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ON EQUIAFFINE WEINGARTEN SURFACES

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0. The H- and K-theorems for surfaces in the equiaffine space A^3 are known, see [1]–[3]. In the spirit of these investigations, I am going to prove the following

Theorem. *Let $M \subset \mathbb{R}^2$ be a bounded connected domain, ∂M its boundary. Let $m: M \to A^3$ be an elliptic surface with the mean curvature* (= die mittlere Affinkrümmung) *H and the curvature K* (= das affine Krümmungsmass). *Let $\Phi(x, y)$ be a function on $\mathbb{R}^2$ satisfying*

(0.1) $$\Phi_x^2 + 4x\Phi_x\Phi_y + 4y\Phi_y^2 > 0\,.$$

Suppose: (i) *on $m(M)$, we have*

(0.2) $$\Phi(H, K) = 0\,;$$

(ii) *the points of $m(\partial M)$ are umbilical. Then $m(M)$ is an affine sphere.*

1. Let $M \subset R^2$ be a bounded domain, ∂M its boundary. Consider a surface $m: M \to A^3$, A^3 being the 3-dimensional equiaffine space. To each point m of our surface, let us associate an equiaffine frame $\{m; v_1, v_2, v_3\}$ such that v_1, v_2 span the tangent plane at m. Then

(1.1) $$\mathrm{d}m = \omega^1 v_1 + \omega^2 v_2\,, \qquad \mathrm{d}v_1 = \omega_1^1 v_1 + \omega_1^2 v_2 + \omega_1^3 v_3\,,$$
$$\mathrm{d}v_2 = \omega_2^1 v_1 + \omega_2^2 v_2 + \omega_2^3 v_3\,, \quad \mathrm{d}v_3 = \omega_3^1 v_1 + \omega_3^2 v_2 + \omega_3^3 v_3$$

with

(1.2) $$\omega_1^1 + \omega_2^2 + \omega_3^3 = 0\,,$$

(1.3) $$\mathrm{d}\omega^i = \omega^j \wedge \omega_j^i\,, \quad \mathrm{d}\omega_i^j = \omega_i^k \wedge \omega_k^j\,.$$

From

(1.4) $$\omega^3 = 0\,,$$

we have

(1.5) $$\omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = 0$$

and the existence of functions g_1, g_2, g_3 such that

(1.6) $$\omega_1^3 = g_1\omega^1 + g_2\omega^2\,, \quad \omega_2^3 = g_2\omega^1 + g_3\omega^2\,.$$

Let us *suppose that our surface is elliptic,* i.e., $g_1 g_3 - g_2^2 > 0$. Then we are able to specialize the frames in such a way that $g_1 = g_3 = 1$, $g_2 = 0$, i.e.,

$$(1.7) \qquad \omega_1^3 = \omega^1 , \quad \omega_2^3 = \omega^2 .$$

From that,

$$(1.8) \qquad (2\omega_1^1 - \omega_3^3) \wedge \omega^1 + (\omega_1^2 + \omega_2^1) \wedge \omega^2 = 0 ,$$
$$(\omega_1^2 + \omega_2^1) \wedge \omega^1 + (2\omega_2^2 - \omega_3^3) \wedge \omega^2 = 0 ,$$

and we have the existence of functions $a, \ldots, d$ such that

$$(1.9) \qquad 2\omega_1^1 - \omega_3^3 = a\omega^1 + b\omega^2 , \quad \omega_1^2 + \omega_2^1 = b\omega^1 + c\omega^2 ,$$
$$2\omega_2^2 - \omega_3^3 = c\omega^1 + d\omega^2 .$$

It may be seen that

$$(1.10) \qquad G := (\omega^1)^2 + (\omega^2)^2$$

is the invariant *equiaffine metric form*. Introduce the 1-form

$$(1.11) \qquad \omega := \tfrac{1}{2}(\omega_1^2 - \omega_2^1) ;$$

then

$$(1.12) \qquad d\omega^1 = -\omega^2 \wedge \omega , \quad d\omega^2 = \omega^1 \wedge \omega ,$$

and we have, from (1.9_2) and (1.11),

$$(1.13) \qquad \omega_1^2 = \tfrac{1}{2}(b\omega^1 + c\omega^2) + \omega , \quad \omega_2^1 = \tfrac{1}{2}(b\omega^1 + c\omega^2) - \omega .$$

From (1.1). (1.9) and (1.13),

$$dm = v_1\omega^1 + v_2\omega^2 ,$$
$$dv_1 - v_2\omega = \{\tfrac{1}{8}[(3a - c) v_1 + (5b + d) v_2] + v_3\} \omega^1 +$$
$$+ \tfrac{1}{8}[(3b - d) v_1 + (a - 3c) v_2] \omega^2 ,$$
$$dv_2 + v_1\omega = \tfrac{1}{8}[(3b - d) v_1 + (a - 3c) v_2] \omega^1 +$$
$$+ \{\tfrac{1}{8}[(a + 5c) v_1 + (3d - b) v_2] + v_3\} \omega^2 ,$$

i.e., the *equiaffine normal vector* is

$$(1.14) \qquad y := \tfrac{1}{2}\Delta m = \tfrac{1}{4}(a + c) v_1 + \tfrac{1}{4}(b + d) v_2 + v_3 .$$

Let us specialize the frames by the condition $y = v_3$. Then $a + c = b + d = 0$ and (1.2) + (1.9) reduce to

$$(1.15) \qquad \omega_1^1 = -\tfrac{1}{2}(c\omega^1 - b\omega^2) , \quad \omega_2^2 = \tfrac{1}{2}(c\omega^1 - b\omega^2) , \quad \omega_3^3 = 0 ,$$
$$\omega_1^2 + \omega_2^1 = b\omega^1 + c\omega^2 .$$

From (1.15_3),

$$(1.16) \qquad \omega_3^1 \wedge \omega^1 + \omega_3^2 \wedge \omega^2 = 0$$

and

$$(1.17) \qquad \omega_3^1 = \alpha\omega^1 + \beta\omega^2 , \quad \omega_3^2 = \beta\omega^1 + \gamma\omega^2 .$$

Using this, the differentiation of $(1.15_{1,2,4})$ yields

(1.18) $$(db - 3c\omega) \wedge \omega^1 + (dc + 3b\omega) \wedge \omega^2 = (\gamma - \alpha)\, \omega^1 \wedge \omega^2 ,$$
$$-(dc + 3b\omega) \wedge \omega^1 + (db - 3c\omega) \wedge \omega^2 = 2\beta\omega^1 \wedge \omega^2 ,$$

and we get the existence of new functions B, C such that

(1.19) $$db - 3c\omega = (B + \beta)\, \omega^1 + (C + \alpha)\, \omega^2 ,$$
$$dc + 3b\omega = (C + \gamma)\, \omega^1 - (B - \beta)\, \omega^2 .$$

From (1.17),

(1.20) $$(d\alpha - 2\beta\omega) \wedge \omega^1 + \{d\beta + (\alpha - \gamma)\, \omega\} \wedge \omega^2 = \{\tfrac{1}{2}b(\alpha - \gamma) + c\beta\}\, \omega^1 \wedge \omega^2 ,$$
$$\{d\beta + (\alpha - \gamma)\, \omega\} \wedge \omega^1 + (d\gamma + 2\beta\omega) \wedge \omega^2 = \{\tfrac{1}{2}c(\alpha - \gamma) - b\beta\}\, \omega^1 \wedge \omega^2 ,$$

and we get the existence of functions $\alpha_1, \ldots, \gamma_2$ satisfying

(1.21) $$d\alpha - 2\beta\omega = \alpha_1\omega^1 + \alpha_2\omega^2 , \quad d\beta + (\alpha - \gamma)\, \omega = \beta_1\omega^1 + \beta_2\omega^2 ,$$
$$d\gamma + 2\beta\omega = \gamma_1\omega^1 + \gamma_2\omega^2 ;$$

(1.22) $$\beta_1 - \alpha_2 = \tfrac{1}{2}b(\alpha - \gamma) + c\beta , \quad \gamma_1 - \beta_2 = \tfrac{1}{2}c(\alpha - \gamma) - b\beta .$$

Finally, from (1.19),

(1.23) $$\{dB - 2(2C + \alpha + \gamma)\, \omega\} \wedge \omega^1 + (dC + 4B\omega) \wedge \omega^2 =$$
$$= (3\varkappa c + \beta_2 - \alpha_1)\, \omega^1 \wedge \omega^2 ,$$
$$(dC + 4B\omega) \wedge \omega^1 - \{dB - 2(2C + \alpha + \gamma)\, \omega\} \wedge \omega^2 =$$
$$= (-3\varkappa b + \gamma_2 - \beta_1)\, \omega^1 \wedge \omega^2 ;$$

here, $\varkappa$ is the *Gauss curvature* of G (1.10) defined by

(1.24) $$d\omega = -\varkappa\omega^1 \wedge \omega^2$$

in accord with (1.12). From (1.23),

(1.25) $$dB - 2(2C + \alpha + \gamma)\, \omega = B_1\omega^1 + B_2\omega^2 , \quad dC + 4B\omega = C_1\omega^1 + C_2\omega^2 ;$$

(1.26) $$C_1 - B_2 = 3\varkappa c + \beta_2 - \alpha_1 , \quad B_1 + C_2 = 3\varkappa b + \beta_1 - \gamma_2 .$$

2. In our notation, we get the following *invariant forms*

(2.1) $$A := -\tfrac{1}{2}\{c(\omega^1)^3 - 3b(\omega^1)^2\, \omega^2 - 3c\omega^1(\omega^2)^2 + b(\omega^2)^3\} ,$$
$$B := -\omega^1\omega_3^1 - \omega^2\omega_3^2 = -\{\alpha(\omega^1)^2 + 2\beta\omega^1\omega^2 + \gamma(\omega^2)^2\} ,$$

the *Pick invariant*

(2.2) $$J = \tfrac{1}{2}(b^2 + c^2)$$

and the *mean curvature* and the *affine curvature*

(2.3) $$H = -\tfrac{1}{2}(\alpha + \gamma) , \quad K = \alpha\gamma - \beta^2$$

resp. A point m of our surface is called *umbilical* if

(2.4) $$H^2 - K = \tfrac{1}{4}(\alpha - \gamma)^2 + \beta^2 = 0$$

at m. From (1.11) and (1.24),

$$\varkappa = \tfrac{1}{2}(b^2 + c^2 - \alpha - \gamma) = J + H, \tag{2.5}$$

this being the theorema egregium.

Suppose that all points of $m(M)$ are umbilical. Then $\alpha - \gamma = \beta = 0$, and (1.21) + + (1.22) implies $\alpha_1 = \ldots = \gamma_2 = 0$. Thus $\alpha = \gamma = $ const., and $m(M)$ is an *affine sphere.*

3. The analytic backround is given, see [4] or [5], by the following result:

On M, introduce coordinates (u, v), and consider the system

$$\begin{aligned} a_{11}\frac{\partial f}{\partial u} + a_{12}\frac{\partial f}{\partial v} + b_{11}\frac{\partial g}{\partial u} + b_{12}\frac{\partial g}{\partial v} &= c_{11}f + c_{12}g, \\ a_{21}\frac{\partial f}{\partial u} + a_{22}\frac{\partial f}{\partial v} + b_{21}\frac{\partial g}{\partial u} + b_{22}\frac{\partial g}{\partial v} &= c_{21}f + c_{22}g; \end{aligned} \tag{3.1}$$

$a_{11} = a_{11}(u, v), \ldots, c_{22} = c_{22}(u, v)$; for the functions $f = f(u, v)$, $g = g(u, v)$. Suppose that the system (3.1) is elliptic, i.e., the quadratic form

$$\begin{aligned} Q := (a_{12}b_{22} - a_{22}b_{12})\,\xi^2 + (a_{11}b_{21} - a_{21}b_{11})\,\eta^2 - \\ - (a_{11}b_{22} - a_{21}b_{12} + a_{12}b_{21} - a_{22}b_{11})\,\xi\eta \end{aligned} \tag{3.2}$$

is definite. If f, g are its solutions satisfying $f = g = 0$ on ∂M, then $f = g = 0$ in M.

On M, we may introduce coordinates (u, v) such that the metric form (1.10) is $G = (r\,du)^2 + (s\,dv)^2$, i.e.,

$$\omega^1 = r\,du, \quad \omega^2 = s\,dv; \quad r = r(u, v) \neq 0, \quad s = s(u, v) \neq 0. \tag{3.3}$$

It is easy to see, from (1.2), that

$$\omega = -s^{-1}r_v\,du + r^{-1}s_u\,dv. \tag{3.4}$$

4. Let us suppose (0.2). Then

$$\Phi_x\,dH + \Phi_y\,dK = 0 \tag{4.1}$$

with, see (2.3) and (1.21),

$$\begin{aligned} dH &= -\tfrac{1}{2}(\alpha_1 + \gamma_1)\,\omega^1 - \tfrac{1}{2}(\alpha_2 + \gamma_2)\,\omega^2, \\ dK &= (\alpha\gamma_1 + \gamma\alpha_1 - 2\beta\beta_1)\,\omega^1 + (\alpha\gamma_2 + \gamma\alpha_2 - 2\beta\beta_2)\,\omega^2. \end{aligned} \tag{4.2}$$

Inserting these into (4.1), we get

$$\begin{aligned} (\Phi_x - 2\gamma\Phi_y)\,\alpha_1 + (\Phi_x - 2\alpha\Phi_y)\,\gamma_1 + 4\beta\Phi_y\beta_1 = 0, \\ (\Phi_x - 2\gamma\Phi_y)\,\alpha_2 + (\Phi_x - 2\alpha\Phi_y)\,\gamma_2 + 4\beta\Phi_y\beta_2 = 0. \end{aligned} \tag{4.3}$$

From (1.21),

$$\begin{aligned} d(\alpha - \gamma) - 4\beta\omega = (\alpha_1 - \gamma_1)\,\omega^1 + (\alpha_2 - \gamma_2)\,\omega^2, \\ d\beta + (\alpha - \gamma)\,\omega = \beta_1\omega^1 + \beta_2\omega^2. \end{aligned} \tag{4.4}$$

Using (3.3) and (3.4),

$$\frac{\partial(\alpha-\gamma)}{\partial u} = r(\alpha_1-\gamma_1)+(\cdot)\,\beta\,, \quad \frac{\partial(\alpha-\gamma)}{\partial v} = s(\alpha_2-\gamma_2)+(\cdot)\,\beta\,, \tag{4.5}$$

$$\frac{\partial\beta}{\partial u} = r\beta_1+(\cdot)(\alpha-\gamma)\,, \quad \frac{\partial\beta}{\partial v} = s\beta_2+(\cdot)(\alpha-\gamma)\,.$$

From this and (1.22),

$$\beta_1 = r^{-1}\frac{\partial\beta}{\partial u}+(\cdot)(\alpha-\gamma)\,, \quad \beta_2 = s^{-1}\frac{\partial\beta}{\partial v}+(\cdot)(\alpha-\gamma)\,, \tag{4.6}$$

$$\alpha_2 = r^{-1}\frac{\partial\beta}{\partial u}+(\cdot)(\alpha-\gamma)+(\cdot)\,\beta\,, \quad \gamma_1 = s^{-1}\frac{\partial\beta}{\partial v}+(\cdot)(\alpha-\gamma)+(\cdot)\,\beta\,,$$

$$\alpha_1 = r^{-1}\frac{\partial(\alpha-\gamma)}{\partial u}+s^{-1}\frac{\partial\beta}{\partial v}+(\cdot)(\alpha-\gamma)+(\cdot)\,\beta\,,$$

$$\gamma_2 = -s^{-1}\frac{\partial(\alpha-\gamma)}{\partial v}+r^{-1}\frac{\partial\beta}{\partial u}+(\cdot)(\alpha-\gamma)+(\cdot)\,\beta\,.$$

Inserting these into (4.3), we get, for

$$f=\alpha-\gamma\,, \quad g=\beta\,, \tag{4.7}$$

a system of the form (3.1) with

$$\begin{aligned} &a_{11}=r^{-1}(\Phi_x-2\gamma\Phi_y)\,, \quad a_{12}=0\,, \quad b_{11}=4r^{-1}\beta\Phi_y\,,\\ &b_{12}=2s^{-1}(\Phi_x-\alpha\Phi_y-\gamma\Phi_y)\,,\\ &a_{21}=0\,, \quad a_{22}=-s^{-1}(\Phi_x-2\alpha\Phi_y)\,, \quad b_{21}=2r^{-1}(\Phi_x-\alpha\Phi_y-\gamma\Phi_y)\,,\\ &b_{22}=4s^{-1}\beta\Phi_y\,. \end{aligned} \tag{4.8}$$

The associated form (3.2) is then

$$\begin{aligned} Q &= 2(\Phi_x-\alpha\Phi_y-\gamma\Phi_y)\,.\\ &\cdot\{s^{-2}(\Phi_x-2\alpha\Phi_y)\,\xi^2-4r^{-1}s^{-1}\beta\Phi_y\xi\eta+r^{-2}(\Phi_x-2\gamma\Phi_y)\,\eta^2\}\,. \end{aligned} \tag{4.9}$$

Its discriminant is

$$\Delta = 4r^{-2}s^{-2}(\Phi_x+2H\Phi_y)^2\,(\Phi_x^2+4H\Phi_x\Phi_y+4K\Phi_y^2)\,. \tag{4.10}$$

We have $\Phi_x+2H\Phi_y \neq 0$. Indeed, $\Phi_x+2H\Phi_y=0$ would mean

$$\Phi_x^2+4H\Phi_x\Phi_y+4K\Phi_y^2 = -4(H^2-K)\,\Phi_y^2 \leqq 0\,,$$

a contradiction to (0.1). Thus $\Delta>0$, the form Q is definite and we have $\alpha=\gamma$, $\beta=0$ in M. Our proof is finished.

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