $\varepsilon_s \leqq \varepsilon < 2\varepsilon_s$. (Constants ε_s and functions ψ_s are defined in § 3 of Chapter III. Let us recall that for $x = (x_1, \ldots, x_m)$ the symbol $\Psi_s(x)$ denotes $(\psi_s(x_1), \ldots, \psi_s(x_m))$.) Let us define the function $F_2 \colon \langle 0, 1\rangle^m \to SY(n)$ by $F_2(x) = G(\Psi_s(x))$ for $x \in \langle 0, 1\rangle^m$. Lemma III.3.2 implies that F_2 belongs to $KZ(\varepsilon_{s+1})$. We shall show that $\|F_1 - F_2|_{\partial(\langle 0,1\rangle^m)}\| \leqq \sigma$. The function F_1 belongs to $KZ(\varepsilon)$, i.e. also to $KZ(\varepsilon_s)$. Therefore by Lemma III.3.2 again, $F_1(x) = F_1(\Psi_s(x))$ for all $x \in \partial(\langle 0, 1\rangle^m)$. Thus the inequality $\|E - G\| \leqq \sigma$ yields

$$\sigma \geqq \|E|_{\partial(\langle 0,1\rangle^m)} - G|_{\partial(\langle 0,1\rangle^m)}\| = \|F_1 - G|_{\partial(\langle 0,1\rangle^m)}\| =$$
$$= \|F_1 - G\Psi_s|_{\partial(\langle 0,1\rangle^m)}\| = \|F_1 - F_2|_{\partial(\langle 0,1\rangle^m)}\| \,.$$

Both functions $F_1 \colon \partial(\langle 0, 1\rangle^m) \to SY(n)$ and $F_2 \colon \langle 0, 1\rangle^m \to SY(n)$ belong to $KZ(\varepsilon_{s+1})$, therefore to $KZ(\varepsilon/4)$, and have continuous second derivtives w.r.t. their domains.

Theorem 2.1 implies that there exists a function $F \colon \langle 0, 1\rangle^m \to SY(n)$ from $KZ(\varepsilon/8)$, which extends F_1 and has continuous second derivatives.

Theorem 2.3. *Let $\frac{1}{10} > \varepsilon > 0$ and $L > 1$. There exists a constant $A = A(L, \varepsilon, m)$ (depending on L, ε and m), such that the following holds.*

Let $F \colon \partial(\langle 0, 1\rangle^m) \to SY(n)$ be a continuous function from $KZ(\varepsilon)$ which can be extended to a continuous function from $\langle 0, 1\rangle^m$ to $SY(n)$, and which has continuous first and second derivatives w.r.t. $\partial(\langle 0, 1\rangle^m)$ satisfying

$$\left\|\frac{\partial F}{\partial x_i}\right\| \leqq L \quad \textit{for} \quad i = 1, \ldots, m\,, \tag{2.10}$$

$$\left\|\frac{\partial^2 F}{\partial x_1 \, \partial x_i}\right\| \leqq L \quad \textit{for} \quad i = 1, \ldots, m\,. \tag{2.11}$$

Then there exists a function $H \colon \langle 0, 1\rangle^m \to SY(n)$ which extends F, belongs to

$KZ(\varepsilon/16)$ and has continuous first and second derivatives satisfying

$$\left\|\frac{\partial H}{\partial x_i}\right\| \leqq A \quad \textit{for} \quad i = 1, \ldots, m\,, \tag{2.12}$$

$$\left\|\frac{\partial^2 H}{\partial x_1\, \partial x_i}\right\| \leqq A \quad \textit{for} \quad i = 1, \ldots, m\,. \tag{2.13}$$

Proof. Let us assume that the theorem is false. Let $F_k\colon \partial(\langle 0, 1\rangle^m) \to SY(n)$, $k \in \mathbb{N}$, be a sequence of functions satisfying the conditions of the theorem and such that there are no extensions of F_k to functions from $\langle 0, 1\rangle^m$ to $SY(n)$ which belong to $KZ(\varepsilon/16)$ and have continuous first and second derivatives, while all the first derivatives and the second derivatives by x_1 and x_i, $i = 1, \ldots, m$, have norms bounded by k.

Since all functions F_k satisfy (2.10), we can select a Cauchy subsequence from them. We shall assume that already $\{F_k; k \in \mathbb{N}\}$ is a Cauchy sequence. Let k_0 be a natural number such that for each $k \geqq k_0$, $\|F_k - F_{k_0}\| \leqq \sigma$. By the previous theorem there is a function $E\colon \langle 0, 1\rangle^m \to SY(n)$ which belongs to $KZ(\varepsilon/8)$, extends F_{k_0} and has continuous first and second derivatives. Let us denote by $V \geqq L$ the constant which bounds norms of all first and second derivatives of the function E. Let $Q = Q(\varepsilon/8, m)$ be the constant from Theorem 2.1. Let $k \geqq k_0$. Both functions $F_k\colon \partial(\langle 0, 1\rangle^m) \to SY(n)$ and $E\colon \langle 0, 1\rangle^m \to SY(n)$ belong to $KZ(\varepsilon/8)$, have continuous second derivatives w.r.t. their domains and norms of their first derivatives are bounded by V. Further,

$$\left\|\frac{\partial^2 F_k}{\partial x_1\, \partial x_i}\right\| \leqq V \quad \text{and} \quad \left\|\frac{\partial^2 E}{\partial x_1\, \partial x_i}\right\| \leqq V \quad \text{for} \quad i = 1, \ldots, m\,. \tag{2.14}$$

Since $E|_{\partial(\langle 0,1\rangle^m)} = F_{k_0}$, the inequality $\|F_k - E|_{\partial(\langle 0,1\rangle^m)}\| \leqq \sigma$ holds.

By Theorem 2.1, for each $k \geqq k_0$ there is a function $H_k\colon \langle 0, 1\rangle^m \to SY(n)$ which has continuous second derivatives, belongs to $KZ(\varepsilon/16)$, extends F_k and satisfies (cf. (2.14))

$$\left\|\frac{\partial H_k}{\partial x_i}\right\| \leqq QV \quad \text{and} \quad \left\|\frac{\partial^2 H_k}{\partial x_1\, \partial x_i}\right\| \leqq Q(V^2 + 2V) \quad \text{for} \quad i = 1, \ldots, m\,.$$

This is a contradiction since for some k we have $k > Q(V^2 + 2V)$, therefore H_k is an extension of F_k with the properties we assumed it can not have.

3. Now we shall prove two lemmas about extensions of functions defined on $\partial(\langle 0, 1\rangle^m)$ and with values in $SY(n)$ to the whole $\langle 0, 1\rangle^m$ for $m = 2, 3$. We shall use the following facts about homotopy groups π_1 and π_2 of manifolds $SY(n)$ (see[HU]):

$$\pi_1(SO(n)) = Z_2 \quad \text{and} \quad \pi_1(SU(n)) = \pi_2(SU(n)) = \pi_2(SO(n)) = 0\,.$$

(Recall that $n > 2$ in the real case and $n > 1$ in the complex case.)

Let $\gamma_1 \colon \langle 0, 1\rangle \to \partial(\langle 0, 1\rangle^2)$ be the function defined as follows (see Fig. 4):

$$\gamma_1(x) = \begin{cases} (3x, 0) & \text{for} \quad x \in \langle 0, \tfrac{1}{3}\rangle\,, \\ (1, 3x - 1) & \text{for} \quad x \in (\tfrac{1}{3}, \tfrac{2}{3}\rangle\,, \\ (3 - 3x, 1) & \text{for} \quad x \in (\tfrac{2}{3}, 1\rangle\,. \end{cases}$$

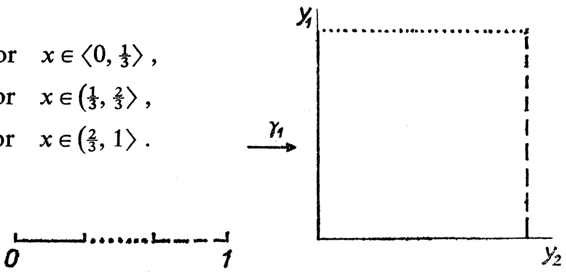

Fig. 4

The function γ_1 is a continuous and bijective mapping of $\langle 0, 1\rangle$ onto $\partial(\langle 0, 1\rangle^2) - \{(y_1, y_2);\ y_1 = 0,\ y_2 \in (0, 1)\}$. This is the domain of the inverse function γ_1^{-1} which is also continuous.

Let $\gamma\colon \langle 0, 1\rangle^2 \to \langle 0, 1\rangle^2$ be defined as follows (see Fig. 5):

$$\gamma(1, x) = \gamma_1(x) \quad \text{for} \quad x \in \langle 0, 1\rangle\,,$$

$$\gamma(0, x) = \left(0, \frac{1}{3} + \frac{x}{3}\right) \quad \text{for} \quad x \in \langle 0, 1\rangle\,,$$

$$\gamma(\beta, x) = (1 - \beta)\,\gamma(0, x) + \beta\gamma(1, x) \quad \text{for} \quad \beta, x \in \langle 0, 1\rangle\,.$$

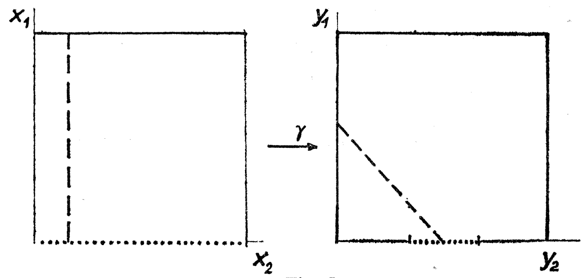

Fig. 5

Then γ and γ^{-1} are continuous bijective mappings of $\langle 0, 1\rangle^2$ onto $\langle 0, 1\rangle^2$.

Lemma 3.1. *Let $F\colon \partial(\langle 0, 1\rangle^2) \to SY(n)$ be a continuous function such that $F(0, y_2) = I$ for $y_2 \in \langle 0, \frac{1}{3}\rangle \cup \langle \frac{1}{3}, 1\rangle$, and such that $F\gamma_1$ is homotopic with the function $F_2\colon \langle 0, 1\rangle \to SY(n)$, where $F_2(x) = F(0, (x + 1)/3)$ for $x \in \langle 0, 1\rangle$. Then F can be extended to a continuous function with the domain $\langle 0, 1\rangle^2$ and with values in $SY(n)$.*

Proof. Let $H\colon \langle 0, 1\rangle^2 \to SY(n)$ be a homotopy of F_{γ_1} and F_2, i.e.

$$H(1, x) = F_{\gamma_1}(x) \quad \text{and} \quad H(0, x) = F_2(x) = F(0, (x + 1)/3) \quad \text{for} \quad x \in \langle 0, 1\rangle\,,$$
$$H(\beta, 0) = H(\beta, 1) = I \quad \text{for} \quad \beta \in \langle 0, 1\rangle\,.$$

Since γ^{-1} is continuous, $H\gamma^{-1}$ is a continuous extension of F.

Lemma 3.2. *Let $F\colon \partial(\langle 0, 1\rangle^3) \to SY(n)$ be a continuous function such that $F(0, y_2, y_3) = I$ for $y_2, y_3 \in \langle 0, 1\rangle$. Then F can be extended to a continuous function with the domain $\langle 0, 1\rangle^3$ and with values in $SY(n)$.*

Proof. Since the proof is similar to that of the previous lemma, we shall only sketch it. We define mappings η_1 and η analogously to γ_1 and γ; η_1 is sketched in Fig. 6

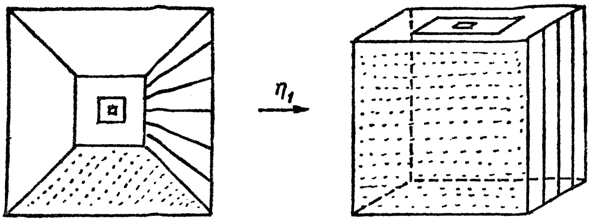

Fig. 6

and η: $\langle 0, 1\rangle^3 \to \langle 0, 1\rangle^3$ is defined as follows:

$$\eta(1, x_1, x_2) = \eta_1(x_1, x_2) \quad \text{for} \quad (x_1, x_2) \in \langle 0, 1\rangle^2 ,$$

$$\eta(0, x_1, x_2) = \left(0, \frac{x_1 + 1}{3}, \frac{x_2 + 1}{3}\right) \quad \text{for} \quad (x_1, x_2) \in \langle 0, 1\rangle^2 ,$$

$$\eta(\beta, x_1, x_2) = (1 - \beta)\, \eta(0, x_1, x_2) + \beta\eta(1, x_1, x_2)$$
$$\text{for} \quad \beta \in \langle 0, 1\rangle \quad \text{and} \quad \langle x_1, x_2\rangle \in \langle 0, 1\rangle^2 .$$

Then η and η^{-1} are continuous bijective mappings of $\langle 0, 1\rangle^3$ onto $\langle 0, 1\rangle^3$. Since $\pi_2(SY(n))$ is trivial, the function $F\eta_1$ is homotopic with the function F_2: $\langle 0, 1\rangle^2 \to \to SY(n)$,

$$F_2(x_1, x_2) = I = F\left(0, \frac{x_1 + 1}{3}, \frac{x_2 + 1}{3}\right).$$

Let H be a homotopy of $F\eta_1$ and F_2. The function $H\eta^{-1}$ is the desired extension of F.

4. Now we shall prove that $SY(n)$ belongs to $EP(1)$. Let us denote by g_0: $\langle 0, 1\rangle \to \to SY(n)$ the function such that $g_0(x) = I$ for each $x \in \langle 0, 1\rangle$. Let g_1: $\langle 0, 1\rangle \to SO(n)$ be a function which satisfies $g_1(0) = g_1(1) = I$, is not homotopic with g_0, belongs to $KZ(\frac{1}{10})$ and has a continuous second derivative; let $\Gamma > 0$ be such that

$$\left\|\frac{dg_1}{dx}\right\| \leqq \Gamma , \quad \left\|\frac{d^2g_1}{dx^2}\right\| \leqq \Gamma . \tag{4.1}$$

By the above mentioned property of the group π_1 for $SY(n)$ we see that every function g: $\langle 0, 1\rangle \to SY(n)$, $g(0) = g(1) = I$, is in the complex case homotoppic with g_0, and in the real case homotopic either with g_0 or with g_1. The next lemma follows again from the fact that $\pi_1(SY(n))$ is either 0 or Z_2.

Lemma 4.1. a) *Let* g: $\langle 0, 1\rangle \to SY(n)$, $g(0) = g(1) = I$. *If* f *is defined by* $f(x) = = g(1 - x)$ *for each* $x \in \langle 0, 1\rangle$, *then* f *is homoropic with* g.

b) *Let* g: $\langle 0, m\rangle \to SY(n)$ $(m \in \mathbb{N})$ *be a function such that* $g(k) = I$ *for each* $k \in \{0, 1, \ldots, m\}$. *Then the number of all* $k < m$ *for which* $g|_{\langle k,k+1\rangle}$ *is homotopic with* g_1 *is even iff the function* $g(x/m)$ *is homotopic with* g_0.

Let $L \geqq 1$ and let $g: \langle 0, 1\rangle \to SY(n)$, $g(0) = g(1) = I$, be a function with a continuous second derivative, which is homotopic with g_0 and satisfies $\|dg/dx\| \leqq L$. Let l be a natural number such that

$$l - 1 < L \leqq l . \tag{4.2}$$

Let us denote by $\bar{g}: \langle 0, l\rangle \to SY(n)$ the function defined by $\bar{g}(x) = g(x/l)$. Then $\bar{g}$ has a continuous second derivative and

$$\left\|\frac{d\bar{g}}{dx}\right\| \leqq 1 . \tag{4.3}$$

We shall transform $\bar{g}$ in three steps to the function identically equal to I. First we approximate $\bar{g}$ by a function which is coordinatewise constant in some neighbourhood of integers (see Ch. II, § 4), using the function Ψ_j with a suitable j (see Ch. III, § 3). Let j be a natural number such that

$$\varepsilon_j < \sigma/2 . \tag{4.4}$$

(In this section we shall use only $\varepsilon_j < \sigma$. (4.4) as it is will be needed in the proof that $SY(n) \in EP(2)$.)

Let $G: \langle 0, 1\rangle \to SY(n)$ be defined as follows:

$$G(x) = \bar{g}([x] + \Psi_j(\{x\})) . \tag{4.5}$$

The function G has a continuous second derivative and due to (4.3) and (III.3.2) the following inequality holds:

$$\left\|\frac{dG}{dx}\right\| \leqq P_j . \tag{4.6}$$

By Lemma III.3.2 the function G belongs to $KZ(\varepsilon_{j+1})$. Moreover, $G(0) = I = G(l)$.

Further, we have by (4.3) and (4.5) for each $x \in \langle 0, l\rangle$:

$$\|G(x) - \bar{g}(x)\| \leqq \|\Psi_j(\{x\}) - \{x\}\| .$$

therefore by (4.4) and Lemma III.3.1

$$\|G - \bar{g}\| \leqq \varepsilon_j < \sigma . \tag{4.7}$$

Hence we have $\|\bar{g}(x)\, G^*(x) - I\| < \sigma$ for each $x \in \langle 0, l\rangle$. Therefore we can define the function $T_1: \langle 0, 1\rangle \times \langle 0, l\rangle \to SY(n)$ by

$$T_1(\beta, x) = \mathscr{L}(\psi_j(\beta), \bar{g}(x)\, G^*(x))\, G(x) . \tag{4.8}$$

For $x \in \langle 0, l\rangle$ we have

$$T_1(1, x) = \bar{g}(x) \quad \text{and} \quad T_1(0, x) = G(x) \tag{4.9}$$

and also

$$T_1(\beta, 0) = T_1(\beta, l) = I \quad \text{for} \quad \beta \in \langle 0, 1\rangle . \tag{4.10}$$

The function T_1 has continuous second derivatives and the following estimates hold:

$$(4.11) \qquad \left\|\frac{\partial T_1}{\partial \beta}\right\| \leqq SP_j \leqq 3S^2P_j^2\,,$$

$$\left\|\frac{\partial T_1}{\partial x}\right\| \leqq S(P_j + 1) + P_j \leqq 3S^2P_j^2\,,$$

$$\left\|\frac{\partial^2 T_1}{\partial x\,\partial \beta}\right\| \leqq P_jS^2(P_j + 1) + P_jSP_j \leqq 3S^2P_j^2\,.$$

Let $x \in R$, $\zeta = \min\{\varepsilon_{j+2}, \sigma/(2P_j)\}$, $\varphi: R \to \langle 0, \infty)$ of class $C^{(2)}$, $\operatorname{supp}\varphi \subset \subset \langle -1/2, 1/2\rangle$, $\int_R \varphi(x)\,dx = 1$, $|d\varphi/dx| \leqq \varkappa$, $|d^2\varphi/dx^2| \leqq \varkappa$. Put $G(x) = I$ for $x < 0$ and for $x > l$,

$$(4.12) \qquad G_2(x) = \zeta^{-1}\int_R G(y)\,\varphi(\zeta^{-1}(x - y))\,dy \quad \text{for} \quad x \in \langle 0, l\rangle\,.$$

By the choice of ζ, (4.6) and (4.12) we have $\|G_2(x) - G(x)\| \leqq \sigma$ for $x \in \langle 0, l\rangle$. Moreover, $G_2(0) = I = G_2(l)$ and $G_2 \in KZ(\varepsilon_{j+2})$, since $G \in KZ(\varepsilon_{j+1})$ and $\zeta \leqq \varepsilon_{j+2}$. From (4.12) we obtain

$$(4.13) \qquad \left\|\frac{dG_2}{dx}\right\| \leqq P_j\,, \quad \left\|\frac{d^2G_2}{dx^2}\right\| \leqq \varkappa^2\zeta^{-2}\,.$$

Put

$$(4.14) \qquad T_2(\beta, x) = \mathscr{L}(\psi_{j+1}(\beta), G(x)\,G_2^*(x))\,G_2(x)\,.$$

We have $T_2 \in KZ(\varepsilon_{j+2})$,

$$(4.15) \qquad T_2(1, x) = G(x)\,, \quad T_2(0, x) = G_2(x) \quad \text{for} \quad x \in \langle 0, l\rangle\,,$$

$$T_2(\beta, 0) = I = T_2(\beta, l) \quad \text{for} \quad \beta \in \langle 0, 1\rangle\,,$$

$$(4.16) \qquad \left\|\frac{\partial T_2}{\partial \beta}\right\| \leqq SP_j \leqq 3S^2P_j^2\,,$$

$$\left\|\frac{\partial T_2}{\partial x}\right\| \leqq S(P_j + 1) + P_j \leqq 3S^2P_j^2\,,$$

$$\left\|\frac{\partial^2 T_2}{\partial \beta\,\partial x}\right\| \leqq P_jS^2(P_j + 1) + P_jSP_j \leqq 3S^2P_j^2\,.$$

Next we shall transform G_2 to a function which is equal to I for each natural number from $\langle 0, l\rangle$. We shall define $T_3: \langle 0, 1\rangle \times \langle 0, l\rangle \to SY(n)$ (see Fig. 7):

$$(4.17) \qquad T_3(1, x) = G_2(x) \quad \text{for} \quad x \in \langle 0, l\rangle\,,$$

$$(4.18) \qquad T_3(\beta, 0) = T_3(\beta, l) = I \quad \text{for} \quad \beta \in \langle 0, 1\rangle\,,$$

$$(4.19) \qquad T_3(0, x) = I \quad \text{for} \quad x \in \langle k, k + \tfrac{1}{3}\rangle \cup \langle k + \tfrac{2}{3}, k + 1\rangle \quad \text{and} \quad k \in \{0, 1, \ldots, l - 1\}\,.$$

For $k \in \{1, \ldots, l - 1\}$ let $T_3(\beta, k): \langle 0, 1\rangle \to SY(n)$ be a function transforming $G_2(k)$

into I by Lemma III.4.3, i.e. $T_3(1, k) = G_2(k)$ and $T_3(0, k) = I$, and $T_3(\beta, k)$ as a continuous second derivative and the norm of the first and second derivatives are bounded by $6P_0^2\pi^2$. Finally, for $x \in \langle k + \frac{1}{3}, k + \frac{2}{3}\rangle$, where $k \in \{0, 1, \ldots, l - 1\}$, we define $T_3(0, x)$ according to the homotopy class of the function $f_k: \langle 0, 1\rangle \to \to SY(n), f_k(x) = T_3(\gamma_1(x) + (0, k))$:

$$T_3(0, x) = I = g_0(3x - 1 - 3k)\,, \quad \text{if } f_k \text{ is homotopic with } g_0\,,$$
$$g_1(3x - 1 - 3k)\,, \quad \text{if } f_k \text{ is homotopic with } g_1\,. \tag{4.20}$$

$T_3(0, x)$ as a function of x belongs to $KZ(\frac{1}{10})$, has a continuous second derivative and the norm of the first derivative is bounded by 3Γ because of (4.1).

Up to now, we have defined T_3 for those points from $\langle 0, 1\rangle \times \langle 0, l\rangle$ which have at least one integer coordinate. T_3 has continuous second derivatives w.r.t. this domain, belongs to $KZ(\varepsilon_{j+2})$ and the norms of its first and second derivatives are bounded by the constant $\max\{\varkappa^2\zeta^{-2}, 3S^2P_j^2, 6P_0^2\pi^2, 3\Gamma\}$. Due to (4.20), the function T_3 can be extended, by Lemma 3.1, on each square $\langle 0, 1\rangle \times \langle k, k + 1\rangle$, where $k \in \{0, 1, \ldots, l - 1\}$. Therefore T_3 can be extended on each such square, by Theorem 2.3, so that the resulting function $T_3: \langle 0, 1\rangle \times \langle k, k + 1\rangle \to SY(n)$ belongs to $KZ((\varepsilon_{j+2})/16)$, has continuous second derivatives and when denoting by $A_0 = = A(\max\{\varkappa^2\zeta^{-2}, 3S^2P_j^2, 6P_0^2\pi^2, 3\Gamma\}, \varepsilon_{j+2}, 2)$ the constant from Theorem 2.3, we have

$$\left\|\frac{\partial T_3}{\partial \beta}\right\|, \left\|\frac{\partial T_3}{\partial x}\right\|, \left\|\frac{\partial^2 T_3}{\partial \beta\, \partial x}\right\| \leqq A_0\,. \tag{4.21}$$

The function T_3, being an element of $KZ((\varepsilon_{j+2})/16)$, does not depend on x in some neighbourhoods of the points (β, k) (on the common boundaries of two squares). Therefore T_3 has continuous second derivatives and satisfies (4.21) on the whole domain $\langle 0, 1\rangle \times \langle 0, l\rangle$.

Now we shall consider the function $t: \langle 0, l\rangle \to SY(n)$, $t(x) = T_3(0, x)$. The function $t(x/l)$ is homotopic with g_0 (a homotopy between them can be obtained using T_1, T_2 and the homotopy between g and g_0), therefore by Lemma 4.1 b) the number of intervals $\langle k, k + 1\rangle$, $k \in \{0, 1, \ldots, l - 1\}$, such that $t|_{\langle k,k+1\rangle}$ is homotopic with g_1, is even. We shall use this property to construct a function $T_4: \langle 0, 1\rangle \times \times \langle 0, 1\rangle \to SY(n)$, which will transform t to the function identically equal to I. The construction of T_4 is similar to that of T_3. We define

$$T_4(1, x) = t(x) \quad \text{for} \quad x \in \langle 0, l\rangle\,, \tag{4.22}$$

$$T_4(0, x) = I \quad \text{for} \quad x \in \langle 0, l\rangle \tag{4.23}$$

and for $k \in \{0, 1, \ldots, l\}$ and $\beta \in \langle 0, 1\rangle$

(4.24) $T_4(\beta, k) = I$, if the number of $m < k$ such that $t|_{\langle m,m+1\rangle}$ is homotopic with g_1 is even,
$g_1(\beta)$ otherwise.

Due to the above mentioned property of the function t, we have

(4.25) $$T_4(\beta, 0) = T_4(\beta, l) = I \quad \text{for} \quad \beta \in \langle 0, 1 \rangle .$$

We have defined T_4 for those points from $\langle 0, 1 \rangle \times \langle 0, l \rangle$ which have at least one integer coordinate. T_4 has continuous second derivatives, belongs to $KZ(\frac{1}{10})$ and the norms of its first derivatives are bounded by 3Γ.

By (4.22) and (4.24) we see that for each $k \in \{0, 1, \ldots, l - 1\}$ the function h_k: $\langle 0, 1 \rangle \to SY(n)$, $h_k(x) = T_4(\gamma_1(x) + (0, k))$, is homotopic with g_0. Similarly as with T_3, using Lemma 3.1 and Theorem 2.3, we get an extension T_4: $\langle 0, 1 \rangle \times \langle 0, l \rangle \to SY(n)$, which belongs to $KZ(\frac{1}{160})$, has continuous second derivatives, and if $A_1 = A(3\Gamma, \frac{1}{10}, 2)$ is the constant from Theorem 2.3 then the following estimates hold:

(4.20) $$\left\| \frac{\partial T_4}{\partial \beta} \right\| , \left\| \frac{\partial T_4}{\partial x} \right\| , \left\| \frac{\partial^2 T_4}{\partial \beta \, \partial x} \right\| \leqq A_1 .$$

Now we can define a homotopy H: $\langle 0, 1 \rangle \times \langle 0, 1 \rangle \to SY(n)$ of the functions g and g_0:

$$\begin{aligned} H(\beta, x) = \; & T_1(4\beta - 3, lx) && \text{for} \quad \beta \in \langle \tfrac{3}{4}, 1 \rangle , \\ & T_2(4\beta - 2, lx) && \text{for} \quad \beta \in \langle \tfrac{1}{2}, \tfrac{3}{4} \rangle , \\ & T_3(4\beta - 1, lx) && \text{for} \quad \beta \in \langle \tfrac{1}{4}, \tfrac{1}{2} \rangle , \\ & T_4(4\beta, lx) && \text{for} \quad \beta \in \langle 0, \tfrac{1}{4} \rangle . \end{aligned}$$

Then we have by (4.9) and (4.23):

$$\begin{aligned} H(1, x) &= T_1(1, lx) = \bar{g}(lx) = g(x) && \text{for} \quad x \in \langle 0, 1 \rangle , \\ H(0, x) &= T_4(0, lx) = I = g_0(x) && \text{for} \quad x \in \langle 0, 1 \rangle , \end{aligned}$$

and by (4.10), (4.15), (4.18) and (4.25), $H(\beta, 0) = H(\beta, 1) = I$ for each $\beta \in \langle 0, 1 \rangle$.

The function T_2 belongs to $KZ(\varepsilon_{j+2})$, T_3 belongs to $KZ((\varepsilon_{j+2})/16)$ and T_4 belongs to $KZ(\frac{1}{160})$. Therefore these functions do not depend on β in some neighbourhoods of points whose β-coordinate is either 0 or 1. From (4.8) and (III.3.3) we see that T_1 has the same property. Hence H does not depend on β in some neighbourhoods of points whose β-coordinate is $\frac{1}{4}$ or $\frac{1}{2}$ or $\frac{3}{4}$. Consequently, the fact that T_1, T_2, T_3 and T_4 have continuous second derivatives implies that also H has continuous second derivatives.

Considering the definition of H and (4.11), (4.16), (4.21) and (4.26) we get the following estimates:

$$\left\| \frac{\partial H}{\partial \beta} \right\| \leqq 4 \max \{A_0, A_1, 3S^2P_j^2\} , \quad \left\| \frac{\partial H}{\partial x} \right\| \leqq l \max \{A_0, A_1, 3S^2P_j^2\}$$

and

$$\left\| \frac{\partial^2 H}{\partial \beta \, \partial x} \right\| \leqq 4l \max \{A_0, A_1, 3S^2P_j^2\} .$$

By (4.2), $2L \geqq l$. Let us denote $c = 8 \max \{A_0, A_1, 3S^2P_j^2\}$. We have shown that for any $L \geqq 1$ and any function g: $\langle 0, 1 \rangle \to SY(n)$, $g(0) = g(1) = I$, which is homotopic with g_0, has a continuous second derivative and the norm of the first derivative bounded by L, there exists a homotopy H of functions g and g_0 of the class $C^{(2)}$, satisfying

$$\left\|\frac{\partial H}{\partial \beta}\right\| \leqq c \quad \text{and} \quad \left\|\frac{\partial H}{\partial x}\right\|, \left\|\frac{\partial^2 H}{\partial \beta\, \partial x}\right\| \leqq cL.$$

Therefore $SY(n)$ has the property $EP(1)$.

5. This section contains the proof that $SY(n)$ belongs to $EP(2)$. We shall only sketch it since it is analogous to the proof that $SY(n)$ belongs to $EP(1)$.

Let us denote by $G_0: \langle 0, 1\rangle^2 \to SY(n)$ the function identically equal to I. Since $\pi_2(SY(n)) = 0$, each function $g: \langle 0, 1\rangle^2 \to SY(n)$, $g(x) = I$ for $x \in \partial(\langle 0, 1\rangle^2)$, is homotopic with G_0.

Let $L \geqq 1$ and let $g: \langle 0, 1\rangle^2 \to SY(n)$, $g(x) = I$ for $x \in \partial(\langle 0, 1\rangle^2)$, be a function of the class $C^{(2)}$ satisfying $\|\partial g/\partial x_i\| \leqq L$ for $i = 1, 2$. Let l be the natural number such that

(5.1) $$l - 1 < L \leqq l,$$

Recall that for $x = (x_1, x_2)$, $x/l = (x_1/l, x_2/l)$. Let us denote by $\bar{g}: \langle 0, l\rangle^2 \to SY(n)$ the function defined by $\bar{g}(x) = g(x/l)$. Then $\bar{g}$ is again of the class $C^{(2)}$ and

(5.2) $$\left\|\frac{\partial \bar{g}}{\partial x_i}\right\| \leqq 1 \quad \text{for} \quad i = 1, 2.$$

Given $\bar{g}$, we define $G: \langle 0, l\rangle^2 \to SY(n)$ by (4.5) and $T_1: \langle 0, 1\rangle \times \langle 0, l\rangle^2 \to SY(n)$ by (4.8) (we have $\|G(x) - \bar{g}(x)\| \leqq \sigma$). These functions have continuous second derivatives and satisfy

(5.3) $$\left\|\frac{\partial G}{\partial x_i}\right\| \leqq P_j \quad \text{for} \quad i = 1, 2,$$

(5.4) $$T_1(1, x) = \bar{g}(x) \quad \text{for} \quad x \in \langle 0, l\rangle^2,$$

(5.5) $$T_1(0, x) = G(x) \quad \text{for} \quad x \in \langle 0, l\rangle^2,$$

(5.6) $$T_1(\beta, x) = I \quad \text{for} \quad x \in \partial(\langle 0, l\rangle^2) \quad \text{and} \quad \beta \in \langle 0, 1\rangle,$$

(5.7) $$\left\|\frac{\partial T_1}{\partial \beta}\right\|, \left\|\frac{\partial T_1}{\partial x_i}\right\|, \left\|\frac{\partial^2 T_1}{\partial \beta\, \partial x_i}\right\| \leqq 3S^2P_j^2 \quad \text{for} \quad i = 1, 2.$$

Moreover, $G \in KZ(\varepsilon_{j+1})$, $G(x) = I$ for $x \in \partial(\langle 0, l\rangle^2)$. Therefore we may put $G(x) = I$ for $x \in \mathbb{R}^2 - \langle 0, l\rangle^2$. Define

$$G_2(x) = \zeta^{-2} \int_{\mathbb{R}^2} G(y)\, \varphi(\zeta^{-1}(x_1 - y_1))\, \varphi(\zeta^{-1}(x_2 - y_2))\, dy_1\, dy_2 \quad \text{for} \quad x \in \langle 0, l\rangle^2.$$

Analogously as in the previous section we get

$$\|G_2(x) - G(x)\| \leqq \sigma \quad \text{for} \quad x \in \langle 0, l\rangle^2,$$

(5.8) $$\left\|\frac{\partial G_2}{\partial x_i}\right\| \leqq P_j, \quad \left\|\frac{\partial G_2}{\partial x_i\, \partial x_h}\right\| \leqq \varkappa^2 \zeta^{-2}, \quad i, h \in \{1, 2\}.$$

Moreover, $G_2 \in KZ(\varepsilon_{j+2})$. Define T_2 by (4.14). We have again $T_2 \in KZ(\varepsilon_{j+2})$, $T_2(1, x) = G(x)$, $T_2(0, x) = G_2(x)$ for $x \in \langle 0, l\rangle^2$,

$$T_2(\beta, x) = I \quad \text{for} \quad \beta \in \langle 0, 1\rangle\,, \quad x \in \partial(\langle 0, l\rangle^2)\,, \tag{5.9}$$

$$\left\|\frac{\partial T_2}{\partial \beta}\right\|, \left\|\frac{\partial T_2}{\partial x_i}\right\|, \left\|\frac{\partial^2 T_2}{\partial \beta\, \partial x_i}\right\| \leqq 3S^2 P_j^2\,, \quad i = 1, 2\,. \tag{5.10}$$

Let us recall that S_1^2 is the set of all $x \in \mathbb{R}^2$ such that at least one coordinate of x is an integer (see § 1 in Chapter II). We shall define functions T_3 and T_4: $\langle 0, 1\rangle \times (\langle 0, l\rangle^2 \cap S_1^2) \to SY(n)$. First, let T_3 be defined as follows (see Fig. 7):

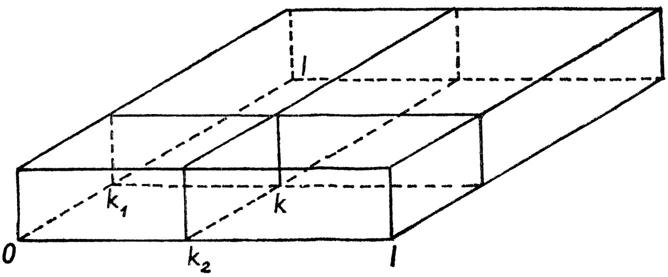

Fig. 7

$$T_3(1, x) = G_2(x),\ T_3(0, x) = I \quad \text{for} \quad x \in \langle 0, l\rangle^2\,, \tag{5.11}$$

$$T_3(\beta, x) = I \quad \text{for} \quad x \in \partial(\langle 0, l\rangle^2) \quad \text{and} \quad \beta \in \langle 0, 1\rangle\,; \tag{5.12}$$

for $k = (k_1, k_2)$, where $k_1, k_2 \in \{1, \ldots, l - 1\}$, let $T_3(\beta, k)$ be the transformation of $G(k)$ to I from Lemma III.4.3. For a fixed $k_1 \in \{1, \ldots, l - 1\}$ we extend T_3 on $\langle 0, 1\rangle \times \{k_1\} \times \langle 0, l\rangle$ and for a fixed $k_2 \in \{1, \ldots, l - 1\}$ we extend T_3 on $\langle 0, 1\rangle \times \times \langle 0, l\rangle \times \{k_2\}$ in the same way as we extended T_2 on $\langle 0, 1\rangle \times \langle 0, l\rangle$ in the previous section.

Let $k_1, k_2 \in \{0, 1, \ldots, l\}$ and let τ: $\langle 0, k_1 + k_2\rangle \to SY(n)$ be the function defined as

$$\tau(x) = T_3(0, x, k_2) \quad \text{for} \quad x \in \langle 0, k_1\rangle\,,$$
$$T_3(0, k_1, k_1 + k_2 - x) \quad \text{for} \quad x \in \langle k_1, k_2\rangle\,.$$

The function $\tau(x/(k_1 + k_2))$ is homotopic with g_0 (we can find a homotopy of these functions using T_3, T_2, T_1 and a homotopy of g and g_0). By Lemma 4.1 b), the number of all $m < k_1 + k_2$ for which $\tau|_{\langle m, m+1\rangle}$ is homotopic with g_1, is even. Define t_1, t_2: $\langle 0, l\rangle \to SY(n)$, $t_1(x) = T_3(0, k_1, x)$ and $t_2(x) = T_3(0, x, k_2)$. By Lemma 4.1 a) we see that the number of $m < k_1$ for which $t_2|_{\langle m, m+1\rangle}$ is homotopic with g_1, is even, iff the number of $m < k_2$ for which $t_1|_{\langle m, m+1\rangle}$ is homotopic with g_1, is even.

Hence we can define:

$$T_4(1, x) = T_3(0, x) \quad \text{for} \quad x \in \langle 0, l\rangle^2 \cap S_1^2\,, \tag{5.13}$$

(5.14) $T_4(0, x) = I$ for $x \in \langle 0, l\rangle^2 \cap S_1^2$,

(5.15) $T_4(\beta, x) = I$ for $x \in \partial(\langle 0, l\rangle^2)$ and $\beta \in \langle 0, 1\rangle$,

and extend the function T_4 for a fixed $k_1 \in \{1, \ldots, l - 1\}$ on the whole $\langle 0, 1\rangle \times \{k_1\} \times \langle 0, l\rangle$, and for a fixed k_2 on the whole $\langle 0, 1\rangle \times \langle 0, l\rangle \times \{k_2\}$ by the same method by which we extended T_4 in the previous section on the whole $\langle 0, 1\rangle \times \langle 0, l\rangle$.

The functions T_3 and T_4 are defined on $\langle 0, 1\rangle \times (\langle 0, l\rangle^2 \cap S_1^2)$, have continuous second derivatives w.r.t. their domains, belong to $KZ((\varepsilon_{j+2})/16)$ and satisfy:

(5.16)
$$\left\|\frac{\partial T_k}{\partial \beta}\right\|, \left\|\frac{\partial T_k}{\partial x_i}\right\|, \left\|\frac{\partial^2 T_k}{\partial \beta\, \partial x_i}\right\| \leqq \max(A_0, A_1)$$
$$\text{for } k = 2, 3 \text{ and } i = 1, 2.$$

Now we shall define the function T_5 for those points from $\langle 0, 1\rangle \times \langle 0, l\rangle^2$, which have at least one integer coordinate:

(5.17)
$$\begin{aligned} T_5(1, x) &= G(x) && \text{for } x \in \langle 0, l\rangle^2, \\ T_5(0, x) &= I && \text{for } x \in \langle 0, l\rangle^2, \\ T_5(\beta, x) &= T_3(2\beta - 1, x) && \text{for } \beta \in \langle \tfrac{1}{2}, 1\rangle \text{ and } x \in \langle 0, l\rangle^2 \cap S_1^2, \\ & \ T_4(2\beta, x) && \text{for } \beta \in \langle 0, \tfrac{1}{2}\rangle \text{ and } x \in \langle 0, l\rangle^2 \cap S_1^2. \end{aligned}$$

T_4 is a continuous function, belongs to $KZ(\varepsilon_{j+2}/(2 \,.\, 16))$, has continuous second derivatives w.r.t. its domain and by (5.16) and (5.3) the following estimates hold for $i = 1, 2$:

(5.18)
$$\left\|\frac{\partial T_5}{\partial \beta}\right\|, \left\|\frac{\partial^2 T_5}{\partial \beta\, \partial x_i}\right\| \leqq 2 \max\{A_0, A_1\},$$
$$\left\|\frac{\partial T_5}{\partial x_i}\right\| \leqq \max\{A_0, A_1, P_j\}, \quad \left\|\frac{\partial^2 T_5}{\partial x_j\, \partial x_l}\right\| \leqq \varkappa^2 \zeta^{-2}.$$

By Lemma 3.2 we can extend the function T_5 for each $k_1, k_2 \in \{0, 1, \ldots, l - 1\}$ on the whole $\langle 0, 1\rangle \times \langle k_1, k_1 + 1\rangle \times \langle k_2, k_2 + 1\rangle$. Using Theorem 2.3 we can get an extension $T_5\colon \langle 0, 1\rangle \times \langle 0, l\rangle^2 \to SY(n)$, which belongs to $KZ(\varepsilon_{j+2}/(2.16.16))$, has continuous second derivatives and if we put $A_2 = A(2 \max\{A_0, A_1, P_j, \varkappa^2\zeta^{-2}\}, (\varepsilon_{j+2})/(2 \,.\, 16), 3)$, the following estimates hold:

(5.19)
$$\left\|\frac{\partial T_5}{\partial \beta}\right\|, \left\|\frac{\partial T_5}{\partial x_i}\right\|, \left\|\frac{\partial^2 T_5}{\partial \beta\, \partial x_i}\right\| \leqq A_2 \quad \text{for } i = 1, 2.$$

Now we can define a homotopy of g and G_0. Let $H\colon \langle 0, 1\rangle \times \langle 0, 1\rangle^2 \to SY(n)$,

$$\begin{aligned} H(\beta, x) &= T_1(3\beta - 2 \,.\, lx) && \text{for } \beta \in \langle \tfrac{2}{3}, 1\rangle, \ x \in \langle 0, 1\rangle^2, \\ & \ T_2(3\beta - 1, lx) && \text{for } \beta \in \langle \tfrac{1}{3}, \tfrac{2}{3}\rangle, \ x \in \langle 0, 1\rangle^2, \\ & \ T_5(3\beta, lx) && \text{for } \beta \in \langle 0, \tfrac{1}{3}\rangle, \ x \in \langle 0, 1\rangle^2. \end{aligned}$$

By (5.4), (5.10) and (5.17) we have

$$H(1, x) = T_1(1, lx) = \bar{g}(lx) = g(x) \quad \text{for} \quad x \in \langle 0, 1 \rangle^2 ,$$
$$H(0, x) = T_5(0, lx) = I = G_0(x) \quad \text{for} \quad x \in \langle 0, 1 \rangle^2 ,$$

and by (5.17), (5.15), (5.12), (5.9) and (5.6),

$$H(\beta, x) = I \quad \text{for} \quad \beta \in \langle 0, 1 \rangle \quad \text{and} \quad x \in \partial(\langle 0, 1 \rangle^2) .$$

By (5.19), (5.10) and (5.7) we can estimate

$$\left\| \frac{\partial H}{\partial \beta} \right\| \leqq 3 \max \{3S^2 P_j^2, A_2\} , \quad \left\| \frac{\partial H}{\partial x_i} \right\| \leqq l \max \{3S^2 P_j^2, A_2\} \quad \text{and}$$
$$\left\| \frac{\partial^2 H}{\partial \beta \, \partial x_i} \right\| \leqq 3l \max \{3S^2 P_j^2, A_2\} \quad \text{for} \quad i = 1, 2 .$$

Since $l \leqq 2L$, the constant $c = 6 \max \{3S^2 P_j^2, A_2\}$ is the desired constant $c(SY(n), 2)$ from Definition I.2.3, therefore $SY(n) \in EP(2)$.

LIST OF SYMBOLS

A^*	I.1	$\mathcal{L}$	A.2
$\|A\|$	I.1	Matr (n)	I.1
$AP(n)$	I.1	$\mathbb{N}$	I.1
$AP_{\text{sol}}(n)$	I.1	$O(n)$	I.1
$a(x)$	II.1	$P(n, r, l, q)$	I.4
$\tilde{b}$	II.2	$P\,P(p)$	I.2
$\hat{b}$	II.3	$\mathcal{P}(r)$	II.1
$\hat{b}_t(x)$	II.5	$\mathcal{P}_j(r)$	II.1
c_j, c_0	III.4	$\mathcal{P}_{\geq j}(r)$	II.1
$C(M, j)$	I.3	p_n	III.5
D	A.1	$QP(n, r)$	I.2
D_j^r	III.3	$QP_{\text{sol}}(n, r)$	I.2
$D^r(n)$	III.3	S	A.2
$EP(j)$	I.2	S_j^r	II.1
$E(n)$	II.5	$SO(n)$	I.1
F	II.1	$SU(n)$	I.1
$f(x, \alpha)$	II.1	$SY(n)$	I.1
g_0, g_1	A.4	$Tr(C)$	I.4
G_0	A.5	$U(n)$	I.1
I	I.1	$V(n, r)$	I.4
$\mathbb{K}$	I.1	$\mathcal{W}$	A.1
$K(r, j)$	II.3	$W(n, r)$	I.4
$KZ(\varepsilon)$	II.4	$[x]$	II.1

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