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CLOSURE OPERATORS ON THE LATTICE
OF RADICAL CLASSES OF LATTICE ORDERED GROUPS

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The notion of radical class of lattice ordered groups was introduced in [7]; cf. also [8], [1], [9], [4], [10]. In this note there will be investigated a question proposed by M. Darnel [4] concerning permutability of certain closure operators on the lattice of radical classes of lattice ordered groups.

1. PRELIMINARIES

We recall the basic notions and some notation.

Let \mathcal{G} be the class of all lattice ordered groups. When considering a subclass X of \mathcal{G} we always assume that the zero group $\{0\}$ belongs to X and that X is closed with respect to isomorphisms.

A subclass R of \mathcal{G} is said to be a *radical class* [7] if it is closed with respect to convex l-subgroups and with respect to joins of convex l-subgroups. It is known that every variety of lattice ordered groups is a radical class [6].

Let \mathcal{R} be the collection of all radical classes; \mathcal{R} is partially ordered by inclusion. Then \mathcal{R} is a "complete lattice" in the sense that if \mathcal{R}_1 is a subcollection of \mathcal{R} , then $\sup \mathcal{R}_1$ and $\inf \mathcal{R}_1$ do exist in \mathcal{R} . (Cf. [7].)

1.1. Theorem. (Cf. [4], Thm. 5.1.) *For any radical class R , there exist unique minimal radical classes R^s and R^h , closed with respect to l-subgroups and l-homomorphic images, respectively, that contain R . Moreover, the collection of s-closed and h-closed radical classes form complete lattices under inclusion.*

It is clear that the mappings $R \rightarrow R^s$ and $R \rightarrow R^h$ are closure operators on the lattice \mathcal{R} .

In [4] it is remarked that in a surprising number of cases (though not all), R^{hs} and R^{sh} are varieties and that this indicates that the s-closure and the h-closure might be strongly linked in some way. Next, the question is raised in [4], whether or not the relation

$$(1) \quad R^{sh} = R^{hs}$$

is valid for each radical class R .

Let us denote by \mathcal{R}_1 the collection of all radical classes \mathcal{R} for which the relation (1) fails to hold. In this paper it will be shown that the collection \mathcal{R}_1 is rather large. Namely, the following result will be established:

1.2. Theorem. *There exists an injective mapping of the class of all cardinals into the collection \mathcal{R}_1 .*

2. A CONSTRUCTION

In this section a linearly ordered group G will be constructed which will be applied below in proving Theorem 1.2.

If G_1 is any linearly ordered group, then each subgroup of G_1 is linearly ordered by the induced linear order.

The additive group of all reals (all rational numbers) with the natural linear order will be denoted by R_0 (or by R'_0 , respectively).

For each $i \in R'_0$ let $A_i = R_0$. Next, let A^0 be the lexicographic product

$$A^0 = \Gamma A_i \quad (i \in R'_0)$$

(cf. [5]). The elements of A^0 will be written in the form $a = \langle \dots, a_i, \dots \rangle$ ($i \in R'_0$). The support $S(a)$ of the element a is defined by

$$S(a) = \{i \in R'_0 : a_i \neq 0\}.$$

Let A be the subgroup of A^0 consisting of all elements of A^0 with finite support.

Let $B = R'_0$. For $a \in A$ and $b \in B$ we denote

$$a^b = \langle \dots, a'_i, \dots \rangle \quad (i \in R'_0),$$

where $a'_i = a_{i-b}$ for each $i \in R'_0$.

Let B_0 be the set of all pairs (b, a) with $b \in B$ and $a \in A$. For $(b_i, a_i) \in B_0$ ($i = 1, 2$) we put $(b_1, a_1) \leq (b_2, a_2)$ if either $b_1 < b_2$, or $b_1 = b_2$ and $a_1 \leq a_2$. We define the operation $+$ on B_0 by putting

$$(b_1, a_1) + (b_2, a_2) = (b_1 + b_2, a_1^{b_2} + a_2).$$

Then B_0 turns out to be a linearly ordered group. Let

$$A^{01} = \{(b, a) \in B_0 : b = 0\}.$$

The following assertion follows immediately from the definition of B_0 .

2.1. Lemma. *A^{01} is an l -ideal of B_0 . If K is an l -ideal in B_0 with $\{0\} \neq K \neq B_0$, then $K = A^{01}$.*

Let α be a cardinal, $\alpha > \aleph_0$. Let I_α be the first ordinal with $\text{card } I_\alpha = \alpha$ and let J_α be a linearly ordered set dual to I_α . For each $j \in J_\alpha$ let $C_j = R_0$. Put

$$C_0 = \Gamma C_j \quad (j \in J_\alpha).$$

Let C be the subgroup of C_0 consisting of all elements of C_0 having a finite support.

Put

$$G_0 = C \circ B_0,$$

where \circ denotes the operation of lexicographic product. The elements of G_0 can be written as triples $g = (c, b, a)$ with $c \in C$, $b \in B$ and $a \in A$. Denote

$$f(g) = \sum a_i + \sum c_j \quad (i \in R'_0, j \in J_\alpha).$$

Put

$$(2) \quad G = \{g \in G_0 : f(g) \text{ is an integer}\}.$$

Then G is a subgroup of G_0 ; thus G is a linearly ordered group.

Let J_1 be a subset of J_α such that either $J_1 = \emptyset$ or J_1 is an ideal of the linearly ordered set J_α . Denote

$$\begin{aligned} G^1(J_1) &= \{g = (c, b, a) \in G : c_j = 0 \text{ for each } j \in J_\alpha \setminus J_1\}, \\ G^2 &= \{g = (c, b, a) \in G : c = 0 \text{ and } b = 0\}. \end{aligned}$$

From 2.1 we obtain:

2.2. Lemma. *Both $G^1(J_1)$ and G^2 are l -ideals of G . If K is an l -ideal of G with $\{0\} \neq K \neq G$, then either $K = G^1(J_1)$ for some J_1 or $K = G^2$.*

Also, in view of the definition of G we have:

2.3. Lemma. *Let K_1 be a convex subgroup of G . Then some of the following conditions is satisfied:*

- (i) $K_1 = G^1(J_1)$ for some J_1 .
- (ii) K_1 is a convex subgroup of G^2 .

Lemma 2.3 implies:

2.4. Lemma. *Let G' be a linearly ordered group. Suppose that there exist subgroups G'_i ($i \in I$) of G' such that*

- (i) $G = \bigcup_{i \in I} G'_i$,
 - (ii) for each G'_i there exists a convex subgroup of G which is isomorphic to G'_i .
- Then G' is isomorphic to a convex subgroup of G .*

Let R be the radical class of lattice ordered groups generated by the linearly ordered group G .

From 2.4 and Theorem 3.4, [8] we infer:

2.5. Lemma. *The radical class R is the class of all lattice ordered groups which can be expressed (up to isomorphism) as direct sums of some convex subgroups of G .*

Now we shall construct a linearly ordered group H_2 belonging to R^{sh} .

Denote

$$H = \{g = (c, b, a) \in G : b = 0\}.$$

Then H is a subgroup of G , whence $H \in R^s$. Let I be an ideal of the linearly ordered set R'_0 such that $I \neq R'_0$. Put

$$H_1 = \{g = (c, b, a) \in H : c = 0 \text{ and } a_i = 0 \text{ for each } i \in R'_0 \setminus I\}.$$

H_1 is an l-ideal of the linearly ordered group H . In view of (2) we obtain:

2.6. Lemma. *The linearly ordered group $H_2 = H/H_1$ is isomorphic to the linearly ordered group*

$$C = \Gamma A_i \quad (i \in R'_0 \setminus I).$$

For any subclass X of \mathcal{G} we denote by

Sub X – the class of all l-subgroups of lattice ordered groups belonging to X ;

Hom X – the class of all homomorphic images of lattice ordered groups belonging to X .

Let Y be the class of all linearly ordered groups K having the property that K is isomorphic to some convex subgroup of G . From the construction of G and from 2.6 we obtain

2.7. Lemma. *The linearly ordered group H_2 does not belong to the class Sub Hom Y .*

Clearly $H_2 \in R^{sh}$. Moreover, H_2 contains a strong unit (cf. [5]).

3. THE RADICAL CLASS R^{hs}

3.1. Lemma. *Let K_m ($m \in M$) be lattice ordered groups and let $K = \sum_{m \in M} K_m$. Let K_0 be an l-ideal of K and for each $m \in M$ let K_{0m} be the projection of K_0 into K_m . Then the lattice ordered group K/K_0 is isomorphic to the direct sum $\sum_{m \in M} K_m/K_{0m}$.*

The proof is easy.

From 3.1 and 2.5 we obtain:

3.2. Lemma. *The class Hom R is the class of all lattice ordered groups which can be expressed (up to isomorphism) as direct sums of linearly ordered groups belonging to Hom Y .*

For any lattice ordered group L we denote by $c(L)$ the system of all convex l-subgroups of L ; the system $c(L)$ is partially ordered by inclusion. In fact, $c(L)$ is a complete lattice. The lattice operations in $c(L)$ will be denoted by \bigvee^c and \bigwedge^c . (The operation \bigwedge^c coincides with the set-theoretic intersection.)

Let H_2 be as in Section 2.

3.3. Lemma. *Let K_m ($m \in M$) be linearly ordered groups belonging to $c(H_2)$ such that $\bigvee_{m \in M}^c K_m = H_2$. Then there is $m \in M$ such that $K_m = H_2$.*

For any $X \subseteq \mathcal{G}$ we denote by X_r the radical class generated by X . From 3.2 and Proposition 5.5, [4] it follows:

3.4. Lemma. $R^h = (\text{Hom } Y)_r$.

Now since Hom Y is a class of linearly ordered groups, $(\text{Hom } Y)_r$ can be obtained by means of Thm. 3.4 in [8]; from this theorem, from 3.3 and 2.7 we infer:

3.5. Lemma. *The linearly ordered group H_2 does not belong to the radical class R^h .*

3.6. Lemma. Let $D_i (i \in I)$ be lattice ordered groups. Suppose that K is an l -subgroup of the direct sum $\sum_{i \in I} D_i$ such that (i) K is a linearly ordered group, and (ii) K has a strong unit. Then there exists $i \in I$ such that the projection $k \rightarrow k_i$ is an isomorphism of K into D_i .

Proof. Let e be a strong unit in K . Put $I_1 = \{i \in I : e_i \neq 0\}$. The set I_1 is finite. For each $k \in K$ there exists a positive integer n such that $-ne \leq k \leq ne$. Hence if $i \notin I_1$, then $k_i = 0$. Therefore K is an l -subgroup of $\sum D_i (i \in I_1)$.

There exists a minimal subset I_2 of I_1 having the property that the mapping

$$(3) \quad k \rightarrow \langle \dots, k_i, \dots \rangle \quad (i \in I_2)$$

is an isomorphism. Assume that $\text{card } I_2 \geq 2$. Choose $i_2 \in I_2$. Then the mapping

$$k \rightarrow k_{i_2}$$

is a homomorphism of K into D_{i_2} , but it fails to be an isomorphism. Thus there is $0 < k' \in K$ such that

$$(4) \quad k'_{i_2} = 0.$$

Put $I_3 = I_2 \setminus \{i_2\}$. The mapping

$$(5) \quad k \rightarrow \langle \dots, k_i, \dots \rangle \quad (i \in I_3)$$

is a homomorphism of K into $\sum D_i (i \in I_3)$, but (in view of the minimality of I_2) the mapping (5) fails to be an isomorphism. Hence there is $0 < k'' \in K$ such that

$$(6) \quad k''_i = 0 \quad \text{for each } i \in I_3.$$

We distinguish two cases.

a) $k'' \leq k'$. Then from (4) we infer that $k''_{i_2} = 0$, hence $k'_i = 0$ for each $i \in I_2$; in view of (3) we arrive at a contradiction.

b) $k' < k''$. Then according to (6) we have $k'_i = 0$ for each $i \in I_3$ and hence $k'_j = 0$ for each $j \in I_2$. This contradicts the fact that the mapping (3) is an isomorphism. Therefore $\text{card } I_2 = 1$, which completes the proof.

Put $Q = R^h$. Let Q'_1 be the class of all l -subgroups of elements of Q and let $Q_1 = (Q'_1)_r$. Define $Q'_2, Q_2, Q'_3, Q_3, \dots$ analogously. Then we have (cf. [4], Section 5)

$$(7) \quad Q^s = \bigvee Q_i \quad (i = 1, 2, \dots).$$

Denote $Q_0 = Q$.

Let us denote by R_d the class of all lattice ordered groups having the property that each upper bounded disjoint subset is finite.

In view of Thm. 3.4, [8] we have $R \subseteq R_d$. Then from 3.4 we obtain $R^h \subseteq R_d$. Finally, from [4], Lemma 5.4 and from (7) we get that the following lemma is valid:

3.7. Lemma. $Q^s \subseteq R_d$.

3.8. Lemma. Let $K' \in R_d$. Let K and $K_i (i \in I)$ be elements of $c(K')$ such that $K = \bigvee_{i \in I} K_i$. Suppose that K has a strong unit. Then there exists a finite subset $\{K_j\}_{j \in J}$ of $c(K)$ such that (i) for each $j \in J$ there exists $i \in I$ with $K_j \in c(K_i)$, and (ii) $K = \sum_{j \in J} K_j$.

Proof. Let e be a strong unit of K . Then there exists a finite subset I_1 of I and for each $i \in I_1$ there exists $0 < e_i \in K_i$ such that $e = \sum_{i \in I_1} e_i$. Then we have $K = \bigvee_{i \in I_1}^c K_i$. For each $i \in I_1$ let K'_i be the convex l-subgroup of K_i generated by the element e_i . The relation

$$(8) \quad K = \bigvee_{i \in I_1}^c K'_i$$

is valid and for each $i \in I_1$, e_i is a strong unit in K'_i .

Let $i \in I_1$. Then each disjoint subset of K'_i is finite. Hence according to [2] (cf. also [5], Chap. V, Section 6) K_i can be expressed as a direct sum of a finite number of lattice ordered groups K''_{it} such that each K''_{it} is a nontrivial lexico extension, i.e., $K''_{it} = \langle K'_{it} \rangle$, $K''_{it} \neq K'_{it}$.

Let us consider two such l-groups $K'_{i_1 t_1}$ and $K'_{i_2 t_2}$. Both of them belong to $c(K)$. In view of [3], Propos. 2.9 we have two possibilities:

- (i) $K'_{i_1 t_1}$ is comparable with $K'_{i_2 t_2}$,
- (ii) $K'_{i_1 t_1} \cap K'_{i_2 t_2} = \{0\}$.

Hence we can choose a finite number of these l-subgroups K'_i which will be denoted as K'_j ($j \in J$; J finite) such that (cf. (8))

$$K = \bigvee_{j \in J}^c K'_j$$

and the system $\{K'_j\}_{j \in J}$ is disjoint. This implies that $K = \sum_{j \in J} K'_j$.

3.9. Lemma. H_2 does not belong to Q^s .

Proof. By way of contradiction, assume that H_2 belongs to Q^s . Hence in view of (7) and Lemma 3.3 there exists a positive integer i such that $H_2 \in Q_i$. Let i be the least positive integer having this property.

Suppose that $i = 1$. According to 3.3 and Lemma 5.4 in [4] we must have $H_2 \in Q'_1$. Hence there is $K \in Q_0 = Q$ such that H_2 is isomorphic to an l-subgroup H'_2 of K ; without loss of generality we can suppose that $H'_2 = H_2$. Let K_1 be the convex subgroup of K generated by the element e . Clearly $K_1 \in Q$.

According to 3.2 and 3.4 the radical class Q is generated by a class of linearly ordered groups. Thus in view of Propos. 3.4, [8] K_1 is a direct sum of linearly ordered groups D_i ($i \in I$). Moreover, H_2 is an l-subgroup of K_1 . Each D_i belongs to Q . From 3.6 we conclude that there is $i \in I$ such that H_2 is isomorphic to D_i . Therefore H_2 belongs to Q , which is a contradiction (cf. 3.5).

Now suppose that $i > 1$. Then according to 3.3 and Lemma 5.4 in [4] we have $H_2 \in Q'_i$. Thus there exists $K \in Q_{i-1}$ such that $H_2 \in \text{Sub}\{K\}$. We may suppose that H_2 is an l-subgroup of K . Let K_1 be the convex l-subgroup of K generated by the element e . Because Q_{i-1} is a radical class, we have $K_1 \in Q_{i-1}$. At the same time, H_2 is an l-subgroup of K_1 .

There exist $K_i \in Q'_{i-1}$ ($i \in I$) such that $K_1 = \bigvee_{i \in I} K_i$, $K_i \in c(K_1)$ for each $i \in I$. We apply 3.7 and 3.8; let K_j ($j \in J$) be as in 3.8. Then all K_j belong to Q'_{i-1} as well. According to 3.8 and 3.6 there exists $j \in J$ such that H_2 is isomorphic to an l-subgroup of K_j . At the same time, there exists $K'_j \in Q_{i-2}$ such that K_j is an l-subgroup

of K'_j . Hence H_2 is isomorphic to an l-subgroup of K'_j and therefore $H_2 \in Q'_{i-1}$, which is a contradiction with respect to the minimality of i .

Since $Q^s = R^{hs}$, from 3.9 and from the relation $H_2 \in R^{sh}$ we obtain:

3.10. Proposition. $R^{sh} \neq R^{hs}$.

In view of the construction introduced in Section 2, the linearly ordered group G and the radical class R were defined by means of a cardinal α , $\alpha > \aleph_0$. Let us now write $G(\alpha)$ and $R(\alpha)$ instead of G and R .

By using Lemma 5.4 in [4] we can easily verify that if α and β are cardinals with $\aleph_0 < \alpha < \beta$, then $G(\beta)$ does not belong to $R(\alpha)$. As a corollary we obtain:

3.11. Proposition. *Let α and β be cardinals, $\aleph_0 < \alpha < \beta$. Then $R(\alpha) \neq R(\beta)$.*

Let C be the class of all cardinals greater than \aleph_0 . In view of 3.11 there exists an injective mapping of the class C into the class \mathcal{R}_1 (cf. Section 1 for the notation); from this we infer that Theorem 1.1 is valid.

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