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TOPOLOGICAL CONVERGENCE AND UNIFORM CONVERGENCE

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1. Introduction. This work was inspired by the recent papers of Beer [1, 2, 3]. Beer studied metric spaces whereas we work in uniform spaces. We make a detailed study of the relationships among uniform convergence (U.C.), uniform convergence on compacta (U.C.C.), pointwise convergence (P.C.) [Kelley [4]], Hasudorff convergence (H.C.) [Beer, [1, 2, 3], Naimpally [6]], Leader convergence (L.C.) [Leader [5], Njåstad [8]], Topological convergence (T.C.) [Beer [1, 2]] proximal convergence (R.C.) [see below]. We provide examples to clarify these relationships and also prove several results.

For General Topology see Kelley [4] and for Proximity Spaces see Naimpally-Warrack [7].

In this paper $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ denote Hausdorff uniform spaces with associated (Efremovič) proximities $\delta_1 = \delta(\mathcal{U})$, $\delta_2 = \delta(\mathcal{V})$ respectively. For the ease in writing proofs, we'll suppose that $\mathcal{U}$, $\mathcal{V}$ contain only symmetric members i.e. $\mathcal{U}$, $\mathcal{V}$ are bases. D denotes a directed set and $(f_n: n \in D)$ a net of functions on X to Y converging to a function $f: X \to Y$. $C(X, Y)$ denotes the set of all continuous functions on X to Y.

1.1. Definition. (Hausdorff Convergence) $f_n \to^{\text{H.C.}} f$ iff for each $U \in \mathcal{U}$, $V \in \mathcal{V}$, there exists an $m \in D$ such that for all $n \geqq m$, and for each $x \in X$, there exist $y, z \in X$ such that $(x. y)$ and (x, z) are both in U and $(f_n(x), f(y))$, $(f(x), f_n(z))$ are both in V. Intuitively H.C. can be looked upon as the convergence of f_n to f in the hyperspace (Hausdorff) uniformity of $X \times Y$ when all functions are viewed as subsets of $X \times Y$, as for example

$$f = \{(x, f(x)): x \in X\} \subset X \times Y .$$

It is easy to show that U.C. implies H.C. and that the converse holds if f is uniformly continuous. In particular, if X is compact, then H.C. = U.C. (For the metric case see Beer [1] and Naimpally [6]).

1.2. Definition. (Leader Convergence) $f_n \to^{\text{L.C.}} f$ iff for each $A \subset X$, $E \subset Y$ if $f(A)$ non δ_2 E, then eventually $f_n(A)$ non δ_2 E.

It is known that U.C. implies L.C. and the converse holds if D is linearly ordered

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or V is totally bounded. (Leader [5], Njåstad [8]). We prove that L.C. implies U.C.C.; in particular, if X is compact then L.C. = U.C.

1.3. Definition. (Proximal Convergence) $f_n \to^{\text{R.C.}} f$ iff for subsets A, B of X, if $f(A)$ non $\delta_2 f(B)$, then eventually $f_n(A)$ non $\delta_2 f_n(B)$.

It is implicit in Leader's proof (see [7]) that L.C. implies R.C. and that R.C. preserves continuity i.e. $f_n \in C(X, Y)$ and $f_n \to^{\text{R.C.}} f$ implies $f \in C(X, Y)$. However, R.C. need not imply P.C. even when $X = Y = \mathbb{R}$ (see (Example 2.4). Obviously, P.C. does not imply R.C.

1.4. Definition. (Topological Convergence) $f_n \to^{\text{T.C.}} f$ iff
(a) for each $x \in X$, there is a net (x_n) such that $x_n \to x$ and $f_n(x_n) \to f(x)$; and
(b) for each subnet $(x_k, f_{n_k}(x_k)) \to (x, y)$, $y = f(x)$.

It is easy to show that H.C. implies T.C. and that T.C. and P.C. are independent. If $X \times Y$ is compact, then T.C. = H.C. = U.C. (for this and further information see Beer [1]).

It is known that if $\{f_n\}$ is eventually equicontinuous and $f_n \to^{\text{P.C.}} f$, then $f_n \to^{\text{U.C.C.}} f$ (Kelley [4]).

2. Examples. In this section we present some examples to clarify the relationships among the various convergences.

2.1. Example. We take $X = Y = \mathbb{R}$ and $f(x) = x^2$. For each $n \in \mathbb{N}$, we set $f_n(x) = (x + n^{-1})^2$. Here f_n converges to f in H.C. and U.C.C. (hence in T.C. and P.C.) and R.C. but not in L.C. or U.C. To see H.C. we observe that the Hausdorff distance between f and f_n is n^{-1} (for $(x, f(x))$ choose $(x - n^{-1}, f_n(x - n^{-1}))$ on f_n and for $(x, f_n(x))$ choose $(x + n^{-1}, f(x + n^{-1}))$ on f). However, $|f_n(n) - f(n)| > 2$ for each $n \in \mathbb{N}$ and so f_n does not converge to f uniformly.

2.2. Example. (Beer [3]). Here $X = \{n^{-1} : n \in \mathbb{N}\} \cup \{0\}$, $Y = [0, 1]$,

$$
\begin{aligned}
f_n(x) &= 1 - k^{-1} \quad \text{for} \quad x = k^{-1}\,, \quad k \leqq n\,, \\
&= 0 \qquad\qquad \text{otherwise}\,. \\
f(k^{-1}) &= 1 - k^{-1}\,, \\
f(0) &= 0\,.
\end{aligned}
$$

Here $f_n \to^{\text{H.C.}} f$ but f is not continuous although each f_n is so. Hence $f_n \not\to^{\text{R.C.}} f$.

2.3. Example. Here we take $X = Y = \mathbb{R}$. For each $n \in \mathbb{N}$, $f_n(x) = nx(1 + n^2x^2)^{-1}$, $f(x) = 0$ for each x. Here $f_n \to^{\text{R.C.}} f$ but f_n does not converge to f in H.C. or T.C. If the limit function is constant, then the convergence is R.C. Since $(n^{-1}, 2^{-1}) \in f_n$ and $\to (0, 2^{-1}) \notin f$, f_n does not converge to f topologically.

2.4. Example. Here we take $X = Y = \mathbb{R}$. For each $n \in \mathbb{N}$, $f_n(x) = x + n$ and $f(x) = x$. Here $f_n \to^{\text{R.C.}} f$ but $f_n \not\to^{\text{P.C.}} f$. Thus R.C. and P.C. are independent.

3. Results. As noted in Section 1, Leader showed that U.C. implies L.C. and that the converse holds if V is totally bounded or f_n is a sequence. Here we show that if $f_n \in C(X, Y)$ and $f_n \to^{\text{L.C.}} f$, then $\{f_n\}$ is eventually equicontinuous. This in turn implies that $f_n \to^{\text{U.C.C.}} f$ and $f_n \to^{\text{T.C.}} f$. So if X is compact, L.C. = U.C. We also show that if X is pseudocompact then on $C(X, \mathbb{R})$, L.C. = U.C.

3.1. Theorem. *Suppose $f_n \in C(X, Y)$ and $f_n \to^{\text{L.C.}} f$; then $\{f_n\}$ is eventually equicontinuous.*

Proof. By Leader's theorem, f is continuous. Let $V \in V$; then there is a $W \in V$ such that $W^4 \subset V$. Since f is continuous at $x \in X$, there is a $U \in U$ such that $f(U(x)) \subset \subset W[f(x)]$. Hence $f(U(x))$ non $\delta_2(Y - W^2[f(x)])$. Since $f_n \to^{\text{L.C.}} f$, eventually $f_n(U(x))$ non $\delta_2(Y - W^2[f(x)])$. So eventually, $f_n(U(x)) \subset W^2[f(x)]$. This in turn implies that eventually, $f_n(U(x)) \subset W^4[f_n(x)] \subset V[f_n(x)]$.

3.2. Corollary. (Kelley [4]). *If $f_n \in C(X, Y)$ and $f_n \to^{\text{L.C.}} f$, then $f_n \to^{\text{U.C.C.}} f$.*

3.3. Remark. Theorem 3.1 shows that if $f_n \to^{\text{L.C.}} f$ then f_n converges to f locally uniformly (or simply uniformly as it is called). Weierstrass proved that if X is compact and f_n converges to f locally uniformly, then $f_n \to^{\text{U.C.}} f$.

3.4. Corollary. *If X is compact, then on $C(X, Y)$,* U.C. = L.C. = H.C.

3.5. Theorem. *If X is pseudocompact, then on $C(X, \mathbb{R})$* U.C. = L.C.

Proof. Suppose $f_n \in C(X, \mathbb{R})$, and $f_n \to^{\text{L.C.}} f$. Then $f \in C(X, \mathbb{R})$ and $f(X) \subset [-p, p]$ for some $p \in \mathbb{R}$. So for $\varepsilon > 0$ there exists a finite set $\{r_i : 1 \leqq i \leqq q\} \subset \mathbb{R}$ such that

$$f(X) \subset \bigcup_{i=1}^{q} S(r_i, \varepsilon/2) .$$

Then $X = \bigcup_{i=1}^{q} A_i$ where $A_i = f^{-1}(S(r_i, \varepsilon/2))$.

Since $f(A_i) = S(r_i, \varepsilon/2)$, eventually $f_n(A_i) \subset S(r_i, \varepsilon)$ as in the proof of 3.1.

So eventually, for each $x \in X$,

$$f_n(x) \in S(f(x), 2\varepsilon) .$$

3.6. Remark. If V is totally bounded, then the above proof can be modified to show that L.C. = U.C. This proof is different from the ones given by Leader [5] or Njåstad [8].

3.7. Theorem. *If $f_n \to^{\text{P.C.}} f$ and $\{f_n\}$ is eventually equicontinuous, then $f_n \to^{\text{T.C.}} f$.*

Proof. P. C. implies 1.4(a). To prove 1.4(b), suppose a subnet $(x_k, f_{n_k}(x_k)) \to (x, y)$. Suppose $V \in V$; then there is a W such that $W^3 \subset V$. Since $\{f_n\}$ is eventually equicontinuous, there is an $m \in D$ and $U \in U$ such that for all $n \geqq m, f_n(U(x)) \subset W[f_n(x)]$ and $f(U(x)) \subset W[f(x)]$.

Since $f_n \to^{\text{P.C.}} f$, we may suppose that for $n \geqq m$, $f_n(x) \in W[f(x)]$. So eventually, $x_k \in U(x)$ and $f_{n_k}(x_k) \in W[y]$, $f_{n_k}(x_k) \in W^2[f(x)]$. So $y \in W^2[f(x)] \subset V[f(x)]$. Since V is arbitrary, $y = f(x)$.

3.8. Corollary. *On* $C(X, Y)$, L.C. *implies* T.C.

3.9. Corollary. *If* X *is locally compact,* $f_n \in C(X, Y)$ *and* $f_n \to^{\text{U.C.C.}} f$, *then* $f_n \to^{\text{T.C.}} f$.

Proof. Follows from the known fact that eventually $\{f_n\}$ is equicontinuous.

3.10. Theorem. *If* X *is discrete, then* P.C. $\Rightarrow$ T.C. *Conversely, if on* $C(X, [0, 1])$ (*or* $C(X, Y)$, *where* Y *contains an arc*) P.C. $\Rightarrow$ T.C., *then* X *is discrete.*

Proof. If X is discrete and $f_n \to^{\text{P.C.}} f$, then $\{f_n\}$ is eventually equicontinuous. So by Theorem 3.4, $f_n \to^{\text{T.C.}} f$. If X is not discrete, there is a net $x_n \to x_0$, $x_n \neq x_0$. For $V \in \mathcal{V}$ if $x_n \in V^2(x_0) - V(x_0)$, then there are functions $h_{n,V}, g_{n,V} \in C(X, Y)$ ($Y = [0, 1]$) such that

$$h_{n,V}(x_0) = 0 \quad \text{and} \quad f_{n,V}(X - V(x_0)) = 1\,,$$
$$g_{n,V}(V(x_0) \cup \{x_n\}) = 0\,, \quad g_{n,V}(X - V^2(x_0)) = 1\,.$$

$f_{n,V} = h_{n,V} - g_{n,V} \to^{\text{P.C.}} f$ where $f(x) = 0$ for each x. But $f_{n,V}(x_n) = 1$, $x_n \to x_0$ and $f(x_0) = 0$. So $f_{n,V} \not\to^{\text{T.C.}} f$.

We conclude with a generalization of Beer's result [2].

3.11. Theorem. *If* X *is locally connected,* Y *rim compact and* $f_n \to^{\text{T.C.}} f$ *in* $C(X, Y)$, *then* $f_n \to^{\text{P.C.}} f$ *and* $\{f_n\}$ *is eventually equicontinuous.*

Proof. Suppose $f_n(x) \not\to^{\text{P.C.}} f$; then there exists a $V \in \mathcal{V}$ such that $f_{n_k}(x) \notin V[f(x)]$ where f_{n_k} is a subnet of f_n. Since Li $f = f$, there is a net $(w_k, f_{n_k}(w_k)) \to (x, f(x))$. Eventually $w_k \in U_k(x)$ which, we may take to be connected and $\{x\} = \bigcap U_k(x)$. Choose $W \in \mathcal{V}$ such that $W \subset V$ and $E = \partial W[f(x)]$ is compact. Eventually, $f_{n_k}(w_k) \in W[f(x)]$; so $f_{n_k}(U_k(x))$ intersects $W(f(x))$ and $Y - \overline{W(f(x))}$. Since $f_{n_k}(U_k(x))$ is connected, eventually Ls $(f_{n_k}(U_k(x)) \cap E \neq \emptyset$. Choose y_0 from the set. Then $(x, y_0) \in$ Ls $f_n - f$, a contradiction.

The above proof is patterned after Beer's; the second part is proved similarly.

3.12. Corollary. *If* X *is locally connected,* Y *is rim compact and* $f_n \to^{\text{T.C.}} f$ *in* $C(X, Y)$, *then* $f_n \to^{\text{U.C.C.}} f$.

3.13. Corollary. *If* X *is a locally connected compact space and* Y *rim compact, then on* $C(X, Y)$ T.C. = U.C.

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