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CRITERIA FOR ABSOLUTE AND STRONG CONVERGENCE OF FOURIER SERIES

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1. Let $C(T)$ denote the class of continuous functions $f\colon T = R/2\pi Z \to C$ and A its subclass of continuous functions with absolutely convergent Fourier series. If (a'_n, b'_n) and (a''_n, b''_n), $n \in N$, are the pairs of (cosine, sine) Fourier coefficients of $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$, respectively, then the Fourier coefficients of the function f are of the form

$$\begin{aligned} (1) \qquad 2\hat{f}(n) &= (a'_n + b''_n) + \mathrm{i}(a''_n - b'_n)\,, \\ 2\hat{f}(-n) &= (a'_n - b''_n) + \mathrm{i}(a''_n + b'_n)\,, \quad n \in N_0\,. \end{aligned}$$

The following result is well known ([5], p. 9).

Theorem A. *If $f \in C(T)$ and $\hat{f}(n) \geqq 0$ for every $n \in Z$, then $f \in A$. Every function in A is a linear combination of functions in $C(T)$ with nonnegative Fourier coefficients.*

N. Artémiades ([1], Th. 1; [2], Th. 1) has generalized this result to

Theorem B. *Let $f \in C(T)$. If there exists $a \in R$ such that $a \leqq \arg \hat{f}(n) \leqq a + \pi/2$ for every $n \in Z$, then $f \in A$. Every $f \in A$ is a linear combination of continuous functions on T with the above property.*

Theorem B can easily be improved to

Theorem 1. *Let $f \in C(T)$. If there exist $a \in R$ and $\delta > 0$ such that $a - \pi/2 + \delta \leqq \arg \hat{f}(n) \leqq a + \pi/2 - \delta$ $(n \in Z)$, then $f \in A$. Every $f \in A$ is a linear combination of functions belonging to $C(T)$ with this property.*

Proof. We may suppose $a = 0$. (Otherwise we would consider the function $g = \mathrm{e}^{-\mathrm{i}a} f$.) Let

$$F(x) = \frac{f(x) + \overline{f(-x)}}{2}$$

for every $x \in T$. Then $F \in C(T)$ and $\hat{F}(n) = \operatorname{Re} \hat{f}(n) \geqq 0$ $(n \in Z)$, by the assumptions. Now $\sum \operatorname{Re} \hat{f}(n) < \infty$ by Theorem A. This and $|\operatorname{Im} \hat{f}(n)| \leqq \operatorname{Re} \hat{f}(n) \,.\, \operatorname{tg}(\pi/2 - \delta)$ $(n \in Z)$ implies $\sum |\operatorname{Im} \hat{f}(n)| < \infty$. Hence $f \in A$. The second assertion is obvious.

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Remark 1. The function

$$f(x) = \sum_{n=2}^{\infty} \left(\frac{\cos nx}{n \log^2 n} + \frac{\sin nx}{\log n} \right)$$

provides an example that the assertion of Theorem 1 does not necessarily hold if we merely suppose $a - \pi/2 < \arg \hat{f}(n) < a + \pi/2$ $(n \in Z)$.

2. If f is a real even function, then Theorem A actually says that its Fourier series is absolutely convergent if the cosine Fourier coefficients are nonnegative. A similar statement for the (necessarily sine) Fourier series of an odd function is false. However. one has an analogue if absolute convergence is replaced by uniform convergence. This is the content of a classical theorem by Paley ([3], p. 277). In a recent work N. Tanović-Miller (see [6], [7]) has shown that it can be interesting to interpolate the study of absolute and uniform convergence of trigonometric and Fourier series by that of strong convergence.

Definition 1. A sequence $\{d_k\}$ is said to be *strongly C_1 summable to a limit d*, and we write $d_k \to d[C_1]$, if

$$\frac{1}{n} \sum_{k=1}^{n} |d_k - d| = o(1) \quad (n \to \infty) .$$

Definition 2. A sequence $\{d_k\}$ is said to be *strongly convergent to a limit d*, and we write $d_k \to d$ [I], if

i) $d_k \to d$ $(k \to \infty)$,

ii) $(1/n) \sum_{k=1}^{n} k|d_k - d_{k-1}| = o(1)$ $(n \to \infty)$, i.e. $k(d_k - d_{k-1}) \to 0$ $[C_1]$.

The notion of strong convergence was introduced by J. M. Hyslop [4], who extended Hardy-Littlewood's concept of strong C_1 summability to Cesàro transformations C_α of order $\alpha \geqq 0$.

Let S denote the class of continuous functions on T with uniformly strongly convergent Fourier series. From the results of [6] (see Th. 1 and Th. 3 therein) one can deduce

Theorem C. $S = \{f \in C(T)\colon (1/n) \sum_{|k| \leqq n} |k \hat{f}(k)| = o(1)\}$.

The strict inclusions $A \subset S \subset U$ hold ([6], Th. 4(ii)), where U is the class of sums of uniformly convergent Fourier series.

Paley's theorem, mentioned above, has been generalized to

Theorem D ([6], The. 6(ii)). *If f is a continuous real odd function with nonnegative Fourier coefficients, then $f \in S$.*

Let us modify Theorem D so as to obtain a form analogous to that of Theorem A.

Theorem 2. *If $f \in C(T)$ has the property $i\hat{f}(n) \geqq 0$, $i\hat{f}(-n) \leqq 0$ $(n \in N)$, then $f \in S$.*

Proof. Looking at the relations (1), we see that the assumptions of the theorem yield

$$(2) \qquad a_n' = b_n'' = 0 \quad \text{and} \quad |a_n''| \leqq b_n' \quad (n \in N)\,.$$

Hence the Fourier series of the function $u = \operatorname{Re} f$ has the form $\sum_{n=1}^{\infty} b_n' \sin nx$, with $b_n' \geqq 0$ for every $n \in N$. By Theorem C and Theorem D this implies $u \in S$, i.e.

$$(3) \qquad \frac{1}{n} \sum_{k=1}^{n} k b_k' = o(1)\,.$$

The Fourier series of $v = \operatorname{Im} f$ is $\sum_{n=1}^{\infty} a_n'' \cos nx$. (2) and (3) yield

$$\frac{1}{n} \sum_{k=1}^{n} k|a_k''| = o(1)$$

and therefore $f \in S$ by Theorem C.

On this result we base the next two theorems.

Theorem 3. *Let $f \in C(T)$. If there exist $a \in R$ and $\delta > 0$ such that $a - \pi + \delta \leqq \arg \hat{f}(n) \leqq a - \delta$, $a + \delta \leqq \arg \hat{f}(-n) \leqq a + \pi - \delta$ $(n \in N)$, then $f \in S$.*

Proof. As in the proof of Theorem 1 we may suppose $a = 0$. Consider the function G defined by

$$G(x) = \frac{f(x) - \overline{f(-x)}}{2}\,.$$

Its Fourier coefficients satisfy the relations $\mathrm{i}\hat{G}(n) = -\operatorname{Im} \hat{f}(n) \geqq 0$, $\mathrm{i}\hat{G}(-n) = -\operatorname{Im} \hat{f}(-n) \leqq 0$ for every $n \in N$. Hence $G \in S$ by Theorem 2, i.e.

$$\frac{1}{n} \sum_{|k| \leqq n} |k \operatorname{Im} \hat{f}(k)| = o(1)\,.$$

If we set $\operatorname{Re} \hat{f}(k)$ instead of $\operatorname{Im} \hat{f}(k)$, this relation remains true, since $|\operatorname{Re} \hat{f}(k)| \leqq |\operatorname{Im} \hat{f}(k)| \boxed{\mathrm{c\,tg}}\, \delta$. Hence

$$\frac{1}{n} \sum_{|k| \leqq n} |k \hat{f}(k)| = o(1) \quad \text{and} \quad f \in S\,.$$

Remark 2. By Zygmund-Paley's theorem (see [3], p. 307 or [8], p. 219) the series

$$\sum \left(\pm \frac{\cos nx}{n^{2/3} \log n} + \frac{\sin nx}{n \log n} \right)$$

is uniformly convergent for almost every choice of signs $\pm$. Therefore its sum f (for an appropriate choice of $\pm$) belongs to $C(T) \setminus S$ since its coefficients do not satisfy the condition of Theorem C. This shows that the assumption of Theorem 3 infolving δ cannot be replaced by $a - \pi < \arg \hat{f}(n) < a$, $a < \arg \hat{f}(-n) < a + \pi$ $(n \in N)$.

Theorem 4. *Let $f \in C(T)$. If there exists $a \in R$ such that $a - \pi/2 \leqq \arg \hat{f}(n) \leqq a$, $a \leqq \arg \hat{f}(-n) \leqq a + \pi/2$ $(n \in N)$, then $f \in S$.*

Proof. Let $a = 0$. Again consider the functions

$$F(x) = \frac{f(x) + \overline{f(-x)}}{2}, \quad G(x) = \frac{f(x) - \overline{f(-x)}}{2}.$$

Then $\hat{F}(n) = \operatorname{Re} \hat{f}(n) \geqq 0$ for every $n \in Z$ and therefore $F \in A \subset S$ by Theorem A. Proceeding in the same way as in the proof of Theorem 3, we see that $G \in S$. Hence $f = F + G \in S$.

Remark 3. The function

$$f(x) = \sum_{n=2}^{\infty} \frac{\sin nx}{n \log n} \in S \setminus A$$

satisfies the conditions of Theorem 4 but not those of Theorem 1. The function

$$g(x) = \sum_{n=2}^{\infty} \left(\pm \frac{\cos nx}{n^{2/3} \log n} \right),$$

with an appropriate choice of sings $\pm$, provides an example that the separation of coefficients $\hat{f}(n)$ and $\hat{f}(-n)$ $(n \in N)$, included in the conditions of Theorem 4, is significant.

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