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A COARSE CONVERGENCE GROUP NEED NOT BE PRECOMPACT

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0. PRELIMINARIES AND THE MAIN RESULT

A convergence space is a pair $(X, \mathscr{C})$ where X is a set and $\mathscr{C}$ is a suitable subset of ${}^{\omega}X \times X$. If $\langle S, x\rangle \in \mathscr{C}$, then we say that the sequence S converges to the point x and denote this fact by $S \to x$ or $\lim S = x$. It is sometimes convenient to express explicitly the values of S; in such a case we write $\langle s_n\colon n \in \omega\rangle \to x$ or $\lim \langle s_n\colon n \in \omega\rangle = x$. If $\langle s_n\colon n \in \omega\rangle \in {}^{\omega}X$ and for some $p \in X$, $s_n = p$ for all $n \in \omega$, then the sequence is said to be *constant* and denoted by (p).

The commonly adopted list of axioms reads as follows.

(i) Constant sequence axiom: For each $p \in X$, $\langle (p), p\rangle \in \mathscr{C}$.

(ii) Subsequence axiom: If $\langle S, x\rangle \in \mathscr{C}$ and $f \in {}^{\omega}\omega$ is strictly increasing, then $\langle S \circ f, x\rangle \in \mathscr{C}$.

(iii) Urysohn axiom: If $S \in {}^{\omega}X$, $x \in X$ and for each strictly increasing $f \in {}^{\omega}\omega$ there is a strictly increasing $g \in {}^{\omega}\omega$ with $\langle S \circ f \circ g, x\rangle \in \mathscr{C}$, then $\langle S, x\rangle \in \mathscr{C}$, too.

(iv) Unique limits axiom: If for some $x, y \in X$ and $S \in {}^{\omega}X$ one has $\langle S, x\rangle \in \mathscr{C}$, $\langle S, y\rangle \in \mathscr{C}$, then $x = y$.

If $(X, +)$ is a group and $\mathscr{C}$ is a convergence structure over X, then $\mathscr{C}$ is compatible with $+$, or $(X, +, \mathscr{C})$ is a *convergence group* if, moreover,

(v) Group convergence axiom: $\langle\langle s_n\colon n \in \omega\rangle, x\rangle \in \mathscr{C}$ and $\langle\langle t_n\colon n \in \omega\rangle, y\rangle \in \mathscr{C}$ implies $\langle\langle s_n - t_n\colon n \in \omega\rangle, x - y\rangle \in \mathscr{C}$ (briefly, $\langle S, x\rangle$, $\langle T, y\rangle \in \mathscr{C}$ implies $\langle S - T, x - y\rangle \in \mathscr{C}$) holds.

Similarly as in the topological spaces, there are many convergence structures over a given set X. $(X, \mathscr{C})$ is said to be *coarser* than $(X, \mathscr{D})$ if $\mathscr{D} \subseteq \mathscr{C}$. In contrast to the topological case, if $(X, \mathscr{C})$ is a convergence space, then there always exists a *coarse* convergence structure $\mathscr{D} \supseteq \mathscr{C}$, ike. $\mathscr{D}$ is a convergence structure and if $\mathscr{D} \subsetneq \mathscr{E} \subseteq {}^{\omega}X \times X$, then $\mathscr{E}$ fails to be a convergence structure. The same holds for the convergence groups and the proof is just a straightforward application of the maximality principle (cf. [FZ]).

Quite recently, I. Prodanov and L. Stoyanov proved the following remarkable theorem.

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Theorem [PS]: *Every minimal Abelian topological group is precompact.*

Looking for an analogue in the convergence setting, obviously "coarse" replaces "minimal", but we need a proper notion corresponding to "precompact". The one below seems to be natural.

Definition. A convergence space $(X, \mathscr{C})$ is called *sequentially compact*, if each infinite subset of X contains a nontrivial convergent sequence, i.e. for each $Y \in [X]^{\geq \omega}$ there is some $\langle\langle a_n: n \in \omega\rangle, x\rangle \in \mathscr{C}$ with $\langle a_n: n \in \omega\rangle \neq (x)$ and $\{a_n: n \in \omega\} \in [Y]^{\omega}$.

A convergence group $(X, +, \mathscr{C})$ is called *precompact*, if it admits an embedding into a sequentially compact convergence group.

Now, we formulate the main result of the paper. Recall that a group $(G, +)$ is called *Boolean* if $x + x = e$ for all $x \in X$, e denoting the zero element. Clearly every Boolean group is Abelian.

Theorem. *Assuming* CH, *there is a coarse convergence Bovlean group which cannot be embedded into any sequentially compact convergence group.*

It is not quite clear what the theorem really says. It either indicates that the Prodanov-Stoyanov theorem fails in convergence groups, or suggests that our definition of being precompact is not quite sound. We leave it to the reader to judge the problem.

1. THE PROOF OF THE THEOREM: Part without CH

In order to keep all matters under control, we shall try to strip all things to bare bones throughout the whole proof. For instance, we shall not be particularly interested whether our example is coarse or not. Let us see first why we are justified to do so.

1.1. Lemma. *Let $(X, +, \mathscr{C})$ be a convergence group and let $(X, +, \mathscr{D})$ be a coarse convergence group with $\mathscr{D} \supseteq \mathscr{C}$. Suppose there is a sequence $Y = \langle y_n: n \in \omega\rangle \in {}^{\omega}X$ such that for each strictly increasing $f \in {}^{\omega}\omega$ there is a strictly increasing $g \in {}^{\omega}\omega$ and $p \neq e$ with $\langle\langle y_{f\circ g(n)} - y_{f\circ g(n+1)}: n \in \omega\rangle, p\rangle \in \mathscr{C}$. Then $(X, +, \mathscr{D})$ cannot be embedded into a sequentially compact convergence group.*

Proof. Suppose the contrary, let $(X, +, \mathscr{D})$ embed into a sequentially compact $(Z, +, \mathscr{E})$. By the sequential compactness of Z, there is some $f \in {}^{\omega}\omega$, f strictly increasing, and $z \in Z$ such that $\langle y_{f(n)}: n \in \omega\rangle$ converges to z. By our assumption, for some non-zero $p \in X$ and some $g \in {}^{\omega}\omega$, $\lim \langle y_{f\circ g(n)} - y_{f\circ g(n+1)}: n \in \omega\rangle = p$ in $\mathscr{C}$, hence in $\mathscr{D}$ as well, provided $\mathscr{C} \subseteq \mathscr{D}$. But $Y \circ f \circ g$ is a subsequence of $Y \circ f$, therefore using axioms (ii) and (v) we can calculate in Z as follows:

$e = z - z = \lim \langle y_{f\circ g(n)} - y_{f\circ g(n+1)}: n \in \omega\rangle = p \neq e$. This contradiction proves the lemma. □

The lemma just proved suggests that we need to find a sequence $\langle y_n: n \in \omega\rangle$ in the space we are looking for, and sufficiently many convergent sequences of the form $\langle y_{f(n)} - y_{f(n+1)}: n \in \omega \rangle$. One may ask whether this suffices; our next lemma confirms that it does, with one exception: the uniqueness of the limits may be lost.

1.2. Lemma and definition. *Let $(X, +)$ be a group, $\mathscr{F} \subseteq {}^{\omega}X \times X$. Then there is a structure $\mathscr{C} \subseteq {}^{\omega}X \times X$ such that $\mathscr{C} \supseteq \mathscr{F}$, $\mathscr{C}$ satisfies* (i), (ii), (iii) *and* (v), *and $\mathscr{C}$ is minimal with respect to the inclusion.*

Moreover, $\mathscr{C}$ can be obtained as follows. For $\mathscr{G} \subseteq {}^{\omega}X \times X$, denote

$$\delta\mathscr{G} = \{\langle S \circ f, x\rangle : \langle S, x\rangle \in \mathscr{G} \text{ and } f \in {}^{\omega}\omega \text{ is strictly increasing}\} \cup \cup \{\langle (x), x\rangle : x \in X\}.$$

$$-\mathscr{G} = \{\langle -S, -x\rangle : \langle S, x\rangle \in \mathscr{G}\},$$

$$\langle \mathscr{G} \rangle = \{\langle S_0 + S_1 + \ldots + S_n,\ x_0 + x_1 + + x_n\rangle : n \in \omega \text{ and for each } i \leqq n,\ \langle S_i, x_i\rangle \in \mathscr{G}\},$$

$$\zeta\mathscr{G} = \{\langle S, x\rangle \in {}^{\omega}X \times X : \text{for each strictly increasing } f \in {}^{\omega}\omega \text{ there is a strictly increasing } g \in {}^{\omega}\omega \text{ with } \langle S \circ f \circ g, x\rangle \in \mathscr{G}\}.$$

Then $\mathscr{C} = \zeta\langle \delta\mathscr{F} \cup \delta(-\mathscr{F})\rangle$.

This $\mathscr{C}$ will be called *the convergence hull of $\mathscr{F}$.*

We shall omit the proof, for the lemma is a mere variation on the similar ones already proved in [Z] or [D].

Let us now describe how our convergence group will look like. Denote by Inc the set of all strictly increasing functions from ω to ω. For $A \subseteq$ Inc, let $X(A) = \omega \cup A$ and let $G(A)$ be the free Boolean group over $X(A)$, that means, $G(A) = ([X(A)]^{<\omega}, \Delta)$, where Δ stands for the symmetric difference: $p \,\Delta q = (p \cup q) \setminus (p \cap q)$ for $p, q \subseteq X(A)$. Thus the zero element in $G(A)$ is just the empty set.

The convergence structure $\mathscr{C}(A)$ is defined as the convergence hull of the family

$$\mathscr{F}(A) = \{\langle\langle\{f(n), f(n+1)\} : n \in \omega\rangle, \{f\}\rangle : f \in A\}.$$

According to the previous lemmas, $(G(A), \Delta, \mathscr{C}(A))$ is the group we are looking for, provided we can show that the limits are unique and verify the condition from 1.1. This definitely does not hold in general, as indicated by easy examples. For instance, if $f_0, f_1 \in A$, where $f_0(n) = n$, $f_1(n) = 2n$, then the uniqueness of the limits. fails. Indeed,

$$\lim \langle\{n, n+1\} : n \in \omega\rangle = \{f_0\}, \quad \lim \langle\{2n, 2n+2\} : n \in \omega\rangle = \{f_1\}$$

and by the subsequence axiom, we have also

$$\lim \langle\{2n, 2n+1\} : n \in \omega\rangle = \lim \langle\{2n+1, 2n+2\} : n \in \omega\rangle = \{f_0\}.$$

Consider the sequence $S = \langle\{2n, 2n+1\}\,\Delta\{2n+1, 2n+2\}\,\Delta\{2n, 2n+2\} : n \in \omega\rangle$. We have $\lim S = e$, for S is just a constant sequence (e); on the other hand, $\lim S = \{f_0\}\,\Delta\,\{f_0\}\,\Delta\,\{f_1\} = \{f_1\}$.

Another simple example shows that 1.1 need not apply. Consider $A = \{f_0\}$.

If $Y = \langle \{n^2\}: n \in \omega \rangle \in {}^{\omega}G(A)$, then there is no $g \in \text{Inc}$ such that $\langle \{(g(n))^2, (g(n+1))^2\}: n \in \omega \rangle$ converges in $(G(A), \Delta, \mathscr{C}(A))$.

So, from now on, all our effort will concentrate on finding a suitable "bug free" subset $A \subseteq \text{Inc}$.

We plan to use the sequence $\langle \{n\}: n \in \omega \rangle \in {}^{\omega}G(A)$ as the Y from 1.1. The reader can immediately check that the assumptions of 1.1 are satisfied for $Y = \langle \{n\}: n \in \omega \rangle$ if and only if

$$(*) \qquad \text{for each} \quad g \in \text{Inc} \quad \text{there is an} \quad h \in \text{Inc} \quad \text{with} \quad g \circ h \in A\,.$$

The property of A which implies the uniqueness of limits is a bit worse to find. The next lemma will simplify our search.

1.3. Lemma. *Let $(X, +)$ be a group, $\mathscr{C} \subseteq {}^{\omega}X \times X$, and let $(X, +, \mathscr{C})$ satisfy (i), (ii), (iii), (v). Then $(X, +, \mathscr{C})$ has unique limits iff for each $x \in X$, $x \neq e$ we have $\langle (e), x \rangle \notin \mathscr{C}$.*

Proof. If $\langle (e), x \rangle \in \mathscr{C}$ for some $x \neq e$, then the sequence (e) has at least two limits, namely x and e.

If there is some $\langle p_n: n \in \omega \rangle$ and $s \neq t$ both being limits of $\langle p: n \in \omega \rangle$, then the constant sequence $(e) = \langle p_n - p_n: n \in \omega \rangle$ has e and $s - t$ as its limits. □

Notice that in the special case $(G(A), \Delta, \mathscr{C}(A))$, according to 1.2, $\lim (e) = x$ iff there are $f_0, f_1, \ldots, f_{k-1} \in A$ (repetitions possible) and $g_0, g_1, \ldots, g_{k-1} \in \text{Inc}$ such that $x = \{f_0\} \Delta \{f_1\} \Delta \ldots \Delta \{f_{k-1}\}$, and, for all $n \in \omega$, $e = \{f_0(g_0(n)), f_0(g_0(n) + 1)\} \Delta \{f_1(g_1(n)), f_1(g_1(n) + 1)\} \Delta \ldots \Delta \{f_{k-1}(g_{k-1}(n)), f_{k-1}(g_{k-1}(n) + 1)\}$.

To abbreviate the notation, for $\langle x_0, \ldots, x_{k-1} \rangle \in {}^{k}G(A)$, let $\Delta_{i \in k} x_i = x_0 \Delta x_1 \Delta \ldots \ldots \Delta x_{k-1}$.

Thus $(G(A), \Delta, \mathscr{C}(A))$ has unique limits iff

$$(**) \qquad \text{for each} \quad k \in \omega \quad \text{and for each} \quad \langle f_0, \ldots, f_{k-1} \rangle \in {}^{k}A\,,$$

if there is some $\langle g_0, \ldots, g_{k-1} \rangle \in {}^{k}\text{Inc}$ such that for all $n \in \omega$, $\Delta_{i \in k} \{f_i(g_i(n)), f_i(g_i(n) + 1)\} = e$, then $\Delta_{i \in k} \{f_i\} = e$.

Since we are not interested in g_i's, let us get rid of them. It can be done as follows: $(G(A), \Delta, \mathscr{C}(A))$ has unique limits provided that

$$(***) \qquad \text{for each} \quad k \in \omega \quad \text{and for each} \quad \langle f_0, \ldots, f_{k-1} \rangle \in {}^{k}A:$$

if for each $n \in \omega$ there is some $\langle a_0, \ldots, a_{k-1} \rangle \in {}^{k}(\omega - n)$ such that $\Delta_{i \in k} \{f_i(a_i), f_i(a_i + 1)\} = e$, then $\Delta_{i \in k} \{f_i\} = e$.

Indeed, $(***)$ implies $(**)$.

There are two expressions dealing with the symmetric difference in $(***)$. The latter, $\Delta_{i \in k} \{f_i\} = e$, is clearly satisfied iff for each $f \in A$ the cardinality of the set $\{i \in k: f = f_i\}$ is an even number. Let us discuss briefly the former.

Suppose $\langle s_i: i \in k \rangle$ is an ordered k-tuple of two-element sets.

Call $\langle s_i: i \in k\rangle$ a *path* if $\underset{i\in k}{\Delta} s_i = \emptyset$ and for each $l < k$ and each permutation $\pi: k \to k$, $\underset{i\in l}{\Delta} s_{\pi(i)} \neq \emptyset$. A path is said to be *trivial* if $k = 2$, otherwise it is *nontrivial.* Obviously, if $\langle s_0, s_1\rangle$ is a trivial path, then $s_0 = s_1$.

It is easy to show that for each k-tuple $\langle s_i: i \in k\rangle$ of two-element sets with $\underset{i\in k}{\Delta} s_i = \emptyset$ there is a permutation $\pi: k \to k$, some $r \in k$ and natural numbers $0 = j_0 < j_1 < \ldots$ $\ldots < j_r = k$ such that for each $l < r$, $\langle s_{\pi(i)}: j_l \leqq i < j_{l+1}\rangle$ is a path. Let us say that $\langle s_i: i \in k\rangle$ is *decomposed* into $\{\langle s_{\pi(i)}: j_l \leqq i < j_{l+1}\rangle: l < r\}$.

Using the notions just introduced, we have the final version of a sufficient condition.

$(G(A), \Delta, \mathscr{C}(A))$ has unique limits provided that

$$(+) \qquad \text{for each } k \in \omega \text{ and for each } \langle f_0, \ldots, f_{k-1}\rangle \in {}^kA\,,$$
$$\left|\{\langle a_0, \ldots, a_{k-1}\rangle \in {}^k\omega: \langle\{f_i(a_i), f_i(a_i + 1)\}: i \in k\rangle \text{ is a nontrivial path}\}\right| < \omega\,.$$

and

for distinct $f, g \in A$,

$$\left|\{(n, m) \in \omega \times \omega: f(n) = g(m) \text{ and } f(n + 1) = g(m + 1)\}\right| < \omega\,.$$

We need to show that $(+)$ implies $({*}{*}{*})$. Let $k \in \omega$, $l \leqq k$, $\pi: k \to k$ a permutation. By $(+)$, when applied to $\langle f_{\pi(0)}, \ldots, f_{\pi(l-1)}\rangle \in {}^lA$, there is some $n(\pi, l) \in \omega$ such that $\langle\{f_{\pi(i)}(a_i), f_{\pi(i)}(a_i + 1)\}: i \in l\rangle$ is never a nontrivial path provided $\max\{a_i: i \in\in l\} \geqq$ $\geqq n(\pi, l)$. Let $n_1 = \max\{n(\pi, l): l \leqq k,\ \pi \in {}^kk$ is a permutation$\}$. Let $n_2 =$ $= \max\{n, m \in \omega$: there are $i < j < k$ such that $f_i \neq f_j$ and $f_i(n) = f_j(m), f_i(n + 1) =$ $= f_j(m + 1)\}$.

Consider $\langle a_0, \ldots, a_{k-1}\rangle \in {}^k(\omega - n)$, where $n > \max\{n_1, n_2\}$. If $\underset{i\in k}{\Delta}\{f_i(a_i),$ $f_i(a_i + 1)\} = e$, then the only possible decomposition of $\langle\{f_i(a_i), f_i(a_i + 1)\}: i \in k\rangle$ is the decomposition into trivial paths, for $n > n_1$. But if $\{f_i(a_i), f_i(a_i + 1)\} =$ $= \{f_j(a_j), f_j(a_j + 1)\}$, then $f_i(a_i) = f_j(a_j)$ and $f_i(a_i + 1) = f_j(a_j + 1)$, thus $f_i = f_j$ for $n > n_2$. So for each $f \in A$, $|\{i < k: f = f_i\}|$ is even, therefore $\underset{i\in k}{\Delta}\{f_i\} = e$, which was to be proved.

2. THE PROOF OF THE THEOREM: Part using CH

Our aim now is to construct, assuming CH, a family $A \subseteq$ Inc such that both $(*)$ and $(+)$ hold for A. This will complete the proof of the theorem.

Enumerate Inc $= \{g_\alpha: \alpha \in \omega_1\}$ and let $f_0 = g_0$, $h_0 = \text{id}$.

Suppose $\alpha < \omega_1$ and let $\{f_\beta: \beta < \alpha\}$ and $\{h_\beta: \beta < \alpha\}$ be found. Our induction assumptions are as follows:

(j) for each $\beta < \alpha$, $f_\beta, h_\beta \in$ Inc;

(jj) for each $\beta < \alpha$, $f_\beta = g_\beta \circ h_\beta$;

(jjj) $(+)$ holds for $A_\alpha = \{f_\beta: \beta < \alpha\}$.

Since $\alpha < \omega_1$, reenumerate $\{f_\beta : \beta < \alpha\}$ as $\{F_n : n \in \omega\}$. Proceed by induction to find the values of f_α and h_α in the manner described below. Let $h_\alpha(0) = 0$, $f_\alpha(0) = g_\alpha(h_\alpha(0))$. If $f_\alpha(n) = g_\alpha(h_\alpha(n))$ is already known. let us define $n + 1$ subsets of ω by the following rule:

$B_0 = \{n\} \cup \{k \in \omega$: there are $q, j \leqq n$ such that $f_\alpha(q) = F_j(k)\}$, and for $1 \leqq p \leqq n$.

$B_p = B_{p-1} \cup \{k \in \omega$: there are $i, j \leqq n$ and $q \in B_{p+1}$ such that

$$F_i(k) \in \{F_j(q - 1, F_j(q), F_j(q + 1)\}\} .$$

Evidently the set B_n is finite, so let $t = \max\{F_j(q + 1): q \in B_n, j \leqq n\}$. There is some $m > h_\alpha(n)$ such that $g_\alpha(m) > t$, hence define $h_\alpha(n + 1) = m$, $f_\alpha(n + 1) = g_\alpha(h_\alpha(n + 1))$.

Since it is now obvious that (j) as well as (jj) hold true, let us check (jjj) for $A_{\alpha+1}$.

Let $\langle f_{\alpha_0}, \ldots, f_{\alpha_{k-1}} \rangle \in {}^k A_{\alpha+1}$.

If $f_\alpha \notin \{f_{\alpha_0}, \ldots, f_{\alpha_{k-1}}\}$, then (jjj) follows by the induction hypothesis.

If $f_\alpha \in \{f_{\alpha_0}, \ldots, f_{\alpha_{k-1}}\}$, then there is some $m \in \omega$ such that $\{f_{\alpha_0}, \ldots, f_{\alpha_{k-1}}\} \subseteq \{f_\alpha, F_0, \ldots, F_m\}$. Let $n > \max\{m, k\}$ be arbitrary. We claim that there is no nontrivial path $\langle\{f_{\alpha_i}(a_i), f_{\alpha_i}(a_i + 1)\} : i \in k\rangle$ with $\{f_\alpha(n), f_\alpha(n + 1)\}$ occurring in it. This immediately follows from our definition of $f_\alpha(n + 1)$ – it is too big to be reached by an overlapping sequence of sets $\{F_j(i), F_j(i + 1)\}$ with $f_\alpha(n)$ belonging to some of them, provided the length of the sequence is at most n. But in the case under consideration it is even less, namely k. Therefore $\{\langle a_0, \ldots, a_{k-1}\rangle \in {}^k\omega : \langle\{f_{\alpha_i}(a_i), f_{\alpha_i}(a_i + 1)\} : i \in k\rangle$ is a nontrivial path$\} \subseteq {}^k\max\{m, k\}$.

The proof of the second statement follows immediately from the fact that $n \in B_0$ in our construction and we shall omit the details.

Therefore $A = \{f_\alpha : \alpha < \omega_1\}$ is the set of functions we needed and the proof of the theorem is complete.

3. CONCLUDING REMARKS

a) It should be noted that only minor modifications of the proof given in the second part are needed to show the validity of the theorem under the assumption "there exists a 2^ω-scale". However, the existence of the set $A \subseteq {}^\omega\omega$ satisfying $(*)$ and $(+)$ in ZFC alone is still an open question.

b) R. Frič and F. Zanolin proposed in [FZ] another definition of precompactness of convergence groups. They call a convergence group G *precompact* if each sequence in G contains a Cauchy subsequence. It is easy to check that our group fails to be precompact under the latter definition, either, so both definitions still may coincide.

References

[D] *M. Dolcher:* Topologie e strutture di convergenza, Ann. Scuola Norm. Sup. Pisa *14* (1960), 63—92.

[FZ] *R. Frič, F. Zanolin:* Coarse convergence groups, in: Convergence Structures 1984, Proc. of the Conference on Convergence, Bechyně 1984, Mathematical Research 24, Akademie-Verlag, Berlin 1985, 107—114.

[PS] *I. Prodanov, L. Stoyanov:* Every minimal Abelian group is precompact, C.R. Acad. Bulgar. Sci. *37* (1984), 23—26.

[Z] *F. Zanolin:* Solution of a problem of Josef Novák about convergence groups, Bollettino Un. Mat. Ital. (*5*) 14-A (1977), 375—381.

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